Chapter 5

Two-person zero-sum game through artificial neural network structures*

Game theory has a tremendous scope in decision making process; and consequently decision makers' hesitant characters play an important role in it. In this chapter, a game situation is clarified under artificial neural network through logic-gate switching circuit in hesitant fuzzy environment with a suitable example; and this concept can be applied in future for real-life situations.

5.1 Motivation

Introducing neuro-fuzzy concept in decision making problems, make a new way in artificial intelligence and expert systems. Sometimes, neural networks are used to optimize certain performances. In general, knowledge acquisition becomes difficult when problem's variables, constraints, environment, decision maker's attitude and complex behavior are encountered with. A sense of fuzziness prevails in these situations, sometimes numerically and sometimes linguistically. Neural networks (or neural nets) help to overcome this problem. Neural networks are explicitly and implicitly hyped to draw out fuzzy rules from numerical information and linguistic information. Logic-gate and switching circuit mobilize the fuzzy data in crisp environment and can be used in artificial neural network (ANN), also. Motivated by these facts, we reconstruct two-person zero-sum game under ANN structure.

5.2 Introduction

Artificial neural frameworks or artificial neural nets are physical cell frameworks generally able to acquire, store and use exploratory information towards knowledge. The learning is acquired in stable mappings inserted in network framework. Neurons or nodes are the basic unit or element of net. In brain-neuron system, i.e., in neural net systems, activity starts at networks' polarization, then the firing rate of neuron is considered through a set of input connections using synapses on cells and the corresponding dendrite; then neurons are given internal resting space and consequently neuron's axonal projections are done. In ANNs, every processing element is marked

* Some selective parts of this chapter are communicated to an International Journal.

by an activity level, an output cost or value, a group of input links, a bias cost, i.e., an artificial resting stage of correspopnding neuron, and a bunch of output links. Each of these characteristics of the unit is expressed mathematically by means of real numbers. Thus, every connection possesses weight, may be positive or negative, i.e., synaptic influence which decides the impact of the approaching contribution on the enactment level of the unit. Weights determine excitatory or inhibitory initiation. ANN may be classified as the generalization of brain-style computational methods in mathematical sciences, mainly in applied sciences. McCulloch and Pitts [94], Hebb [50] originated the idea of brain-style computation. Minsky and Papert [97] proposed artificial intelligence as symbol processing and it became a dominant theme in artificial intelligence. Because of the vulnerability, imprecision attributes of frameworks in question, and the vagueness, ambiguity of adjudications of decision players, we understand inclusion of aversion, hesitance environments in game problems. Neural nets have been applied in fuzzy logic-systems, soft-computing, function approximation, fuzzy modeling, etc., but hybrid-neural net has not been applied in matrix game using logic-gate switching circuits. The main aims of this study are-to construct a game model using artificial neural nets, to apply switching circuit gates in neural nets, to compute a quick geometric way for defuzzification of a set of hesitant fuzzy elements.

5.3 Basic Concepts

In this section, we recall the basic concepts of **Definition 3.3.2** and arithmetic operations of TIFNs. In fuzzy sets, t-norm and t-conorm are two sorts of operations. They are otherwise called as triangular-norm and triangular-conorm, respectively. In this section, triangular norm and conorm, complement of fuzzy set, and hesitant fuzzy set with definitions, properties are discussed.

Definition 5.3.1 ***Triangular norm*** *[43]: A mapping T is a triangular norm such that,* $T : [0,1] \times [0,1] \rightarrow [0,1],\ \forall\, x,\ y,\ x_1,\ y_1,\ z \in [0,1],$ *with the following conditions as axioms:*

(i) $T(x,0) = 0, T(x,1) = x;$ *Boundary condition.*

(ii) $T(x,y) = T(y,x);$ *Condition for symmetricity.*

(iii) $T(x,T(y,z)) = T(T(x,y),z);$ *Condition for associativity.*

(iv) $T(x,y) \leq T(x_1,y_1)$ *if* $x \leq x_1, y \leq y_1;$ *Condition for monotonicity.*

Definition 5.3.2 ***Triangular co-norm*** *[43]: A mapping T is a triangular conorm such that,* $T : [0,1] \times [0,1] \rightarrow [0,1],\ \forall\, x,\ y,\ x_1,\ y_1,\ z \in [0,1],$ *with the following conditions as axioms:*

(i) $T(x,0) = x, T(x,1) = 1;$ *Boundary condition.*

(ii) $T(x,y) = T(y,x);$ *Condition for symmetricity.*

(iii) $T(x,T(y,z)) = T(T(x,y),z);$ *Condition for associativity.*

(iv) $T(x,y) \leq T(x_1,y_1)$ *if* $x \leq x_1, y \leq y_1;$ *Condition for monotonicity.*

Definition 5.3.3 ***Complement of a fuzzy set:*** *Considering a fuzzy sentence p; we describe its complement as some sentences fulfilling the uniformity: $M^c(p) = W - M(p)$, where $M^c(p)$ means $M(p)$'s complementary set; W is the entire set of sentences; M is a membership function that partners p with the members of $M(p)$.*

