Chapter 3

Price-discount and delay-in-payments in an inventory model

3.1 Introduction

Deterioration of products is one of the major part to be considered in manufacturing industry. In real life, it is too difficult to maintain the deterioration of items such as fruits and vegetables. These types of items, which are decayed due to the time factor, are not in a good condition to fulfill the demand of customers. Therefore, the effect of deterioration cannot be disregarded in production lot-size. Deterioration is generally taken to be a function of the on-hand inventory. Dave and Pandya (1985) presented two inventory models i.e., infinite and finite horizon models in which deterioration is assumed to be a constant fraction of on hand inventory. They constructed under the assumptions of instantaneous delivery and no shortages. Heng *et al.* (1991) formulated a production model for lot-size, order-level inventory system with the effect of decay. They minimized total cost by optimizing optimal lot-size and order level. Skouri and Papachristos (2002) derived a continuous review inventory model by highlighting deterioration, shortage, and opportunity cost. Demand rate is depicted as a concave function of time. Exponentially decreasing partial backlogging rate which is also time-dependent function is discussed in their model. Teng and Chang (2005) observed an EPQ model for deteriorating items. It is assumed that lot of stocks provides a negative impression on buyer and the amount of display-space is limited. By utilizing this idea, demand can be treated as stock dependent and also selling-price-dependent. Sana (2010) studied an ordering inventory model with perishable items and price-dependent demand. Shortage takes places at starting of the inventory cycle and deterioration is time-proportional. He also formulated the criterion for optimal solution to obtain replenishment schedule and showed the optimal ordering policy is unique. Sarkar and Sarkar (2013) deduced an inventory model with time-dependent deteriorating products, where demand is inventory dependent. Bhunia *et al.* (2014) deals with an inventory model with deteriorating item for two different warehouses having several preserving facilities. Shortages are provided and partially backlogged is considered as waiting time-dependent. Different realistic cases, sub cases, and scenarios corresponding problems have been considered as non-linear constrained optimization problems along with the solution process demand. Wu *et al.* (2016) extended earlier research works regarding inventory models with trapezoidal-type demand rate. They presented two inventory systems one with shortages and another for without shortages. Deterioration rate is taken to be as time-dependent and time value of money is included in their model.

In traditional inventory models, it is considered that the demand rate is constant over the whole time. EOQ model for constant demand was first derived by Harris (1913). But, all assumptions are not valid in real-life situation. After the pioneering attempt of Harris, some notable research models about linear-trend demand in demand were done by many researchers. Goswami and Chaudhuri (1991) explained the inventory replenishment policy for a deteriorating item. They obtained number of reorders, interval between two successive reorders, and shortage intervals over a finite time-horizon. By introducing the concept of exponentially deteriorating items, Hariga (1997) proposed two computationally efficient solution methods which develop the optimal replenishment schedules for perishable products with fixed lifetime. Demand is measured as time-dependent for both models, which are constructed under the assumption of discrete opportunities for replenishment over a fixed planning horizon. Hsu and Li (2006) surveyed a non-linear mathematical programming model to optimize a delivery service strategy for online shopping by considering time-dependent consumer demand. They determined optimal number and duration of service cycles for discriminating strategy and also maximized profit subject to demand and supply interaction. Their results proved that discriminating service strategy is a more beneficial strategy with respect to uniform strategy. Sarkar *et al.* (2011) derived an EMQ model for single type of items with time-dependent demand under the impact of both inflation as well as time-value of money. During long-run manufacturing process, machine may produces defective items. The production of defective items increases along with time and depends on reliability of system. By considering this point of view, they introduced the concept of defective items and reliability of system into their model. Khanra *et al.* (2013) examined an inventory model with finite time horizon and quadratic time-dependent demand. The idea of delay-in-payments is used to their model. Shortages are considered after some variable time. Their model is formulated under three different circumstances depending on the time of occurrence of shortages, credit-period, and cycle time. Li (2015) generated a distributor's delivery strategy problem with carbon-emissions, retailers time-dependent demands, and supply interactions. He proved that models with demand-supply interactions can result maximum profit and market share rather than without demandsupply interactions. Zhao *et al.* (2016) described an integrated multi-stage supply chain for time-dependent demand over a finite planning horizon. Their model obtained a production-inventory policy which measured as a weighted directed acyclic graph.

Selling-price is one of the important factor in success of business. Most of the products are available in the market, where the demand rate of products and the selling-price are closely related. It is observed that at certain time of product cycle, the monotonically decreasing price-pattern of any items results increasing in sales. Whenever price of any product reduces, customers are more affective to that product i.e., demand of that product increases directly. This pattern confirms our price-dependent demand model. Avinadav *et al.* (2013) discussed an inventory model for perishable items with jointly price and time-dependent demand. They considered that items have a fixed shelf-life and the demand rate diminishes linearly with respect to selling-price. In addition, demand rate decreases polynomially over the time after replenishment, until it disappears either at the reservation price or at expiration time. Alfares and Ghaithan (2016) developed an inventory model by assuming the variability of the demand rate, unit holding cost, and unit purchase cost. They presented a selling-price-dependent demand rate, a storage time-dependent holding cost, and an order size-dependent purchase cost based on all-units quantity discount.

Earlier, it is assumed that retailers pays the amount instantly for products which they purchase from supplier. Now a days, supplier offers retailer a fixed time-period to pay the purchasing amount. This fixed time-period is called trade-credit period. On the other hand, retailer offers their customer a partial trade-credit. Interest is charged to customers if they are not able to settle the purchasing amount. For this reason, retailer can delay the payment up to the last moment of credit-period. Hence, retailer can gain more profits. Considering the well-known trade-credit policy, respective researchers invented different inventory models with trade-credit financing. The viewpoint of credit-linked demand and two-levels of trade-credit policy was introduced by Jaggi *et al.* (2008) to reflect the real-life situations. They depicted an easy-to-use algorithm which obtain the optimal credit and replenishment policy. Sarkar *et al.* (2014) considered the business strategy that suppliers offer credit-period to motivate customers for buying more items. Their model highlighted this policy along with the production of defective items and inspection policy. The lead time is considered as stochastic in nature. Li *et al.* (2015) extended their model to the situation where retailers delay-in-payments is provided by supplier. They incorporated corresponding inventory game with permissible delay-in-payments, and showed that its core is nonempty. Their model generated a core allocation rule is that can be reached through population monotonic allocation scheme. See Table 3.1 for contribution of several authors.

