# 2022

## 1st Semester Examination

## APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

### Paper : MTM - 101

### (Real Analysis)

Full Marks : 40 Time : Two Hours

*The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.*

1. Answer any ***four*** questions : 2×4=8

(a) Show that subtraction of two complex measurable functions on a measurable set $X$ is measurable.

(b) Let $X$ be a measurable space and $\chi_E : X \to \mathbb{R}$ be a measurable function where $\chi_E(x) = \begin{cases} 1 \text{ } if \text{ } x \in E \\ 0 \text{ } if \text{ } x \notin E \end{cases}$. Is $E$ a measurable set in $X$?

(c) Define Borel set.

(d) Show that the set of all rational numbers is a null subset of $\mathbb{R}$.

(e) For any set $E \subseteq \mathbb{R}, c \in \mathbb{R}$ prove that $m^{*}(E+c) = m^{*}(E)$.

(f) For every $\in > 0$ and $f \in L^{1}(\mu)$, show that $\mu\{x \in X : |f(x)| \geq \in\} \leq \frac{1}{\in}\int f \, d\mu$.

2. Answer any ***four*** questions : 4×4=16

(a) Establish a necessary and sufficient condition for a function $f:[a, b] \to \mathbb{R}$ to be a function of bounded variation on $[a, b]$. 4

(b) Show that the function $f(x)$ defined on [2, 5] by

$$f(x) = \begin{cases} 3, \text{ for all rationals } x \text{ in } [2, 5] \\ 4, \text{ for all irrationals } x \text{ in } [2, 5] \end{cases}$$

is not a function of bounded variation on [2, 5]. 4

(c) If the function $f:[a, b] \to \mathbb{R}$ is continuous and $\phi:[a, b] \to \mathbb{R}$ is bounded variation, show that RS-integral $(RS) - \int_{a}^{b} f(x) d\phi(x)$ exists. Hence show that for any continuous function $f:[a, b] \to \mathbb{R}$ the Riemann integral $(R) - \int_{a}^{b} f(x) dx$ exists. 3+1

(d) Show that every finite sum of real numbers can be expressed as the R-S integral over some interval. 4

(e) Let $f_n : X \to \mathbb{R}^*$ be measurable for $n = 1, 2, 3, ....$ Then show that $\liminf_{n\to\infty} f_n$ and $\inf_{n\to\infty} f_n$ are measurable functions on $X$. 4

(f) If $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, ...,$ and $f(x) = \sum_{n=1}^{\infty} f_n(x), x \in X$, then show that

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu .$$ 4

3. Answer any ***two*** questions : 8×2=16

(a) (i) Let $\mu$ be a measure on a $\sigma$-algebra $\mathfrak{M}$. Then show that $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$. 4

(ii) Let $\mu$ be a measure on a a $\sigma$-algebra of subsets of $X$. Show that the outer measure $\mu^*$ induced by $\mu$ is countably subadditive. 4

(b) (i) Let $f(x)$ be defined as $f(x) = \frac{1}{x^{\frac{4}{5}}}$ if $0 < x \le 1$ and $f(0) = 0$. Show that $f$ is Lebesgue integrable on [0, 1]. Also compute the integral. 6

(ii) Evaluate the following : 2

$$\int_1^4 (2x^2 + 3) d([x] + 2).$$

P.T.O.

(c) (i) Show that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal in this case. 4

(ii) Let $\{E_k\}$ be a sequence of measurable sets in $X$, such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then prove that almost all $x \in X$ lie at most finitely many of the sets $E_k$. 4

(d) (i) Show that every bounded measurable function on $[a, b]$ is Lebesgue integrable on $[a, b]$. 5

(ii) Show that the Cantor set is an uncountable set. 3