Definition 5.3.4 ***Hesitant Fuzzy Set (HFS)*** *[140]: Based on reference set X, A_{HF} is defined to be a hesitant fuzzy set described by the function $h_A(x)$. Here, $h_A(x)$ is applied to X and gives a subset of $[0, 1]$, i.e., $A_{HF} = \{\langle x, h_A(x)\rangle : x \in X\}$ where $h_A(x)$ is named as hesitant fuzzy element (HFE), an essential unit of HFS, is a set fitted with various merits in $[0, 1]$ represent the conceivable membership degrees to component $x \in X$.*

Example 5.3.1 *$A_{HF} = \{\langle x_1, 0.1, 0.4, 0.3\rangle, \langle x_2, 0.3, 0.35\rangle, \langle x_3, 0.2, 0.4, 0.6, 0.69, 0.8\rangle\}$ is a HFS; $\{x_1, x_2, x_3\} \in X$, a reference set and $h_A(x_1) = \{0.1, 0.4, 0.3\}$, $h_A(x_2) = \{0.3, 0.35\}$, $h_A(x_3) = \{0.2, 0.4, 0.6, 0.69, 0.8\}$ are hesitant fuzzy elements.*

Property 5.3.1 *Considering h, h_1 and h_2 as three HFEs, a few tasks are characterized by Torra [140] as pursues:*

(i) $h^c = \{1 - \gamma : \gamma \in h\}$, *complement of* h;

(ii) $h_1 \cup h_2 = \{\gamma_1 \vee \gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$;

(iii) $h_1 \cap h_2 = \{\gamma_1 \wedge \gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$;
Furthermore, in order to aggregate hesitant fuzzy information, Xu and Xia [160] defined some new operations on h, h_1 and h_2 with $\lambda > 0$ as below:

(iv) $h_1 \oplus h_2 = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$;

(v) $h_1 \otimes h_2 = \{\gamma_1\gamma_2 : \gamma_1 \in h_1, \gamma_2 \in h_2\}$;

(vi) $h^\lambda = \{\gamma^\lambda : \gamma \in h\}$;

(vii) $\lambda h = \{1 - (1 - \gamma)^\lambda : \gamma \in h\}$.

To compare magnitudes of HFEs, Xia and Xu [161] defined the following laws of comparison:
For any HFE h, the score function of h is $S(h) = \sum_{\gamma \in h}^{e_h} \frac{\gamma}{|e_h|}$, where e_h is the set of all elements in h, $|e_h|$ denotes the cardinality of e_h.
For any two HFEs, h_1 and h_2, $S(h_1) > S(h_2)$ implies that $h_1 > h_2$; $S(h_1) < S(h_2)$ implies that $h_2 > h_1$; otherwise, $S(h_1) = S(h_2)$ implies that $h_1 = h_2$.
This type of variables, called *linguistic variables*, was introduced by Zadeh [168].

5.4 Mathematical Model

In this section, mathematical model related to ANN based fuzzy matrix game is discussed from classical point of view. For this purpose, fuzzy hybrid neural-net, max fuzzy neuron, min fuzzy neuron and fuzzy logic-gate switching circuit are considered.

5.4.1 Classical matrix game

In this part, we describe some basics on classical game theory. A matrix game is communicated as $A = (a_{ij})$ $(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q)$ with components as real numbers and the corresponding matrix is termed as payoff matrix. We think about two players. Players I and II play row i and column j, individually and the results to players I and II are a_{ij} and $-a_{ij}$, respectively in case of zero-sum concept. Strategies that advantage player's individual adjustments are picked by players. Expecting the game with arrangement of unadulterated techniques S_1 and S_2 and that of blended or mixed strategies X and Y for players I and II individually, we characterize, $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_p\},\ S_2 = \{\beta_1, \beta_2, \ldots, \beta_q\}, X = \{(X_1, X_2, \ldots, X_p)^T : \sum_{i=1}^{p} X_i = 1,\ X_i \geq 0,\ i = 1, 2, \ldots, p\}, Y = \{(Y_1, Y_2, \ldots, Y_q)^T : \sum_{j=1}^{q} Y_j = 1,\ Y_j \geq 0,\ j = 1, 2, \ldots, q\}$. Here X_i $(i = 1, 2, \ldots, p)$ and Y_j $(j = 1, 2, \ldots, q)$ are probabilities in which the players I and II sort-out their pure strategies $\alpha_i \in S_1$ $(i = 1, 2, \ldots, p)$ and $\beta_j \in S_2$ $(j = 1, 2, \ldots, q)$ individually and game is enunciated as $G \equiv (X, Y, A)$.

Basically, we wish to get the most favourable strategy(ies) for players' and the value of considered game. The characterization of the estimation of a game depends on ensuring the maximum profit to the maximizing player I or the minimum conceivable loss to the minimizing player II; and here the best strategic procedures are utilized by both players. If a player records the most exceedingly awful potential results of all things considering his or her prospective strategies, the individual in question will opt for the most reasonable strategy to be fitted for the concerned person. The idea accords with the principle of minimax and maximin. A saddle point comes to exist while maximin for player I equals to minimax for player II.