Table 3.1: Contribution of several authors

Author(s)	**Time-dependent demand**	**Price-dependent demand**	**Delay -in-payments**	**Price-discount**	**Deterioration**
Dave and Pandya (1985)					√
Heng *et al.* (1991)					√
Goswami and Chaudhuri (1991)					√
Hariga (1997)	√				
Skouri and Papachristos (2002)	√				√
Teng and Chang (2005)		√			√
Hsu and Li (2006)	√				

Author(s)	**Time-dependent demand**	**Price-dependent demand**	**Delay-in-payments**	**Price-discount**	**Deterioration**
Jaggi *et al.* (2008)			√	√	
Sana and Chaudhuri (2008)			√	√	
Sana (2010)		√			√
Sarkar *et al.* (2011)	√				
Avinadav *et al.* (2013)	√	√			
Khanra *et al.* (2013)	√		√		
Sarkar and Sarkar (2013)					√
Sarkar *et al.* (2014)					√
Bhunia *et al.* (2014)					√
Li *et al.* (2015)					√
Li (2015)	√				
Alfares and Ghaithan (2016)		√			
Wu *et al.* (2016)					√
You (2006)		√			
This chapter	√	√	√	√	√

In this chapter, authors extended the research works of Sana and Chaudhuri's (2008) model [Sana, S. and Chaudhuri, K.S. (2008). A deterministic EOQ model with delays in payments and price-discounts offers. *European Journal of Operational Research*, 184, 509-533.] for deteriorating products with different demand functions. The demand function of commodities is taken to be both time and selling-price-dependent. Suppliers allows a fixed trade-credit-period to retailer to pay the purchasing amount. In addition, supplier provides price-discount strategy on the purchasing amount to the retailer. The profit function of retailer is maximized for finite replenishment rate by determining the selling-price per unit and duration of inventory cycle. Numerical examples are designed to illustrate this model.

3.2 Mathematical model

The following notation are considered to develop this model.

Decision variables

P	selling-price per unit (\$/unit)
T	duration of inventory cycle (months)

Parameters

$I_1(t)$	on-hand inventory during the time t $(0 \leq t \leq t_1)$ (units)
$I_2(t)$	on-hand inventory at time t $(t_1 \leq t \leq T)$ (units)
μ	replenishment/supply rate (units)
K	variable delay-period (unit time)

K_i — ith permissible delay-period (unit time)

$D(P,t)$ — demand dependent on time i.e., $D(P,t) = a_1 + b_1 t + c_1 t^2 - hP$, a_1, b_1, c_1, and $h > 0$

$\theta(t)$ — deterioration rate, $0 < \theta(t) < 1$

δ_i — discount rate on the MRP at the ith permissible delay-period

C_P — purchasing cost per unit (\$/unit)

C_M — maximum retail price (MRP) per unit (\$/unit)

C_H — holding cost (\$/unit)

C_1 — ordering cost (\$/order)

U_c — rate of interest gaining due to credit-balance (/\$/unit time)

U_f — rate of interest due to financing inventory (/\$/unit time)

t_1 — duration of the replenishment rate

T^* — optimal duration of inventory cycle

t_1^* — optimal duration of replenishment

AVP_{1i} — average profit of the system when $T \geq K_i$

AVP_{2i} — average profit of the system when $T \leq K_i$

In this chapter, same assumptions as on Sana and Chaudhuri (2008a) are considered except the demand, deterioration, and finite replenishment rate.

1. The inventory system involves single-type of products.

2. The demand rate increases quadratically with time and decreases linearly with selling-price.

3. Replenishment rate is finite.

4. Permissible delay-in-payments are considered to the retailer by the supplier.

6. Time horizon is infinite and lead time is neglected. No shortage and backlogging are allowed.

The inventory cycle starts with zero inventory. The supply rate μ continues till the time t_1 and then it reaches the zero level at time $t = T$ $(T \geq t_1)$ to adjust the demand in the market. To pay the total purchasing cost, the supplier offers different credit-periods K_i, when, $i = 1, 2, 3$ to the retailer. The purchasing cost of different credit-periods are as follows

$$C_P = \left\{ \begin{array}{ll} C_M(1-\delta_1) & \text{when } K = K_1 \\ C_M(1-\delta_2) & \text{when } K = K_2 \\ C_M(1-\delta_3) & \text{when } K = K_3 \\ \infty & \text{when } K > K_3. \end{array} \right\}$$

where K_i's are the ith permissible delay-period at which the discount rate to the retailer is δ_i. Also C_P tends to ∞ at $K \geq K_3$, i.e, at infinite purchasing cost, the retailer never purchase any item from the supplier and must prefer the discount rate δ_i.

Considering this policy, two cases may arise, which are as follows:

Case 1 When $T \geq K$

i.e., inventory cycle length T is larger or equal to the credit-period K.

(See Figure 3.1).

Case 2 When $T \leq K$

i.e., inventory cycle length T is smaller or equal to the credit-period K.

(See Figure 3.2).

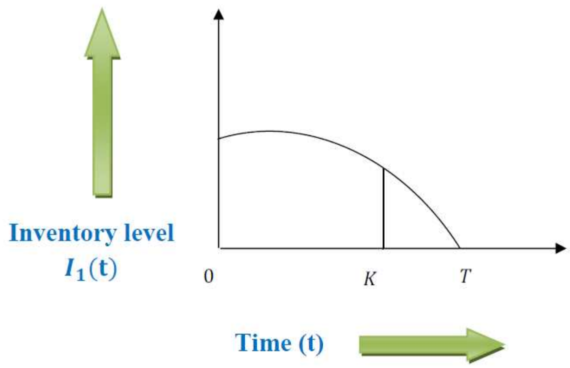

Figure 3.1: Graphical illustration of inventory system while $T \geq K$

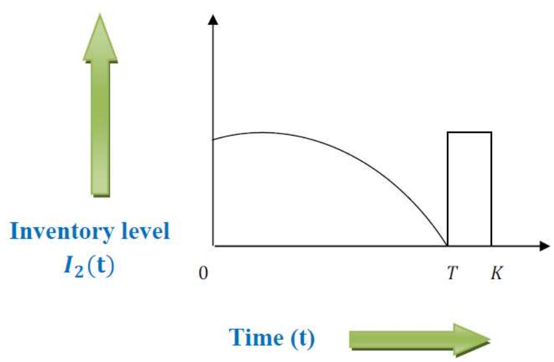

Figure 3.2: Graphical presentation for inventory model when $T \leq K$

Now, the governing differential equations of this model under the presence of deterioration are

$$\frac{dI_1}{dt} = \mu - D(P,t) - \theta I_1 \quad \text{with} \quad I_1(0) = 0, \qquad 0 \le t \le t_1$$

and

$$\frac{dI_2}{dt} = -D(P,t) - \theta I_2 \quad \text{with} \quad I_2(T) = 0, \qquad t_1 \le t \le T$$

From these two governing differential equations, it can be found that

$$I_1(t) = X_1(1 - e^{-\theta t}) - \frac{b_1 t}{\theta} - \frac{c_1 t^2}{\theta} + \frac{2c_1 t}{\theta^2}, \qquad 0 \le t \le t_1,$$

where $X_1 = \left(\frac{\mu - a_1 + hP}{\theta} + \frac{b_1}{\theta^2} - \frac{2c_1}{\theta^3}\right)$.

and

$$I_2(t) = X_2(e^{\theta(T-t)} - 1) + Y_1(Te^{\theta(T-t)} - t) + Y_2(T^2 e^{\theta(T-t)} - t^2), \qquad t_1 \le t \le T,$$

where $X_2 = \left(\frac{a_1 - hP}{\theta} - \frac{b_1}{\theta^2} + \frac{2c_1}{\theta^3}\right)$.

