Chapter 2

Mathematical Methods and Optimization Techniques

2.1 Preliminary

Before studing different solid transportation problems (STPs) in different aspects and forms in different environments, we need to discusse the following preliminary ideas.

2.1.1 Crisp Set Theory

Crisp Set [109]: In our life, we have mostly used the crisp sets. Crisp means yes or no type rather than more-or-less type. In set theory, an object can either a member of a set or not and in optimization problem, a solution is either feasible or not. A classical set, X, is defined by crisp boundaries, i.e., there is no uncertainty in the prescription of the elements of the set. Normally, it is defined as a collection of well defined objects or elements, $x \in X$, where X may be countable or uncountable.

Convex Set [109]: Let x_1 and x_2 be any two points in a subset $S \subset \Re^n$, the subset S is said to be convex set iff $x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S; \quad 0 \leq \lambda \leq 1$.

Convex Combination [109]: For a given set of vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ a linear combination $\alpha = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n$ is called a convex combination of the given vectors provided that $\sum_{i=1}^{n} \lambda_i = 1$.
Convex Function: For any two point $x_1, x_2 \in S$, if a function $f : S \rightarrow \Re$ satisfies the inequality $f\{(1-\lambda)x_1 + \lambda x_2\} \leq (1-\lambda)f(x_1) + \lambda f(x_2)$, for $0 \leq \lambda \leq 1$, then the function is called convex

Quasi-convex Function [109]: For any two number $x_1, x_2 \in S$, if a function $f(x)$ satisfies $f((1-\lambda)x_1 + \lambda x_2) \leq \max\ (f(x_1), f(x_2))$, for $0 \leq \lambda \leq 1$, then it is called quasi-convex. It is noted that a convex function is also quasi-convex since $f((1-\lambda)x_1+\lambda x_2) < (1-\lambda)f(x_1) + \lambda f(x_2) < \max\ (f(x_1), f(x_2))$.

Pseudo-convex Function [109]: For any two point $x_1, x_2 \in S$, a function $f(x)$ is said to be pseudo-convex function provided that $f(x_2) \geq f(x_1)$ implies that $(x_2 - x_1)^T \nabla f(x_1) \geq 0$. The definition of convex functions can be modified for concave functions by replacing '$\leq$' by '$\geq$'. Correspondingly, the definition of quasi-convex functions becomes appropriate for quasi-concave functions by the exchange of '$\leq$' to '$\geq$' and 'max' to 'min'. To get the definition for pseudo-concave function, '$\geq$' is replaced by '$\leq$' in the definition of pseudo-convex functions.

Relationships among Convex, Quasi-convex and Pseudo-convex Functions:

(i) A convex differentiable function is pseudo-convex.

(ii) A convex function is quasi-convex.

(iii) A pseudo-convex function is strictly quasi-convex.

(iv) If $f(x)$ is either positive or negative quasi-convex (quasi-concave)on a subset S of $\Re^n$, $\frac{1}{f(x)}$ is quasi-concave (quasi-convex) on S.

2.1.2 Interval Number

Arithmetic of Interval Number: In this section, real numbers are denoted by lower case letters and a set is denoted by upper case letter. An order pair of brackets defines an interval $A = [a_L, a_U] = \{a : a_L \leq a \leq a_U\}$ where a_U and a_L are right and left limits of A respectively.

Definition-2.1: If A and B are two closed intervals, then $A * B = \{a * b : a \in A, b \in B\}$ defines a binary composition on the set of closed intervals. The operation $*$ may be any one of $\{+, -, \times, \div\}$ defined on $\Re$. In the case of division, it is assumed that $0 \notin B$. The operations on intervals used here may be explicitly calculated from the above definition as