Expect that player I uses any mixed strategy from X. Clearly, player I's normal increase floor is $\min(X^t AY)$ and if shortly be denoted by v, we need to maximize v, state v^* for certain $X^* \in X$, as $v(X^*) = \max(\min\{\sum_{i=1}^{p} a_{ij} X_i : j = 1, 2, \ldots, q\})$. Such X^* and v^*, respectively called player I's maximin strategy and game-value, are obtained from the accompanying LPP in Model 1 as:

Model 1

$$\text{maximize} \quad v \tag{5.1}$$

$$\text{subject to} \quad \sum_{i=1}^{p} a_{ij} X_i \geq v \ \ (j = 1, 2, \ldots, q), \tag{5.2}$$

$$\sum_{i=1}^{p} X_i = 1, \tag{5.3}$$

$$X_i \geq 0 \ \ (i = 1, 2, \ldots, p). \tag{5.4}$$

What's more, with same contention, player II's optimal or minimax strategy, say $Y^* \in Y$, and game value, state w^* are depicted from Model 2 as:

Model 2

$$\text{minimize} \quad w \tag{5.5}$$

$$\text{subject to} \quad \sum_{j=1}^{q} a_{ij} Y_j \leq w \ \ (i = 1, 2, \ldots, p), \tag{5.6}$$

$$\sum_{j=1}^{q} Y_j = 1, \tag{5.7}$$

$$Y_j \geq 0 \ \ (j = 1, 2, \ldots, q). \tag{5.8}$$

5.4.2 Neural network model

Biological network system: A typical neuron or nerve cell belongs to the vertebrate nervous system which contains the nucleus (genetic informer) and offers to two sorts of cell processes, *axon* and *dendron*. Axon acts as transmitting element or output element whereas dendron as input element. Branches of the axon of one neuron communicating with signals to other neuron at a site is called the synapse. Synapses are the elementary signal processing devices. Though the brain with its nervous system makes up for the slow rate of operation with a few factors, the brain is an exceptionally perplexing, non-linear, parallel data handling framework. From early stage of childhood to adult stage, synapses are modified gradually through the learning process. And these motivate the scientists to use neural networks and the related sciences in artificial intelligence like pattern recognition, perception, and motor controlling in fuzzy sets and systems. Thus neural networks motivate to generate fuzzy rules from different examples [62; 136; 149] with different aspects.

Artificial neural network: In mathematics, biological structures of neural systems influence mathematical modeling to construct network functions as forward and backward calculations. And this leads to ANN. ANN was found having its roots almost 75 years ago in the works of McCulloch and Pitts [94] and later by others [20; 21; 68].

Definition 5.4.1 ***Artificial neural network (ANN)*** *[43]: ANNs are physical cell frameworks which can acquire, store and use experimental information, knowledge and complex utilitarian relations by summing up from a restricted amount of preparing data.*

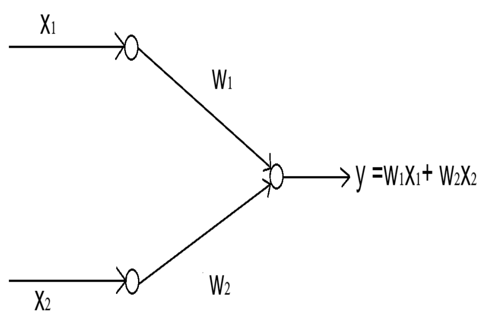

Figure 5.1: A simple neural network model.

Definition 5.4.2 ***Hybrid neural-net*** *[43]: In a simple net, as picturesquely in Fig. 5.1, all input criteria like signals as well as weights are reals. Signals interact with weights and pass through one to another layer using sigmoidal function. This straightforward neural net with increase, expansion, and sigmoidal function is called regular (or standard) neural net. If triangular-norm, triangular-conorm or their combination are employed and used in next layer we call it hybrid neural net.*

Definition 5.4.3 ***Fuzzy hybrid neural-net***: *When weights are crisp and signals are fuzzy then hybrid neural net is termed as fuzzy hybrid neural net. A fuzzy hybrid neural net may not use multiplication, addition and sigmoidal function.*

In ANN, signal flows or transfers on the basis of the net's activity, sometimes, termed as an activation or transfer function. The output of the flow persists if the value of activation function remains greater than some parameters, say, the threshold-level.

Definition 5.4.4 ***Max fuzzy neuron*** *[68]: The signal X_i interfaced with the weight W_i produces $p_i = W_iX_i, i = 1,2$. The input value p_i is aggregated utilizing the most extreme conorm $z = \max\{p_1, p_2\} = \max\{W_1X_1, W_2X_2\}$ and the j-th yield of the neuron is given by $y_j = g_j(f(z - \theta)) = g_j(f(\max\{W_1X_1, W_2X_2\} - \theta))$, where f is an initiation capacity and θ is known as the threshold-level.*

Definition 5.4.5 ***Min fuzzy neuron*** *[68]: The signal X_i communicated with the weight W_i produces $p_i = W_iX_i, i = 1,2$. The input value p_i is amassed utilizing the minimum norm $z = \min\{p_1, p_2\} = \min\{W_1X_1, W_2X_2\}$ and the j-th output of neuron is processed by $y_j = g_j(f(z - \theta)) = g_j(f(\min\{W_1X_1, W_2X_2\} - \theta))$, where f and θ are the same as in* ***Definition 5.4.4***.