[See Appendix A1 for values of Y_1 and Y_2.]

Now from the continuity condition, using $I_1(t_1) = I_2(t_1)$, one can obtain

$$t_1 = \frac{-S + \sqrt{S^2 + 4EF}}{2E}$$

[See Appendix B1 for S, E, and F.]

Case 1 $T \geq K$

While the inventory cycle T is larger or equal to the credit-period K.

Then the holding cost including interest charges is

$$\begin{aligned} &= C_H \int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt \\ &= C_H \Big[X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) - {t_1}^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 {t_1}^3}{3\theta} + \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) (X_2 + Y_1 T + Y_2 T^2) \\ &- X_2(T - t_1) - Y_1 \frac{(T^2 - {t_1}^2)}{2} - Y_2 \frac{(T^3 - {t_1}^3)}{3} \Big] \end{aligned}$$

The profit earns due to credit-balance during the delay-period $[0, K]$ is

$$\begin{aligned} &= U_c P \int_0^{K} (K - t) D(P, t) dt \\ &= U_c P \left((a_1 - hP) \frac{K^2}{2} + b_1 \frac{K^3}{6} + c_1 \frac{K^4}{12} \right) \end{aligned}$$

The interest charged for financing inventory during $[K, T]$ is

$$\begin{aligned} &= U_f C_P \int_K^{T} I_2(t)dt \\ &= U_f C_P \Big[(X_2 + Y_1 T + Y_2 T^2) \left(\frac{e^{\theta(T-K)} - 1}{\theta} \right) - X_2(T - K) - Y_1 \frac{(T^2 - K^2)}{2} - Y_2 \frac{(T^3 - K^3)}{3} \Big] \end{aligned}$$

Therefore, the total profit is

$$\begin{aligned} P_{1i} &= [(P - C_P)\mu t_1 + U_c P \{ \int_0^{t_1} (K_i - t) D(P, t) dt + \int_{t_1}^{K_i} (K_i - t) D(P, t) dt \} - C_H \{ \int_0^{t_1} I_1(t) dt \\ &+ \int_{t_1}^{T} I_2(t) dt \} - U_f C_P \int_{K_i}^{T} I_2(t) dt - C_1], \text{for} \quad i \in \{1, 2, 3\} \end{aligned}$$

Hence, the average profit for Case 1 is

$$
\begin{aligned}
AVP_{1i} &= \frac{P_{1i}}{T} \\
&= \frac{1}{T}[U_cP\{\int_0^{t_1}(K_i-t)D(P,t)dt+\int_{t_1}^{K_i}(K_i-t)D(P,t)dt\}+(P-C_P)\mu t_1 \\
&- C_H\{\int_0^{t_1}I_1(t)dt+\int_{t_1}^{T}I_2(t)dt\}-U_fC_P\int_{K_i}^{T}I_2(t)dt-C_1], \quad \text{for} \quad i\in\{1,2,3\}. \\
&= \frac{(P-C_P)\mu t_1}{T}+\frac{PU_c}{T}\left[(a_1-hP)\frac{{K_i}^2}{2}+b_1\frac{{K_i}^3}{6}+c_1\frac{{K_i}^4}{12}\right]-\frac{C_H}{T}\Big(X_1\left(t_1+\frac{e^{-\theta t_1}-1}{\theta}\right) \\
&- \frac{b_1{t_1}^2}{2\theta}-\frac{c_1{t_1}^3}{3\theta}+\frac{c_1{t_1}^2}{\theta^2}\Big)-\frac{C_H}{T}\Big[\left(\frac{e^{\theta(T-t_1)}-1}{\theta}\right)(X_2+Y_1T+Y_2T^2)-X_2(T-t_1) \\
&- Y_1\frac{(T^2-{t_1}^2)}{2}-Y_2\frac{(T^3-{t_1}^3)}{3}\Big]-\frac{U_fC_P}{T}\Big((X_2+Y_1T+Y_2T^2)\left(\frac{e^{\theta(T-K_i)}-1}{\theta}\right) \\
&- X_2(T-K_i)-Y_1\frac{(T^2-{K_i}^2)}{2}-Y_2\frac{(T^3-{K_i}^3)}{3}\Big)-\frac{C_1}{T}, \quad \text{for} \quad i\in\{1,2,3\}
\end{aligned}
$$

The optimal value of T^* and P^* for maximum AVP_{1i} must satisfy following conditions $\frac{\partial^2 AVP_{1i}}{\partial T^2}<0$, $\frac{\partial^2 AVP_{1i}}{\partial P^2}<0$, and $(\frac{\partial^2 AVP_{1i}}{\partial T^2})(\frac{\partial^2 AVP_{1i}}{\partial P^2})-(\frac{\partial^2 AVP_{1i}}{\partial P\partial T})^2>0$.

The expressions of the two derivatives (See Appendix C1) are highly non-linear.

Case 2 When $(T \leq K)$

When, the inventory cycle T is smaller or equal to the credit-period K, the holding cost excluding interest charges is

$$
\begin{aligned}
&= C_H\int_0^{t_1}I_1(t)dt+\int_{t_1}^{T}I_2(t)dt \\
&= C_H\Big[X_1\left(t_1+\frac{e^{-\theta t_1}-1}{\theta}\right)-{t_1}^2\left(\frac{b_1}{2\theta}-\frac{c_1}{\theta^2}\right)-\frac{c_1{t_1}^3}{3\theta}+\left(\frac{e^{\theta(T-t_1)}-1}{\theta}\right)(X_2+Y_1T \\
&+ Y_2T^2)-X_2(T-t_1)-Y_1\frac{(T^2-{t_1}^2)}{2}-Y_2\frac{(T^3-{t_1}^3)}{3}\Big]
\end{aligned}
$$

The profit gains throughout the delay-period $[0, K]$ is

$$
\begin{aligned}
&= U_cP\int_0^{T}(T-t)D(P,t)dt+\mu t_1(K-T) \\
&= U_cP\left[(a_1-hP)\frac{T^2}{2}+b_1\frac{T^3}{6}+c_1\frac{T^4}{12}+\mu t_1(K_i-t)\right].
\end{aligned}
$$