$$
\begin{aligned}
A+B &= [a_L, a_U] + [b_L, b_U] = [a_L + b_L, a_U + b_U] \\
A-B &= [a_L, a_U] - [b_L, b_U] = [a_L - b_U, a_U - b_L] \\
A\times B &= [a_L, a_U].[b_L, b_U] = [min\{a_L b_L, a_L b_U, a_U b_L, a_U b_U\}, max\{a_L b_L, a_L b_U, a_U b_L, a_U b_U\}] \\
\frac{A}{B} &= \frac{[a_L, a_U]}{[b_L, b_U]} = [a_L, a_U].[\frac{1}{b_U}, \frac{1}{a_U}], \text{ where } 0 \notin B \\
\xi A &= \begin{cases} [\xi a_L, \xi a_U], & \text{for } \xi \geq 0 \\ (\xi a_U, \xi a_L), & \text{for } \xi < 0, \quad \text{where } \xi \text{ is a real number.} \end{cases}
\end{aligned}
$$

Order relations between intervals: Here, a order relation represents the decision-maker's preference between interval costs. Besically it is defined for minimization problems. Let intervals A and B represent uncertain costs for two alternatives. It is considered that the cost of each alternative is known only to lie to the corresponding interval.

Definition-2.2: The order relation $\leq_{LR}$ between $A = [a_L, a_U]$ and $B = [b_L, b_U]$ is defined as

$$A \leq_{LR} B \text{ iff } a_L \leq b_L \text{ and } a_U \leq b_U$$
$$A <_{LR} B \text{ iff } A \leq_{LR} B \text{ and } a_U \neq b_U$$

The order relation $\leq_{LR}$ represents the DM's preference for the alternative with the lower minimum cost, that is, if $A \leq_{LR} B$, then A is preferred to B.

Theorem 2.1: The order relation $\leq_{RC}$ satisfies the transitive law.
Proof: Let $A = [a_L, a_U], B = [b_L, b_U]$ and $C = [c_L, c_U]$ such that
$A \leq_{RC} B$ and $B \leq_{RC} C$,
Now $A \leq_{RC} B$ & $B \leq_{RC} C$, which implies that
$a_U \leq b_U$ & $b_U \leq c_U$; and $a_C \leq b_C$ & $b_C \leq c_C$;
$ie, a_U \leq b_U \leq c_U$; and $a_C \leq b_C \leq c_C$
and $2a_C \leq 2b_C \leq 2c_C$
Then subtracting we have $a_L \leq b_L \leq c_L$ and $A \leq_{RC} B$ & $B \leq_{RC} C$,

which implies that $A \leq_{RC} C$

If $A \neq_{RC} B$ and $B \neq_{RC} C$, we have $A \neq_{RC} C$.

If possible $A =_{RC} C$ which indicates that $a_L = c_L$ and $a_U = c_U$.

Now $a_U \leq b_U \leq c_U$ (Since $A \leq_{RC} B$, $B \leq_{RC} C$ & $A \leq_{RC} C$)

So, we have $a_U = b_U = c_U$

Moreover, $a_L = c_L$ and $a_U = c_U$, so it is obiained that

$$a_C = \frac{a_L + a_U}{2} = \frac{c_L + c_U}{2}$$

$\Rightarrow \quad a_C = b_C = c_C$ (Since $a_C \leq b_C \leq c_C$

$A \leq_{RC} B$ & $B \leq_{RC} C$)

As $a_C = b_C, \quad b_C = c_C$

$a_U = b_U, \quad b_U = c_U$

$\Rightarrow a_L = b_L$ and $b_L = c_L$

$\Rightarrow a_L = b_L = c_L$

$\Rightarrow A = B$ & $B = C$,

which contra. Therefore,

$A \neq_{RC} B$ and $B \neq_{RC} C \Rightarrow A \neq_{RC} C$.

2.1.3 Fuzzy Set Theory

In 1962, Prof. Lotfi Zadeh introduced the notion of fuzzy set to formalize the concept of gradedness in class membership for the representation of human knowledge. It was developed to solve and define a complex system which is full of imprecision or uncertainty in the sense of non-statistical nature. A very brief of the fuzzy set theory is given below.