Here, we consider Max fuzzy neuron and Min fuzzy neuron with t-norm and t-conorm in processing of the problem's solution. Max fuzzy neuron and Min fuzzy neuron nets are given in Fig. 5.2.

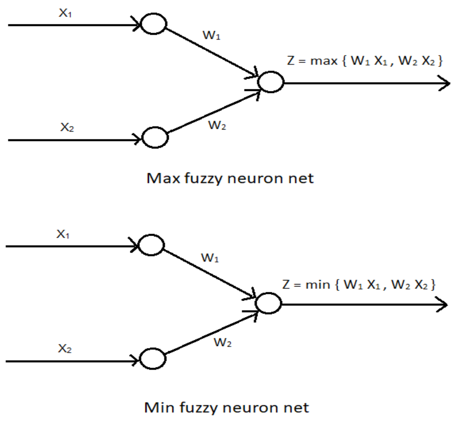

Figure 5.2: Max fuzzy and Min fuzzy neuron nets.

5.4.3 Logic-gate switching circuit

In algebra of switching circuits, electrical or electronic switching circuits are depicted mathematically or planned to get an outline for circuit having some criteria. In switching circuits, we can consider conductor-nonconductor, charged-uncharged, decidedly and contrarily polarized components. These days, semi-conductor components in switching circuits have more importance. In specific situation, switches are meant as so-called *gates*, or combination of gates. This can be treated as emblematic portrayal. In this way a gate (or combination of gates) is a polynomial p which has the elements x_i for each i. We depict the gate as an acknowledgement of a switching function. In the event that, as for worth, $p = 1$ (or 0), we have current (or no current) through switching circuit p. Examples of switching gates with properties as output, are given in Fig. 5.3 (X_is are treated here as input variables).

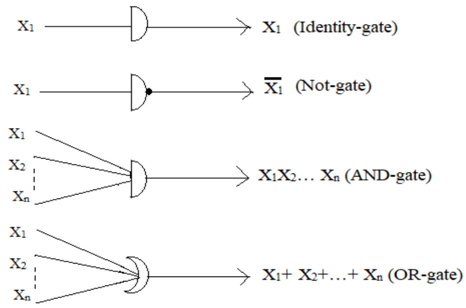

Figure 5.3: Examples of some special gates.

Since, human thinking, nowadays are not confined within 1-0 logical concept, a consequent fuzzy approach tends the switching circuit output towards linguistic variables like fuzzy sets. So the output in switching circuit also collaborates the crisp and fuzzy concept.

5.4.4 Fuzzy logic-gate switching circuit oriented artificial neural network (FLGSCANN) model

Here, we discuss the steps algorithmically to collaborate the fuzzy data through the artificial neural net. From the collected data, the required optimal value is obtained applying the following **Algorithm 5**.

5.4.5 ANN based fuzzy matrix game

In ANNs, weighted interconnections are established by mathematical formulation, termed as rules. Rules are basically governed by two approaches, crisp approach and fuzzy approach. A mathematical model when uses fuzzy set in a way is termed as a fuzzy approach of model rather than the crisp set oriented approach. When ambiguous, uncertain, imprecise conditions are applied in *if-then* relationship between the variables of the rules in ANNs, Fuzzy Neural Networks (FNNs) are originated. FNNs are classified by using two models namely, Mamdani model [92] and Takagi-Sugeno model [136; 138], depend on the structures of *if-then* rules. If the antecedent (*if* part) and the consequent (*then* part) are fuzzy propositions like:
R_i : If x is A_i then y is $B_i, i = 1(1)k$, where A_i and B_i are from linguistic fuzzy sets and k is the

Algorithm 5: Construction of FLGSCANN using fuzzy numbers

Input: Problem data (here, hesitant triangular intuitionistic fuzzy)
Output: Optimal solutions
1 Weight assign with input-data according to problem, if required.
2 Weighted-data summation.
3 Summed data are divided with two switches namely, original and corresponding NOT gate.
4 All combinations are get together.
5 All combined data set forms a geometrical figure, may be any polygon.
6 Each vertex of the polygon is ranked according to their distances from centroid of the polygon.
7 Signal flows through the minimum distance.
8 Optimum vertex is obtained.

number of rules in the model, then the Mamdani model is useful to apply. When knowledge is acquired in quantitative or data-based information, then knowledge can be accessed through the Takagi-Sugeno-Kang (TSK) model.
Here we consider zero-order TSK fuzzy model with four rules and say it as *Modified TSK Model*.

Modified TSK Model:
Assumption: If x and y are inputs then output is z.
Rules: Rules are defined by R_is.

R_1: If x is X_1 and y is Y_1 then z is a_{11}, i.e., the output is $((X_1, Y_1), a_{11})$.
R_2: If x is X_2 and y is Y_1 then z is a_{21}, i.e., the output is $((X_2, Y_1), a_{21})$.
R_3: If x is X_1 and y is Y_2 then z is a_{12}, i.e., the output is $((X_1, Y_2), a_{12})$.
R_4: If x is X_2 and y is Y_2 then z is a_{22}, i.e., the output is $((X_2, Y_2), a_{22})$.