Therefore, the total profit is

$$\begin{aligned} P_{2i} &= [(P - C_P)\mu t_1 + U_c P\{\int_0^T (T-t)D(P,t)dt + \mu t_1(K_i - t)\} - C_H\{\int_0^{t_1} I_1(t)dt \\ &+ \int_{t_1}^T I_2(t)dt\} - C_1], \text{for } i \in \{1,2,3\}. \end{aligned}$$

The average profit for Case 2 is

$$\begin{aligned} AVP_{2i} &= \frac{P_{2i}}{T} \\ &= \frac{1}{T}[U_c P\{\int_0^T (T-t)D(P,t)dt + \mu t_1(K_i - t)\} + (P - C_P)\mu t_1 - C_H\{\int_0^{t_1} I_1(t)dt \\ &+ \int_{t_1}^T I_2(t)dt\} - C_1], \quad \text{for } i \in \{1,2,3\}. \\ &= \frac{(P - C_P)\mu t_1}{T} + \frac{PU_c}{T}\Big((a_1 - hP)\frac{T^2}{2} + b_1\frac{T^3}{6} + c_1\frac{T^4}{12} + \mu t_1(K_i - t)\Big) - \frac{C_H}{T}\Big[X_1\Big(t_1 \\ &+ \frac{e^{-\theta t_1} - 1}{\theta}\Big) - {t_1}^2\left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2}\right) - \frac{c_1 {t_1}^3}{3\theta} + (X_2 + Y_1 T + Y_2 T^2)\left(\frac{e^{\theta(T-t_1)} - 1}{\theta}\right) \\ &- X_2(T - t_1) - Y_1\frac{(T^2 - {t_1}^2)}{2} - Y_2\frac{(T^3 - {t_1}^3)}{3}\Big] - \frac{C_1}{T}, \quad \text{for } \ i \in \{1,2,3\} \end{aligned}$$

Similarly as earlier case, the optimal value of T^* and P^* for maximum AVP_{2i} must fulfil following conditions $\frac{\partial^2 AVP_{2i}}{\partial T^2} < 0$, $\frac{\partial^2 AVP_{2i}}{\partial P^2} < 0$, and $(\frac{\partial^2 AVP_{2i}}{\partial T^2})(\frac{\partial^2 AVP_{2i}}{\partial P^2}) - (\frac{\partial^2 AVP_{2i}}{\partial P \partial T})^2 > 0$.

Appendix D1 provides that expressions of the two derivatives are highly non-linear.

3.3 Numerical examples

By applying the numerical data from Sana and Chaudhuri (2008a) model, the average profit of the system, selling-price per unit, and duration of inventory cycle are calculated.

Example 1(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5$/unit/month, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units,

$C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are $\{AVP_{11} = \$319.101,\ T = 6.8$ months, and $P = \$90.11/\text{unit}\}$. Figure 3.3 indicates the optimality of average profit of the system AVP_{11}.

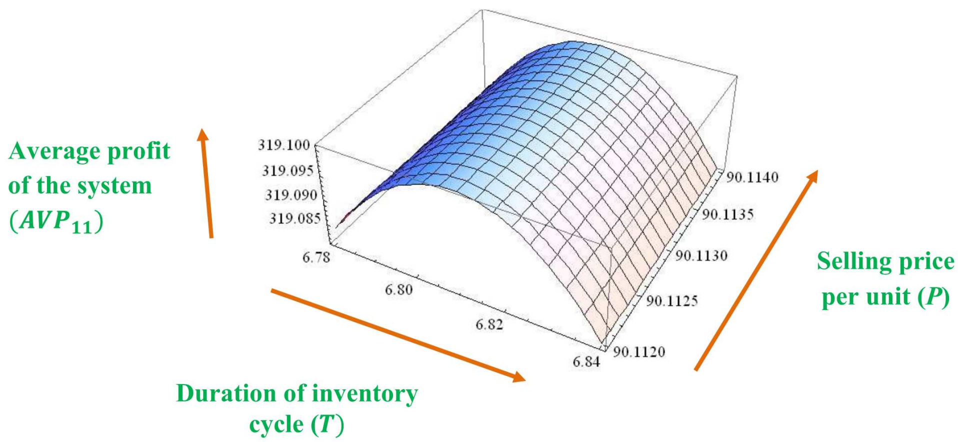

Figure 3.3: Average profit of the system AVP_{11} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 2(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5/\text{unit/month}$, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units, $C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are $\{AVP_{12} = \$452.971,\ T = 7$ months, and $P = \$95.19/\text{unit}\}$. Figure 3.4 indicates the optimality of average profit of the system AVP_{12}.

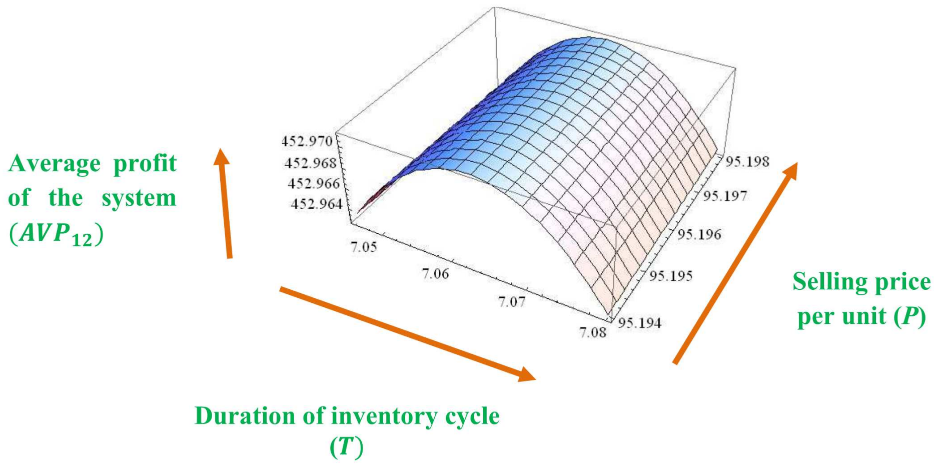

Figure 3.4: Average profit of the system AVP_{12} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 3(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5$/unit/month, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units, $C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are $\{AVP_{13} = \$670.85,\ \ T = 7.39$ months, and $P = \$100.93/\text{unit}\}$. Figure 3.5 indicates the optimality of average profit of the system AVP_{13}.