Fuzzy Set: Let F be an universal collection of elements and x be an object of F, then a fuzzy set $\widetilde{A}$ in F is a set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | \, x \in F\}$, where $\mu_{\widetilde{A}}(x)$ is called membership function of x in $\tilde{A}$ which maps F to the membership space M to be a closed interval [0,u], where $0 < u \leq 1$. So a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not.

Some Properties of Fuzzy Set:

- **Equality [108]:** Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ in F are said to be equal, if and only if

$$\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x), \forall x \in F.$$

- **Containment:** A fuzzy set $\tilde{A}$ in F is a subset of another fuzzy set $\tilde{B}$ in F denoted by $\tilde{A} \subset \tilde{B}$, if and only if

$$\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x), \forall x \in F$$

- **Support [108]:** The support of a fuzzy set $\widetilde{A}$ is a crisp set, denoted by $S(\widetilde{A})$, and it is defined by $S(\widetilde{A}) = \{x \,|\, \mu_{\widetilde{A}}(x) > 0\}$.

- **Crossover Point:** The element in F at which its membership function is 0.5, it is called a crossover point of a fuzzy set.

- **Fuzzy Singleton:** When support of a fuzzy set $\widetilde{A}$ in F is a single point in F with a membership function of one, it is known as a Fuzzy singleton.

- **Core:** The core of a fuzzy set $\widetilde{A}$, denoted by $Core(\widetilde{A})$, is a set of all points of F with unit membership degree in $\widetilde{A}$. So, it is defined as $Core(\widetilde{A}) = \{x \in F \,|\, \mu_{\widetilde{A}}(x) = 1\}$.

- **Normality [108]:** A fuzzy set $\widetilde{A}$ be normal if its core is non-empty, i.e., there exists at least one element $x \in F$, such that $\mu_{\widetilde{A}}(x) = 1$.

- **Convexity [108]:** A fuzzy set $\widetilde{A}$ in F is said to be convex if and only if for any two ellement $x_1, x_2 \in F$. $\mu_{\widetilde{A}}(x)$ satisfies the following inequality

$$\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\} \text{ for } 0 \leq \lambda \leq 1$$

Based on above properties of a fuzzy set, the definition of fuzzy number is given below

Fuzzy Number [108]: A fuzzy subset $\widetilde{A}$ of real number $\Re$ with membership function $\mu_{\tilde{A}} : \Re \to [0, 1]$ is called a fuzzy number if

(a) There exist an element x_0 such that $\mu_{\tilde{A}}(x_0) = 1$, i.e, $\widetilde{A}$ is normal;

(b) $\widetilde{A}$ is convex.

(c) $\mu_{\tilde{A}}$ is upper semi-continuous and

(d) $S(\widetilde{A})$ is bounded, here $S(\widetilde{A}) = cl\{x \in \Re : \mu_{\tilde{A}}(x) > 0\}$, and cl is the closer operator.

Some particular examples of **continuous fuzzy numbers** defined on **real number set** are as follows:

General Fuzzy Number (GFN) [110]: For any fuzzy number $\widetilde{A}$, if there exist four numbers $a_1, a_2, a_3, a_4 \in \Re$ and two functions $f(x) : \Re \to [0, 1]$ and $g(x) : \Re \to [0, 1]$ where f(x) is non decreasing and g(x) is non increasing, such that its membership function $\mu_{\tilde{A}}(x)$ is defined in a following manner (cf. Figure 2.1):

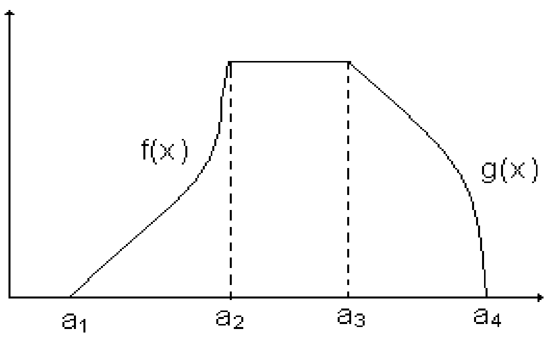

Figure 2.1: Membership function of a GFN

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & \text{for } x < a_1 \\ f(x) & \text{for } a_1 \leq x < a_2 \\ 1 & \text{for } a_2 \leq x \leq a_3 \\ g(x) & \text{for } a_3 < x \leq a_4 \\ 0 & \text{for } a_4 < x < \infty \end{cases} \tag{2.1}$$

The functions f(x) and g(x) be called the left and right functions of the fuzzy number $\widetilde{A}$ respectively.