If all of a_{11}, a_{12}, a_{21} and a_{22} are assumed as fuzzy numbers, the fuzzy game, in matrix form can be written as the following with X_1, X_2 as player I's strategies and Y_1, Y_2 are those for player II,

$$G = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}.$$

Here, for example, the payoff a_{11} emerged as the outcome when player I plays his/her strategy X_1 and player II plays Y_1. Consider player I's strategies have weights w_1 and w_2 and player II's strategies have weights w_3 and w_4 respectively. Therefore, according to the concept of game theory, discussed in Section III, we must have

$$\begin{cases} f_1 = w_1 a_{11} X_1 + w_2 a_{21} X_2 \geq v_I; \\ f_2 = w_1 a_{12} X_1 + w_2 a_{22} X_2 \geq v_I; \\ g_1 = w_3 a_{11} Y_1 + w_4 a_{12} Y_2 \leq v_{II}; \\ g_2 = w_3 a_{21} Y1 + w_4 a_{22} Y_2 \leq v_{II}. \end{cases}$$

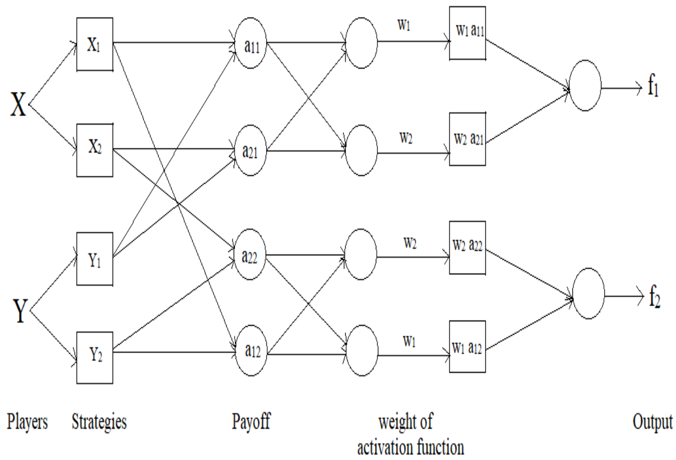

Figure 5.4: Diagrammatic form of player I's problem.

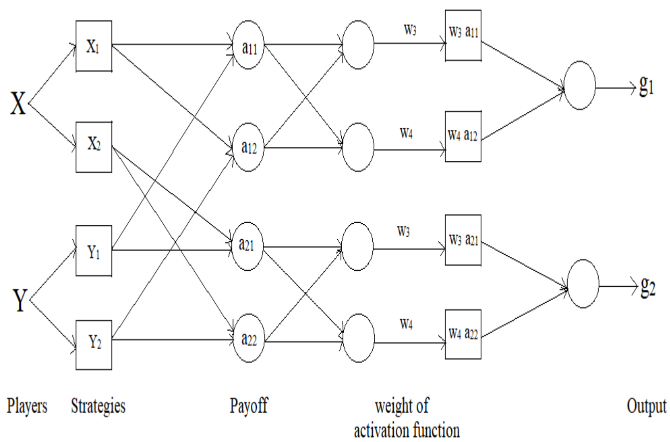

Figure 5.5: Pictorial form of player II's problem.

Assuming that each rectangular game has a solution and assuming v_I and v_{II} to be the game values for players I and II, respectively which are to be optimized. So, in Figs. 5.4 and 5.5, f_1, f_2 and g_1, g_2 are the combined form in Max fuzzy neuron and Min fuzzy neuron respectively according to the existence of the saddle point(s) or can be summed according to the non-existence of saddle point to derive the optimum results through **Algorithm 6**.

We first execute **Algorithm 6** and then consider the algorithms **Algorithm 5** and **Algorithm 6** altogether in **Algorithm 7** to get optimal results.

Algorithm 6: Construction of matrix game solution

Input: Problem matrix

Output: Game strategies and game value

1 Construction of rules of the matrix game according to the strategies of the players.

2 Combination of rules to form the payoffs of the matrix game.

3 Application of the concept of the saddle point or mixed strategy to the matrix game.

4 Achieving optimum strategies and obtaining the game value.

Algorithm 7: Construction of ANN-logic gate-switching circuit oriented matrix game solution

Input: Game Problem

Output: Required solution of game

1 Follow **Algorithm 5**, stepwise.

2 Follow **Algorithm 6**, stepwise.

5.5 Computative Example

Consider the existence of two business houses H_1 and H_2. By selling their products, both these houses aim at increasing their profits in terms of market-shares. One wishes to maximize his gain and the other aims to cut his loss. The two houses consider various strategies. House H_1 considers

X_1: Advertisement,

X_2: Reducing the printed price.

And house H_2 chooses

Y_1: Attracting packaging features,

Y_2: Giving promotion-pack to customers free of cost.