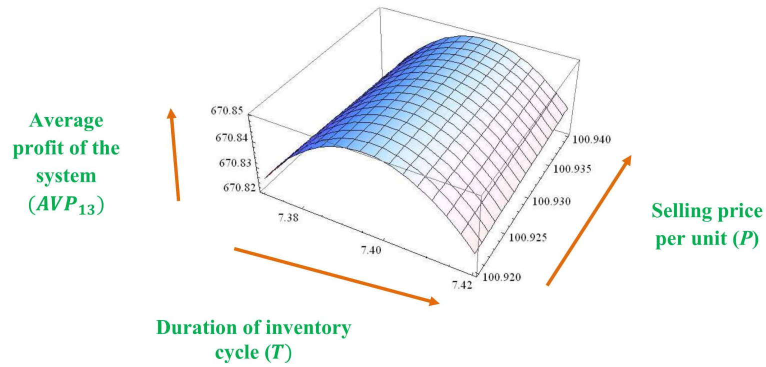

Figure 3.5: Average profit of the system AVP_{13} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 4(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5$/unit/month, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units, $C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are $\{AVP_{21} = \$171.71,\ T = 5.2 \text{ months, and } P = \$84.22/\text{unit}\}$. Figure 3.6 indicates the optimality of average profit of the system AVP_{21}.

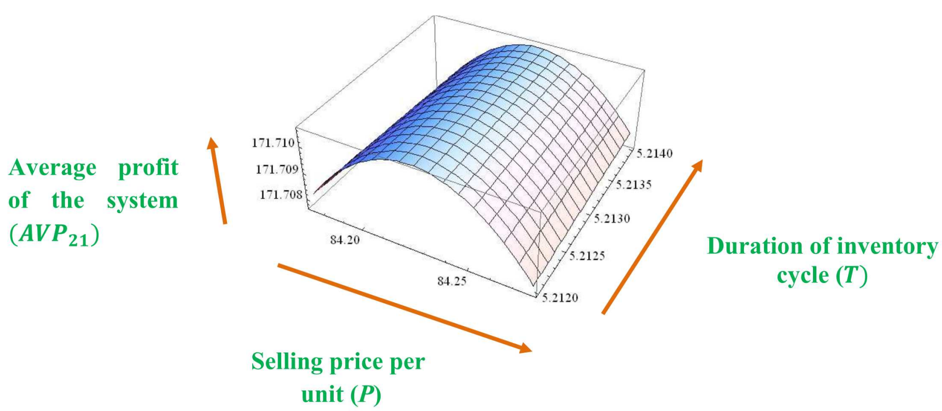

Figure 3.6: Average profit of the system AVP_{21} versus selling-price per unit (P) and duration of inventory cycle (T)

Example 5(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5$/unit/month, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units, $C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are$\{AVP_{22} = \$361.11,\ \ T = 6$ months, and $P = \$91.5/\text{unit}\}$. Figure 3.7 indicates the optimality of average profit of the system AVP_{22}.

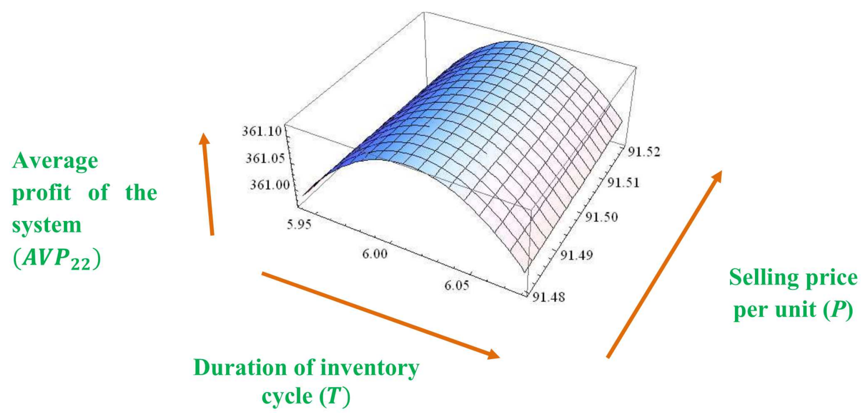

Figure 3.7: Average profit of the system AVP_{22} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 6(a)

Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5$/unit/month, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.71$ units, $C_M = \$120/$ unit, $U_c = \frac{0.13}{12}/\$/\text{month}$, $U_f = \frac{0.16}{12}/\$/\text{month}$, and $\mu = 300$ units. Then the optimal solutions are$\{AVP_{23} = \$634.286,\ \ T = 6.67$ months, and $P = \$98.57/\text{unit}\}$. Figure 3.8 indicates the optimality of average profit of the system AVP_{23}.

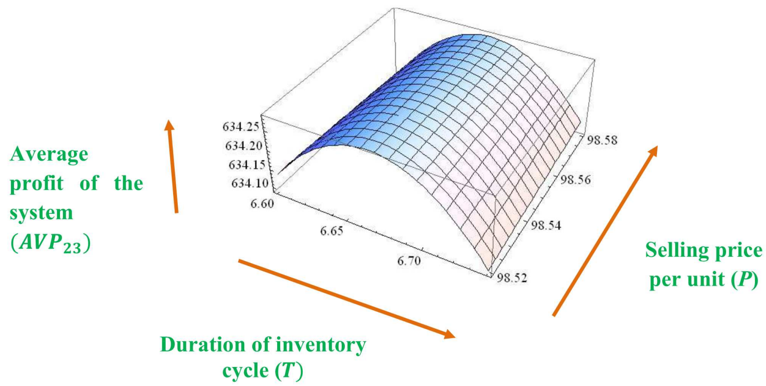

Figure 3.8: Average profit of the system AVP_{23} versus duration of inventory cycle (T) and selling-price per unit (P)

Case Study

This model discussed the concept of delay-in-payments and several price discount policy on purchasing cost to the retailer provided by the supplier. In this model, highlighted factors are delay-in-payments and price discount system. Any marketing companies are practical example of this model. They offer delay-in-payments and several price discount policy to customers for increasing their sales of products. For example, supplier offer 30 days to retailer for settling payment. If the retailer is enable to pay the amount within 5 days, then supplier will provide 20% discount on purchasing cost to retailer. On the other hand, if the retailer takes 15 days to pay that amount, then the supplier will provide 10% discount on purchasing cost. Additionally after 15 days, supplier will not provide any price discount on purchasing cost to retailer.

Numerical examples

Example 1(b)

Let $C_1 = \$25$ per order, $\theta = 0.12$, $C_H = \$0.6$/unit/month, $K_1 = 2$ months, $\delta_1 = 20\%$, $a_1 = 90$ units, $b_1 = 5.4$ units, $c_1 = 0.4$ units, $h = 1.6$ units, $C_M = \$90/$ unit, $U_c = \frac{0.14}{12}/\$/\text{month}$,

$U_f = \frac{0.17}{12}$/\$/month, and $\mu = 210$ units. Then the optimal solutions are $\{AVP_{11} = \$211.68,\ T = 0.5$ months, and $P = \$23.98$/unit$\}$. Figure 3.9 indicates the optimality of average profit of the system AVP_{11}.