Linear Fuzzy Number (LFN) [107]: For any fuzzy number $\widetilde{A}$, if there exist two numbers $a_1, a_2 \in \Re$ and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : \Re \rightarrow [0, 1]$ as follows (cf. Figure 2.2):

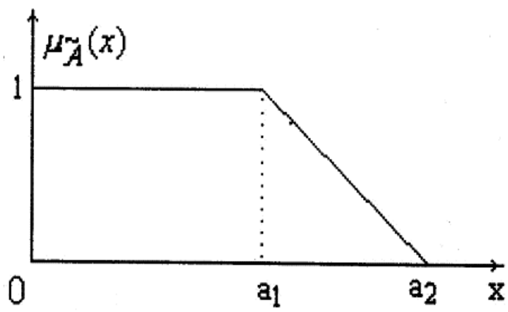

Figure 2.2: Membership function of a one side (left) TFN i.e, LFN

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 & \text{if } x \leq a_1 \\ \dfrac{a_2 - x}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ 0 & \text{if } x \geq a_2 \end{cases} \tag{2.2}$$

Triangular Fuzzy Number (TFN) [21]: For a fuzzy number $\widetilde{A}$, if there exist three numbers $a_1, a_2, a_3 \in \Re$ and it is defined by its continuous membership function $\mu_{\widetilde{A}}(x)$:

$\Re \rightarrow [0,1]$ follows (cf. Figure 2.3), it is known as triangular fuzzy number (TFN). Centraly, it is represented by the triplet (a_1, a_2, a_3)

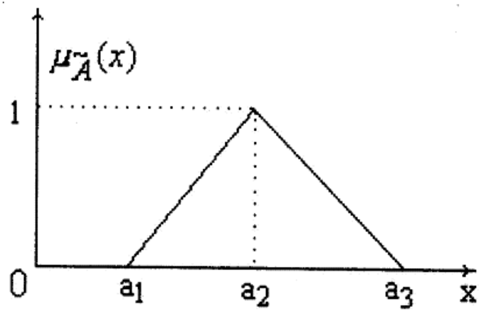

Figure 2.3: Membership function of a TFN

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Parabolic Fuzzy Number (PFN): For a fuzzy number $\widetilde{A}$, if there exist three numbers $a_1, a_2, a_3 \in \Re$ and it is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : \Re \rightarrow [0,1]$ follows (cf. Figure 2.4), it is known as parabolic fuzzy number. Centraly, it is represented by the triplet (a_1, a_2, a_3)

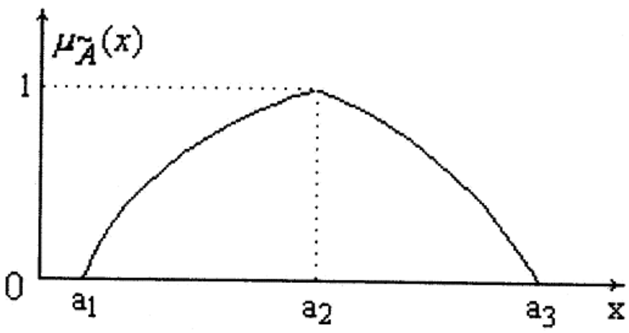

Figure 2.4: Membership function of a PFN

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - (\frac{a_2 - x}{a_2 - a_1})^2 & \text{for } a_1 \leq x \leq a_2 \\ 1 - (\frac{x - a_2}{a_3 - a_2})^2 & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