Again we consider that these two houses have their own managing bodies which call meetings regularly (say, every three months or every six months) to put some weights on their decisions. The decisions after each meeting may vary from the previous meeting's decisions or not. So, hesitant environment arises. Both the houses have efforts to increase their market-shares considering the fact that when one house profits, the other loses. So the outcome, after applying strategies, are the profit-percentage of the houses in terms of market-shares. If we consider this problem as a game issue with two players I and II representing houses H_1 and H_2, respectively, then the payoff matrix is as follows:

$$\check{G} = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{pmatrix} \check{C}_{11} & \check{C}_{12} \\ \check{C}_{21} & \check{C}_{22} \end{pmatrix} \end{array}$$

Here, player I has strategies X_1 and X_2; player II has Y_1 and Y_2. And the payoff elements are hesitant triangular intuitionistic fuzzy elements $\check{C}_{ij},\ i,j = 1,2$ with their corresponding weights, separated by second semicolon, are given below:

$\check{C}_{11} = \{\langle(5.7, 7.7, 9.3); 0.7, 0.2\rangle; 0.4, \langle(5, 7, 9); 0.6, 0.3\rangle; 0.3, \langle(5.7, 7.7, 9); 0.8, 0.1\rangle; 0.3\}$,

$\check{C}_{12} = \{\langle(8, 9, 10); 0.6, 0.3\rangle; 0.5, \langle(8.3, 9.7, 10); 0.7, 0.2\rangle; 0.3, \langle(7, 9, 10); 0.6, 0.2\rangle; 0.2\}$,

$\check{C}_{21} = \{\langle(8.33, 9.67, 10); 0.6, 0.4\rangle; 0.4, \langle(3, 5, 7); 0.6, 0.3\rangle; 0.4, \langle(6.5, 8.6, 10); 0.4, 0.5\rangle; 0.2\}$,

$\check{C}_{22} = \{\langle(6.5, 8.2, 9.3); 0.8, 0.1\rangle; 0.3, \langle(7, 9, 10); 0.7, 0.2\rangle; 0.4, \langle(6.3, 8.3, 9.7); 0.7, 0.2\rangle; 0.3\}$.
Here, $\check{C}_{12} = \{\langle(8, 9, 10); 0.6, 0.3\rangle; 0.5, \langle(8.3, 9.7, 10); 0.7, 0.2\rangle; 0.3, \langle(7, 9, 10); 0.6, 0.2\rangle; 0.2\}$ indicates that if player I assumes X_1 and player II considers Y_2, then the profit will be 90% with minimum 80% to maximum 100% having 6% positive chance and 3% pessimistic chance if the managing body gives 5% weight to their decisions. In the following meeting the decisions remain same and no problem arises but if weight are given 3% then the profit percentage is 97, lying between 83 and 100 having 7% positive chance. The remaining member of the set can be depicted likewise.
Now, using the regular neural net structure, we combine the data in hesitant fuzzy set and using the mean averaging operator, we get from $\check{C}_{11}$, $x_1 = \langle(5.49, 7.49, 9.12); 0.6, 0.3\rangle$. Similarly the others are obtained as:

$$\begin{cases} x_2 = \langle(7.89, 9.21, 10.00); 0.6, 0.3\rangle, \\ x_3 = \langle(5.83, 7.58, 8.80); 0.4, 0.5\rangle, \\ x_4 = \langle(6.64, 8.55, 9.70); 0.7, 0.2\rangle. \end{cases}$$

Since every switching circuit has two inputs as 'on' and 'off', we consider the x_i as 'on' and the $\bar{x}_i$ as 'off' to maintain the circuit rational. This consideration is important on the basis of neural net since in the course of passing signal from one neuron to another, only the predefined/prefixed neuron is in on mode, others are in off mode.

$$\begin{cases} \bar{x}_1 = \langle(0.88, 2.51, 4.51); 0.3, 0.6\rangle, \\ \bar{x}_2 = \langle(0.00, 0.79, 2.11); 0.3, 0.6\rangle, \\ \bar{x}_3 = \langle(1.20, 2.42, 4.17); 0.5, 0.4\rangle, \\ \bar{x}_4 = \langle(0.30, 1.45, 3.36); 0.2, 0.7\rangle. \end{cases}$$

Now, using the multiplication operations on TIFNs, we compute the values of the following sixteen combinations:
$x_1x_2x_3x_4$, $\bar{x}_1x_2x_3x_4$, $x_1\bar{x}_2x_3x_4$, $x_1x_2\bar{x}_3x_4$, $x_1x_2x_3\bar{x}_4$, $\bar{x}_1\bar{x}_2x_3x_4$, $\bar{x}_1x_2\bar{x}_3x_4$, $\bar{x}_1x_2x_3\bar{x}_4$, $x_1\bar{x}_2\bar{x}_3x_4$, $x_1\bar{x}_2x_3\bar{x}_4$, $x_1x_2\bar{x}_3\bar{x}_4$, $\bar{x}_1\bar{x}_2\bar{x}_3x_4$, $\bar{x}_1\bar{x}_2x_3\bar{x}_4$, $x_1\bar{x}_2\bar{x}_3\bar{x}_4$, $\bar{x}_1x_2\bar{x}_3\bar{x}_4$ and $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$.
For example, $\bar{x}_1x_2x_3\bar{x}_4 = \langle(12.14, 254.08, 1333.51); 0.2, 0.7\rangle$ and the others. These set of values of sixteen fuzzy numbers can be assigned as sixteen vertices of a solid figure as in Fig. 5.6. Now,