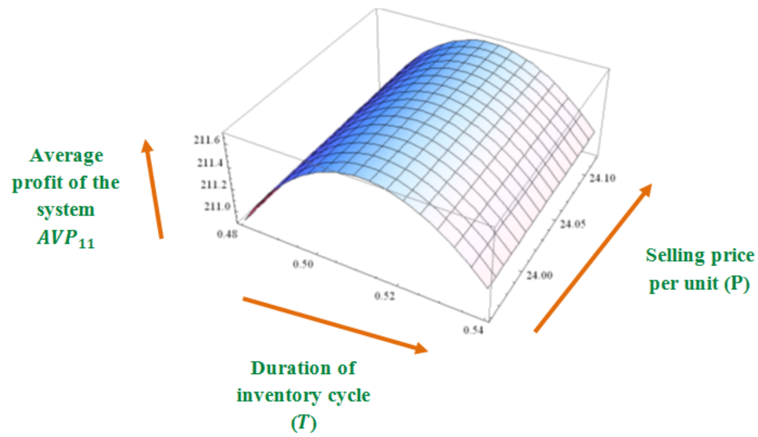

Figure 3.9: Average profit of the system AVP_{11} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 2(b)

Let $C_1 = \$25$ per order, $\theta = 0.11$, $C_H = \$0.6$/unit/month, $K_2 = 4$ months, $\delta_2 = 10\%$, $a_1 = 90$ units, $b_1 = 5.4$ units, $c_1 = 0.4$ units, $h = 1.6$ units, $C_M = \$90$/ unit, $U_c = \frac{0.14}{12}$/\$/month, $U_f = \frac{0.17}{12}$/\$/month, and $\mu = 210$ units. Then the optimal solutions are $\{AVP_{12} = \$370.97,\ T = 0.8$ months, and $P = \$36.8$/unit$\}$. Figure 3.10 indicates the optimality of average profit of the system AVP_{12}.

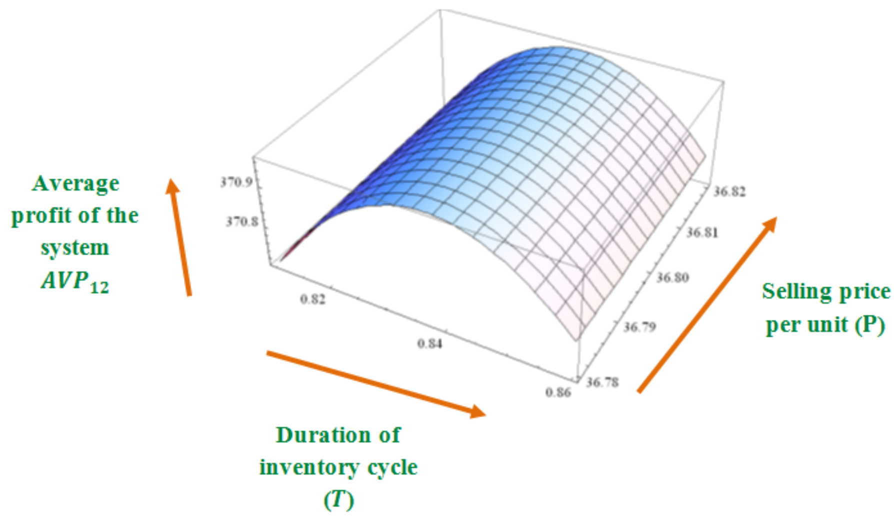

Figure 3.10: Average profit of the system AVP_{12} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 3(b)

Let C_1 = \$25 per order, $\theta = 0.1$, C_H = \$0.6/unit/month, K_3 = 6 months, $\delta_3 = 0\%$, $a_1 = 90$ units, $b_1 = 5.4$ units, $c_1 = 0.4$ units, $h = 1.6$ units, C_M = \$90/ unit, $U_c = \frac{0.14}{12}$/\$/month, $U_f = \frac{0.17}{12}$/\$/month, and $\mu = 210$ units. Then the optimal solutions are $\{AVP_{13}$ = \$658.46, $T = 1.1$ months, and P = \$41.43/unit$\}$. Figure 3.11 indicates the optimality of average profit of the system AVP_{13}.

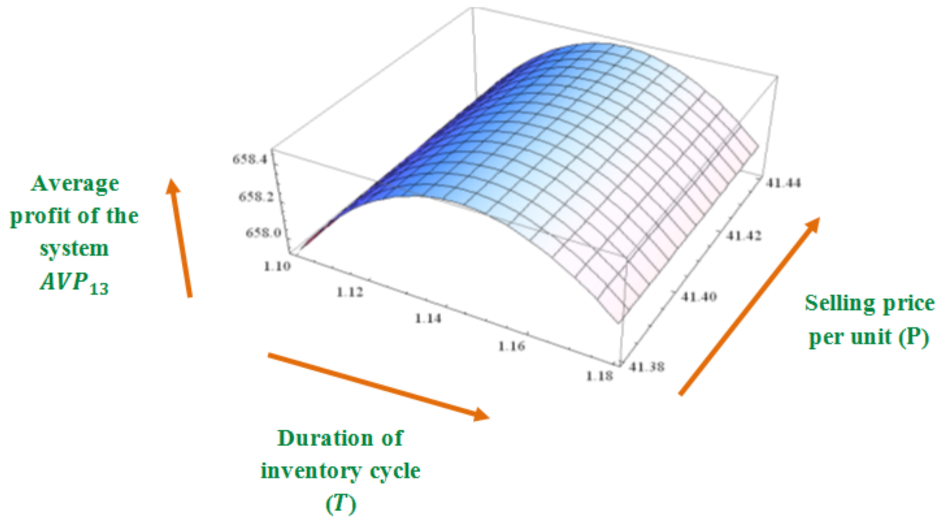

Figure 3.11: Average profit of the system AVP_{13} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 4(b)

Let $C_1 = \$25$ per order, $\theta = 0.23$, $C_H = \$0.6/\text{unit/month}$, $K_1 = 2$ months, $\delta_1 = 20\%$, $a_1 = 90$ units, $b_1 = 5$ units, $c_1 = 0.4$ units, $h = 1.6$ units, $C_M = \$130/$ unit, $U_c = \frac{0.14}{12}/\$/\text{month}$, $U_f = \frac{0.17}{12}/\$/\text{month}$, and $\mu = 200$ units. Then the optimal solutions are $\{AVP_{21} = \$169.50,\ T = 5.2$ months, and $P = \$94.63/\text{unit}\}$. Figure 3.12 indicates the optimality of average profit of the system AVP_{21}.