Trapezoidal Fuzzy Number (TrFN): [110] For a fuzzy number $\widetilde{A}$, if there exist four numbers $a_1, a_2, a_3, a_4 \in \Re$ and it is defined by its continuous membership function $\mu_{\tilde{A}}(x) : \Re \rightarrow [0, 1]$ follows (cf. Figure 2.5), it is known as trapezoidal fuzzy number. Centraly, it is represented by the triplet (a_1, a_2, a_3, a_4)

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\ 1 & \text{for } a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3} & \text{for } a_3 \leq x \leq a_4 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

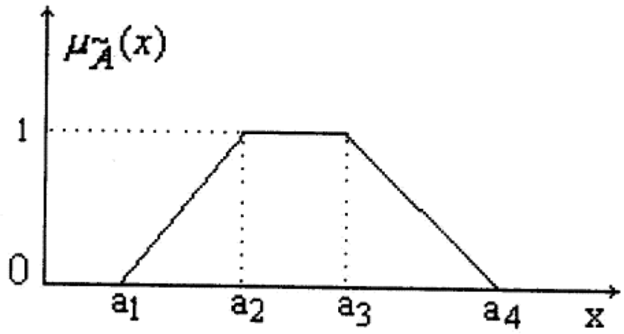

Figure 2.5: Membership function of a TrFN

Arithmetic of Fuzzy Number:

Hsieh [59] presented Function Principle in fuzzy theory for computational model avoiding the complications which are caused by the operations using Extension Principle. The fuzzy arithmetical operations under Function Principle for two trapezoidal fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ are as follows:
Here $\oplus, \ominus, \otimes$ and $\oslash$ are the different fuzzy arithmetical operations by Function Principle.

(i) The addition of $\widetilde{A}$ and $\widetilde{B}$ is $\widetilde{A} \oplus \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any real numbers.

(ii) The multiplication of $\widetilde{A}$ and $\widetilde{B}$ is $\widetilde{A} \otimes \widetilde{B} = (c_1, c_2, c_3, c_4)$, where c_1 =min T, c_2 =min T_1, c_3 =max T_1, c_4 =max T for, $T = \{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $T_1 = \{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$. Also, if a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are all positive real numbers, then $\widetilde{A} \otimes \widetilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$, where $\widetilde{A} \otimes \widetilde{B}$ is a trapezoidal fuzzy number.

(iii) $\widetilde{A}^n = ({a_1}^n, {a_2}^n, {a_3}^n, {a_4}^n)$ if a_1, a_2, a_3 and a_4 are all positive real numbers and n is a natural number.

(iv) $-\widetilde{B} = (-b_4, -b_3, -b_2, -b_1)$, then the subtraction of $\widetilde{A}$ and $\widetilde{B}$ is defined by $\widetilde{A} \ominus \widetilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any real numbers.

(v) $\frac{1}{\widetilde{B}} = (\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1})$, where b_1, b_2, b_3 and b_4 are all positive real numbers. If a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are all positive real numbers, then the division of $\widetilde{A}$ and $\widetilde{B}$ is $\widetilde{A} \oslash \widetilde{B} = (\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1})$, provided $0 \notin B$

(vi) $$\rho \otimes \widetilde{A} = \begin{cases} (\rho a_1, \rho a_2, \rho a_3, \rho a_4) & \text{for } \rho \geq 0 \\ (\rho a_4, \rho a_3, \rho a_2, \rho a_1) & \text{for } \rho < 0. \end{cases}$$

2.1.4 Overview of fuzzy inference process

Human knowledge is often represented imprecisely. In real life problems some vague terms such as high,medium,low etc are used.The target of fuzzy inference process is to form it into natural language expressions of the following type,

IF premise (antecedent) THEN conclusion (consequent)

It is commonly referred to as IF -THEN rule-based form. A fuzzy inference process comprises of three parts(according to Mamdani [91]): (i)Fuzzification of input variables, (ii)Rule-strength and fuzzy output calculation and (iii)Difuzzification of the fuzzy output. **Fuzzification of input variables:**