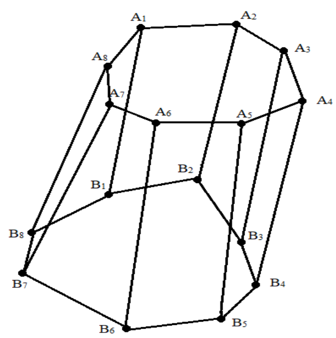

Figure 5.6: A solid three-dimensional figure with sixteen vertices.

inspired from the articles [29; 151], the centroid of the figure is computed using the formulae: $\frac{\sum V_i}{i}, i = 1(1)n$; Here, $n = 16$ and V denote the vertices (In Figs. 5.6 and 5.7, denoted by A_s and B_t; $s, t = 1, \ldots, 8$). The centroid of the figure is the triangular fuzzy number

$\langle(153.25, 624.99, 1747.43); 0.2, 0.7\rangle$. Now, computing the Euclidean distances of all vertices from the centroid, the shortest distance arises for the vertex $\bar{x}_1x_2\bar{x}_3x_4$ and the farthest for the vertex $x_1x_2x_3x_4$. Now considering the vertex $\bar{x}_1x_2\bar{x}_3x_4$, we form the payoff matrix, as below:
Considering player I's strategies have weights $w_1 = 0.5, w_2 = 0.5$ and player II's strategies have

Table 5.1: Payoff matrix of the game problem.

	Strategy Y_1	Strategy Y_2
Strategy X_1	$\langle(0.88, 2.51, 4.51); 0.3, 0.6\rangle$	$\langle(7.89, 9.21, 10); 0.6, 0.3\rangle$
Strategy X_2	$\langle(1.20, 2.42, 4.17); 0.5, 0.4\rangle$	$\langle(6.64, 8.55, 9.70); 0.7, 0.2\rangle$

weights $w_3 = 0.5, w_4 = 0.5$ respectively and using **Definitions 5.4.3, 5.4.4 and 5.4.5**, we derive,

$$\begin{aligned}\text{Max fuzzy neuron \{Min fuzzy neuron\}} &= \max\{\min\{w_3a_{11}, w_4a_{12}\}, \min\{w_3a_{21}, w_4a_{22}\}\}\\ &= (0.5)(2.63),\\ \text{Min fuzzy neuron \{Max fuzzy neuron\}} &= \min\{\max\{w_1a_{11}, w_2a_{21}\}, \max\{w_1a_{12}, w_2a_{22}\}\}\\ &= (0.5)(2.63),\end{aligned}$$

and we get, Max fuzzy neuron {Min fuzzy neuron}=Min fuzzy neuron {Max fuzzy neuron}.
The existence of the saddle point gives the strategy sets for players I and II, respectively X_1 and Y_1 and the value of the game is expressed in TIFN, as, $\langle(0.88, 2.51, 4.51); 0.3, 0.6\rangle$.
But, considering player I's strategies have weights $w_1 = 0.6, w_2 = 0.4$ and player II's strategies have weights $w_3 = 0.5, w_4 = 0.5$ respectively, we obtain,

$$\begin{aligned}\text{Max fuzzy neuron \{Min fuzzy neuron\}} &= \max\{\min\{w_3a_{11}, w_4a_{12}\}, \min\{w_3a_{21}, w_4a_{22}\}\}\\ &= \max\{\min\{1.315, 4.515\}, \min\{1.295, 4.145\}\}\\ &= 1.315,\\ \text{and Min fuzzy neuron \{Max fuzzy neuron\}} &= \min\{\max\{w_1a_{11}, w_2a_{21}\}, \max\{w_1a_{12}, w_2a_{22}\}\}\\ &= \min\{\max\{1.578, 1.036\}, \max\{5.418, 3.316\}\}\\ &= 1.578.\end{aligned}$$

Therefore, Max fuzzy neuron {Min fuzzy neuron}$\neq$Min fuzzy neuron {Max fuzzy neuron}. But we infer that the defuzzified crisp value V of the game satisfies $1.315 \leq V \leq 1.578$.
If we consider the weights $w_1 = 0.9, w_2 = 0.1, w_3 = 0.25, w_4 = 0.75$, then using **Definitions 5.4.3, 5.4.4 and 5.4.5**, we derive,

$$\begin{aligned}\text{Max fuzzy neuron \{Min fuzzy neuron\}} &= \max\{\min\{w_3a_{11}, w_4a_{12}\}, \min\{w_3a_{21}, w_4a_{22}\}\}\\ &= 0.658,\\ \text{Min fuzzy neuron \{Max fuzzy neuron\}} &= \min\{\max\{w_1a_{11}, w_2a_{21}\}, \max\{w_1a_{12}, w_2a_{22}\}\}\\ &= 2.371.\end{aligned}$$

Payoff matrices for player I and player II are diverse because of different weights, earmarked for the strategies of the players, and consequently we get different values of the game. But in each cases, optimal strategies for player I from player I's payoff matrix and optimal strategies for player II from corresponding payoff matrix are: $(X_1^*, X_2^*) = (1, 0), (Y_1^*, Y_2^*) = (1, 0)$, respectively. The optimal value of the game is $\langle(0.88, 2.51, 4.51); 0.3, 0.6\rangle$ with corresponding weight. If we consider different weights to different strategies, we get different resolutions. The whole procedure is picturesquely represented in Fig. 5.7 (Here, $\breve{a}_{ij}$ is hesitant TIFN-based set and ${}^k a_{ij}$ are its members), where the game model is performed through **Algorithm 6**.