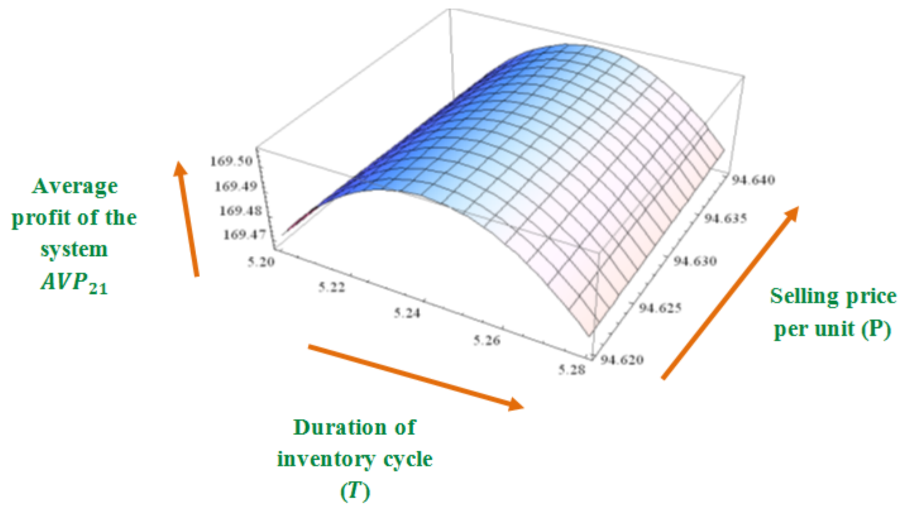

Figure 3.12: Average profit of the system AVP_{21} versus selling-price per unit (P) and duration of inventory cycle (T)

Example 5(b)

Let $C_1 = \$25$ per order, $\theta = 0.2$, $C_H = \$0.6$/unit/month, $K_2 = 4$ months, $\delta_2 = 10\%$, $a_1 = 90$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.6$ units, $C_M = \$130/$ unit, $U_c = \frac{0.14}{12}/\$/\text{month}$, $U_f = \frac{0.17}{12}/\$/\text{month}$, and $\mu = 200$ units. Then the optimal solutions are $\{AVP_{22} = \$267.72,\ T = 5.6$ months, and $P = \$100.84/\text{unit}\}$. Figure 3.13 indicates the optimality of average profit of the system AVP_{22}.

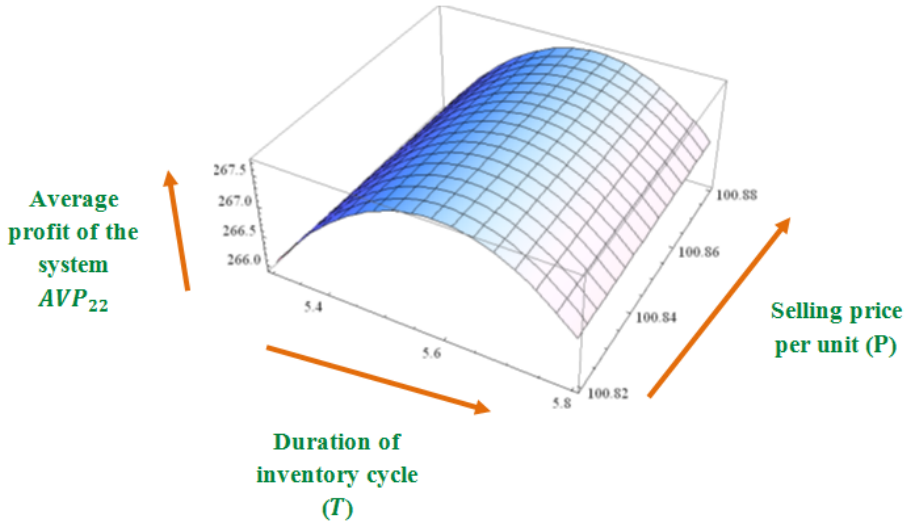

Figure 3.13: Average profit of the system AVP_{22} versus duration of inventory cycle (T) and selling-price per unit (P)

Example 6(b)

Let C_1 = \$25 per order, $\theta = 0.2$, C_H = \$0.6/unit/month, $K_3 = 6$ months, $\delta_3 = 0\%$, $a_1 = 90$ units, $b_1 = 5$ units, $c_1 = 0.5$ units, $h = 1.6$ units, C_M = \$130/ unit, $U_c = \frac{0.14}{12}$/\$/month, $U_f = \frac{0.17}{12}$/\$/month, and $\mu = 200$ units. Then the optimal solutions are$\{AVP_{23}$ = \$514.38, $T = 6.4$ months, and P = \$109.25/unit$\}$. Figure 3.14 indicates the optimality of average profit of the system AVP_{23}.

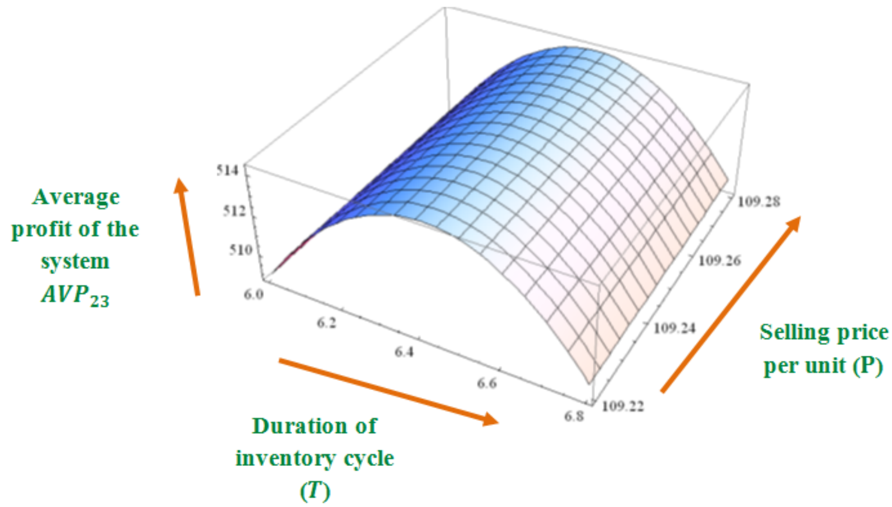

Figure 3.14: Average profit of the system AVP_{23} versus duration of inventory cycle (T) and selling-price per unit (P)

3.4 Concluding remarks and future works

This chapter extended the research work of Sana and Chaudhuri's (2008) model. This chapter discussed about the optimal profit of retailer by obtaining the selling-price per and duration of the inventory cycle. Further, this work can be extended to the multi-item inventory model with probabilistic demand and solving by any meta-heuristic procedure.