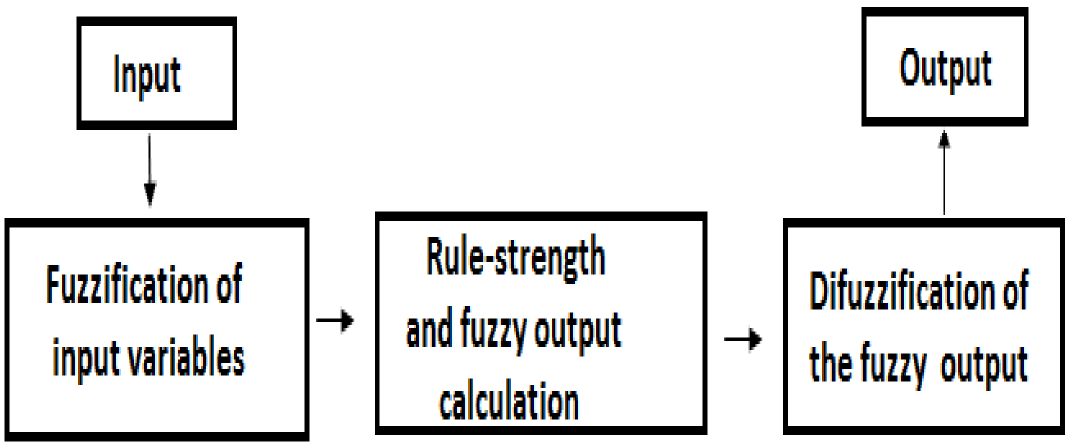

Figure 2.6: Fuzzy logic system

When a crisp value of the transportation costs are given as an input, it must correspond to someone or more antecedent of the rules including some membership grade.

Rule-Strength Calculation:

After fuzzification of the inputs, the degree of belongingness to which every part of the antecedent is satisfied to each rule is known. The degree of a rule is the rule strength of the corresponding rule. If there are more than one antecedent, then the rule strength is calculated by standard fuzzy intersection operator (min operator) as

$$\mu_{R_i} = min\{\mu_A^{R_i}(x), \mu_B^{R_i}(y), ...\}$$

where $\mu_A^{R_i}(x), \mu_B^{R_i}(y)$, ... are the membership values of inputs x, y, ... to the antecedent A, B of the rule R_i. Hence. the output is a single truth value for each rule. This is known as the rule strength of the corresponding rule(where the rule strength lie between 0 and 1), the membership value of fuzzy output is calculated using the relation

$$\underbrace{max}_{\forall i}\{min\{\mu^{R_i}A(x), \mu_{R_i}........\}\}, \forall x \in X$$

where X is the universal set and $\mu_A^{R_i}(x), \mu^{R_i}$ are respectively the membership values of the consequent A to the i-th rule R_i and the rule strength of that rule.

2.1.5 Approximated Value of Triangular Fuzzy Number (TFN)

The approximated value of a triangular fuzzy number $\tilde{a} \equiv (a_1, a_2, a_3)$ is given by $\tilde{a} = \dfrac{a_1 + 2a_2 + a_3}{4}$, according to Kaufmann and Gupta [67].

2.1.6 Zadeh's Extension Principle

One of the basic concepts of fuzzy set theory which is used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle. Let X and Y be two universes and $f : X \longrightarrow Y$ be a crisp function. The extension principle tells us how to induce a mapping $f : P(X) \longrightarrow P(Y)$, where $P(X)$ and $P(Y)$ are the power sets of X and Y respectively.

Following Zadeh [155], the fuzzy extension principle is as follows:

We have a mapping $f : X \longrightarrow Y$, $y = f(x)$ which induce a function $f : \widetilde{A} \longrightarrow \widetilde{B}$ such that $\widetilde{B} = f(\widetilde{A}) = \{((y, \mu_{\tilde{B}}(y))|y = f(x), x \in X)\}$, where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x) & \text{if } f^{-1}(y) \neq \Phi, \\ 0 & \text{otherwise} \end{cases}$$