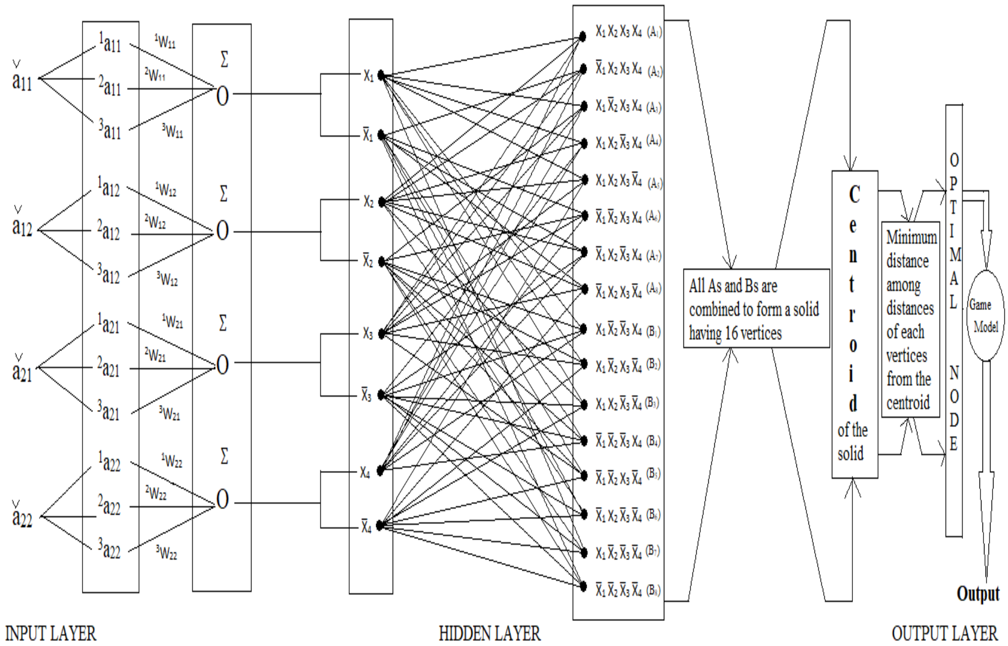

Figure 5.7: Combined model for logic-gate-fuzzy-ANN system.

5.6 Result and Discussion

In this work, we contemplate fuzzy matrix game with respect of ANN and fuzzy logic gate switching circuit. Defuzzification technique using the centroid concept is applied and we achieve a fine result to the matrix game problems.

Here we notice that the weights assigned to the strategies of the players or decision makers, when changed, give an interesting resolution. As the weights are changed, the crisp value of the game are changed, simultaneously. When we consider player I's strategies with weights $w_1 = 0.5, w_2 = 0.5$ and player II's strategies with weights $w_3 = 0.5, w_4 = 0.5$ respectively, we see the crisp value of the game as 1.315 and the profits in terms of market-shares is 25.1% with minimum 8.8% and maximum 45.1% in addition to 3% optimistic and 6% pessimistic chance. But if we apply the weights as $w_1 = 0.6, w_2 = 0.4, w_3 = 0.5, w_4 = 0.5$, we observe that the crisp game value lies between 1.315 and 1.578 with fuzzy value of the game within $\langle(0.88, 2.51, 4.51); 0.3, 0.6\rangle$ and $\langle(1.20, 2.42, 4.17); 0.5, 0.4\rangle$ with corresponding weights. This significantly suggests that the value of a decision, here game, depends upon the decision makers', here players', choices of weights of the alternatives, here strategies, of the game.

5.7 Conclusion

Hesitant fuzzy concept is an important tool to design and to represent the decision makers' hesitance characteristics and has been applied successfully in different aspects [27; 28; 124]. The major objectives of this work, derived as concluding parts, are to explore the potentiality of the neuro-fuzzy systems in modeling game phenomenon and to access its behavioural structures through ANN and logic-gate switching circuits. From the model description, our suggested methodology is unique in the following manners:

- This is (probably) the main endeavour to explain fuzzy matrix game using max fuzzy neuron and min fuzzy neuron in fuzzy hybrid neural network.
- This is a fast approach to combine the hesitant fuzzy elements using neural network.
- The applied defuzzification method is unique in the sense that it can be applied easily.

The analysis of the results indicates that the rendition of FGSCANN model in game theory would be significantly improved if the input data are transformed into the normal or real domain prior to model formulation. The results of the proposed study highly encourage the researchers with a suggestion that ANN is viable for modeling daily life problems in the light of game theory.