3.5 Appendices

Appendix A1

$$Y_1 = \left(\frac{b_1}{\theta} - \frac{2c_1}{\theta^2}\right)$$

$$Y_2 = \frac{c_1}{\theta}$$

Appendix B1

$$\begin{aligned}
S &= X_1\theta + [(\theta + T\theta^2)(X_2 + Y_1T + Y_2T^2)] \\
E &= -\frac{(X_1 + X_2 + Y_1T + Y_2T^2)\theta^2}{2} \\
F &= \left[\left(1 + \frac{\theta^2T^2}{2}\right)(X_2 + Y_1T + Y_2T^2)\right] - X_2
\end{aligned}$$

Appendix C1

The Hessian matrix for AVP_{1i} is $\mathbf{H}$ as follows

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 AVP_{1i}}{\partial T^2} & \frac{\partial^2 AVP_{1i}}{\partial T\partial P} \\ \frac{\partial^2 AVP_{1i}}{\partial P\partial T} & \frac{\partial^2 AVP_{1i}}{\partial P^2} \end{bmatrix}$$

Now, if the leading principal minors $\frac{\partial^2 AVP_{1i}}{\partial T^2} < 0$ and $\frac{\partial^2 AVP_{1i}}{\partial T^2}\frac{\partial^2 AVP_{1i}}{\partial P^2} - (\frac{\partial^2 AVP_{1i}}{\partial P\partial T})^2 > 0$ at the optimal point, then $\mathbf{H}$ is negative definite and the function AVP_{1i} is strictly concave.

Due to highly non-linearity of the principal minors, it can not be shown optimality by analytical method. One can only use the condition of Hessian matrix.

For this case, second order derivatives of the profit function AVP_{1i} are as

$$\begin{aligned}
\frac{\partial^2 AVP_{1i}}{\partial T^2} &= \frac{-2}{T^3}\left[C_1 + C_H X_1\left(t_1 + \frac{e^{-\theta t_1}}{\theta}\right) - C_H\frac{b_1{t_1}^2}{2\theta} - C_H\frac{c_1{t_1}^3}{3\theta} + C_H\frac{c_1{t_1}^2}{\theta^2}\right] + \frac{2C_H}{T^3}\left[X_2(T\right. \\
&- \left. t_1) + \frac{Y_1(T^2 - {t_1}^2)}{2} + \frac{Y_2(T^3 - {t_1}^3)}{3}\right] + \frac{(X_2 + Y_1T + Y_2T^2)}{T}\left[\frac{U_fC_P}{T}\left(\frac{e^{\theta(T-K_i)} - 1}{T\theta}\right.\right. \\
&- \left. e^{\theta(T-K_i)}\right) + \frac{C_H}{T}e^{\theta(T-t_1)} - \frac{C_H}{T^2}\left(\frac{e^{\theta(T-t_1)} - 1}{\theta}\right) - C_H\theta e^{\theta(T-t_1)} - U_fC_Pe^{\theta(T-K_i)}\theta\Big] \\
&+ \frac{2U_fC_P}{T^3}\left[X_2(T - K_i) - \frac{Y_1(T^2 - {K_i}^2)}{2} - \frac{Y_2(T^3 - {K_i}^3)}{3}\right] - \frac{(Y_1 + 2Y_2T)}{T}\Big[C_He^{\theta(T-t_1)} \\
&+ U_fC_Pe^{\theta(T-K_i)}\Big] - \frac{(X_2 + Y_1T + Y_2T^2)}{T^2}(C_H + U_fC_P) + \frac{Y_1}{T^2}\left[C_H\left(\frac{e^{\theta(T-t_1)} - 1}{\theta}\right)\right. \\
&+ \left. U_fC_P\frac{e^{\theta(T-K_i)} - 1}{\theta}\right]
\end{aligned}$$

$$\frac{\partial^2 AVP_{1i}}{\partial P^2} = -\frac{h{K_i}^2U_c}{T},$$

and

$$\frac{\partial^2 AVP_{1i}}{\partial P \partial T} = \frac{(hK_i{}^2 PU_c - \mu t_1)}{T^2} - \frac{U_c}{T^2}\left(\frac{a_1 K_i{}^2}{2} + \frac{b_1 K_i{}^3}{6} + \frac{c_1 K_i{}^4}{12}\right) + \frac{C_H h}{\theta T^2}\left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta}\right) + \frac{C_H h}{\theta} - \frac{U_f C_P h}{T^2 \theta}(1 + K_i).$$

Appendix D1

The Hessian matrix for AVP_{2i} is $\mathbf{H}$ as follows

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 AVP_{2i}}{\partial T^2} & \frac{\partial^2 AVP_{2i}}{\partial T \partial P} \\ \frac{\partial^2 AVP_{2i}}{\partial P \partial T} & \frac{\partial^2 AVP_{2i}}{\partial P^2} \end{bmatrix}$$

Now, if the leading principal minors $\frac{\partial^2 AVP_{2i}}{\partial T^2} < 0$ and $\frac{\partial^2 AVP_{2i}}{\partial T^2}\frac{\partial^2 AVP_{2i}}{\partial P^2} - (\frac{\partial^2 AVP_{2i}}{\partial P \partial T})^2 > 0$ at the optimal point, then $\mathbf{H}$ is negative definite and the function AVP_{2i} is strictly concave.

As the principal minors are highly non-linear, therefore it can not be provide optimality by analytical method.

Second order derivatives of the profit function AVP_{2i} are as

$$\frac{\partial^2 AVP_{2i}}{\partial T^2} = \frac{2(P - C_P)\mu t_1}{T^3} + PU_c\left(\frac{b_1}{3} + \frac{c_1 T}{2}\right) - \frac{2c_1}{T^3} - \frac{C_H Q}{T^3} - \frac{C_H X_2}{T^2} + C_H Y_2\Big[1 - e^{\theta(T-t_1)} + \frac{1}{T}\left(\frac{e^{\theta(T-t_1)} - 1}{\theta}\right)\Big] + C_H\Big(\frac{X_2}{T} + Y_1 + Y_2 T\Big)\Big[\theta e^{\theta(T-t_1)} - \left(\frac{e^{\theta(T-t_1)} - 1}{\theta T^2}\right) + \left(\frac{e^{\theta(T-t_1)}}{T}\right)\Big] - \frac{C_H}{T}(Y_1 + 2Y_2 T)e^{\theta(T-t_1)} + \left(\frac{e^{\theta(T-t_1)} - 1}{\theta}\right)\Big[\frac{C_H}{T^2}(Y_1 + 2Y_2 T) - \frac{2Y_2 C_H}{T}\Big],$$

$$\frac{\partial^2 AVP_{2i}}{\partial P^2} = -\frac{hU_c}{2}(T + 1),$$

and

$$\frac{\partial^2 AVP_{2i}}{\partial P \partial T} = U_c\left[\frac{(a_1 - hP)}{2} + \frac{b_1 T}{3} + \frac{c_1 T^2}{4} - \frac{\mu t_1(K_i - T)}{T^2} - \frac{\mu t_1}{T} - hP\right] - \frac{\mu t_1}{T^2}$$

As the above mentioned profit function is a non-linear equation and second order derivatives of AVP_{2i} with respect to P and T are extremely complicated. Hence the closed form solution cannot be obtained. However, by means of empirical experiments, one can indicate that above equation is concave for small value of P and T.