In general, if f : $X_1 \times X_2 \times ...... \times X_n \longrightarrow$ Y, and $\widetilde{A_1}, \widetilde{A_2}, ..., \widetilde{A_n}$ are fuzzy sets in $X_1, X_2, ..., X_n$, the extension principle is defined as f : $\widetilde{A} \longrightarrow \widetilde{B}$ such that $\widetilde{B} = \{((y, \mu_{\tilde{B}}(y))|y = f(x_1, x_2, ..., x_n), (x_1, x_2, ..., x_n) \in X)\}$, where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1, x_2, ..., x_n) \in f^{-1}(y)} min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2), ..., \mu_{\tilde{A}}(x_n)\} & \text{if } f^{-1}(y) \neq \Phi, \\ 0 & \text{otherwise} \end{cases}$$

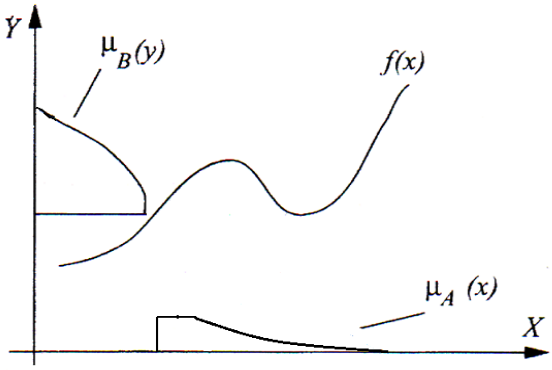

Figure 2.7: Fuzzy Extension Principle

Example 2.1: Let f(x)=x^2 and $\widetilde{A}$ be a symmetric triangular fuzzy number with membership function

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 - \dfrac{|a - x|}{\alpha} & \text{if } |a - x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then using the extension principle we get,

$$\mu_{\widetilde{B}}(y) = \begin{cases} \mu_{\widetilde{A}}(\sqrt{y}) & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

i.e.,

$$\mu_{\widetilde{B}}(y) = \begin{cases} 1 - \dfrac{|a-\sqrt{y}|}{\alpha} & \text{if } |a-\sqrt{y}| \leq \alpha, y = f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

2.1.7 α-cut of Fuzzy Number

The α-level set / α- cut of a fuzzy number $\widetilde{A}$ is a crisp set and it is defined as $\widetilde{A}^{\alpha} = \{x \in \Re : \mu_{\widetilde{A}}(x) \geq \alpha\}$. In other words, the α-level set / α- cut is a non-empty bounded closed interval (Wu [145]), which is denoted by $\widetilde{A}^{\alpha} = [A_L^{\alpha}, A_U^{\alpha}]$, provided $\widetilde{A}$ is a closed fuzzy number. Here, A_L^{α} and A_U^{α} are the lower and upper bounds of the closed interval and

$$A_L^{\alpha} = inf\{x \in \Re : \mu_{\widetilde{A}}(x) \geq \alpha\}$$
$$A_U^{\alpha} = sup\{x \in \Re : \mu_{\widetilde{A}}(x) \geq \alpha\}.$$

Proposition 2.2: (Wu [145]) If $\widetilde{A}$ and $\widetilde{B}$ be two closed fuzzy numbers, then $\widetilde{A} \oplus \widetilde{B}$, $\widetilde{A} \ominus \widetilde{B}$ and $\widetilde{A} \otimes \widetilde{B}$ are also closed fuzzy numbers. These are defined as

$$\begin{aligned} \left(\widetilde{A} \oplus \widetilde{B}\right)^{\alpha} &= \widetilde{A}^{\alpha} \oplus_{int} \widetilde{B}^{\alpha} = [A_L^{\alpha} + B_L^{\alpha}, A_U^{\alpha} + B_U^{\alpha}], \\ \left(\widetilde{A} \ominus \widetilde{B}\right)^{\alpha} &= \widetilde{A}^{\alpha} \ominus_{int} \widetilde{B}^{\alpha} = [A_L^{\alpha} + B_U^{\alpha}, A_U^{\alpha} + B_L^{\alpha}] \\ \left(\widetilde{A} \otimes \widetilde{B}\right)^{\alpha} &= \widetilde{A}^{\alpha} \otimes_{int} \widetilde{B}^{\alpha} \\ &= [min\{A_L^{\alpha}B_L^{\alpha}, A_L^{\alpha}A_U^{\alpha}, A_U^{\alpha}A_L^{\alpha}, A_U^{\alpha}A_U^{\alpha}\}, \\ &\qquad max\{A_L^{\alpha}B_L^{\alpha}, B_L^{\alpha}B_U^{\alpha}, A_U^{\alpha}A_L^{\alpha}, A_U^{\alpha}A_U^{\alpha}\}] \end{aligned}$$

where, $\oplus, \ominus$ and $\otimes$ are binary operations between two fuzzy numbers or one real and another fuzzy number.

2.1.8 Defuzzification Methods

By this method any fuzzy valued function is converted to corresponding crisp valued function. This section, discusses some significant defuzzification methods as follows

(i) The Nearest Interval Approximation:

Here, in this method a fuzzy number is estimated by a corresponding crisp interval. Suppose that $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers whose α-cuts are $[A_L^\alpha, A_U^\alpha]$ and $[B_L^\alpha, B_U^\alpha]$ respectively. Then according to Grzegorzewski [47], the lenth of the space between $\tilde{A}$ and $\tilde{B}$ can be elucidated as:

$$d(\tilde{A}, \tilde{B}) = \sqrt{\int_0^1 (A_L^\alpha - B_L^\alpha)^2 d\alpha + \int_0^1 (A_U^\alpha - B_U^\alpha)^2 d\alpha}$$

In this approach, for a fuzzy number $\tilde{A}$, a closed interval $C_d(\tilde{A})$ is determined in such away that it is the nearest to $\tilde{A}$ on the basis od distance d. Each interval is also a fuzzy number with constant α-cut $\forall\ \alpha \in [0,1]$. Hence $(C_d(\tilde{A}))^\alpha = [C_L^\alpha, C_U^\alpha]$. Now, with respect to C_L^α and C_U^α, the following matric is minimized.

$$d(\tilde{A}, C_d(\tilde{A})) = \sqrt{\int_0^1 (A_L^\alpha - C_L^\alpha)^2 d\alpha + \int_0^1 (A_U^\alpha - C_U^\alpha)^2 d\alpha}$$

In order to minimize $d(\tilde{A}, C_d(\tilde{A}))$, it is sufficient to minimize the function $D(C_L^\alpha, C_U^\alpha) = d^2(\tilde{A}, C_d(\tilde{A}))$. The first partial derivatives are given by

$$\frac{\delta D(C_L^\alpha, C_U^\alpha)}{\delta C_L^\alpha} = -2\int_0^1 A_L^\alpha d\alpha + 2C_L \text{ and } \frac{\delta D(C_L^\alpha, C_U^\alpha)}{\delta C_U^\alpha} = -2\int_0^1 A_U^\alpha d\alpha + 2C_U$$

Solving, $\frac{\delta D(C_L^\alpha, C_U^\alpha)}{\delta C_L^\alpha} = 0$ and $\frac{\delta D(C_L^\alpha, C_U^\alpha)}{\delta C_U^\alpha} = 0$, we get the following

$C_L^* = \int_0^1 A_L^\alpha d\alpha$ and $C_U^* = \int_0^1 A_U^\alpha d\alpha$.

Again since, $\frac{\delta D^2(C_L^*, C_U^*)}{\delta C_L^2} = 2 > 0,\ \frac{\delta D^2(C_L^*, C_U^*)}{\delta C_U^2} = 2 > 0$

and $\frac{\delta D^2(C_L^*, C_U^*)}{\delta C_L^2} . \frac{\delta D^2(C_L^*, C_U^*)}{\delta C_U^2} - \left(\frac{\delta D^2(C_L^*, C_U^*)}{\delta C_L . \delta C_U}\right)^2 = 4 > 0,$

so, $D(C_L^\alpha, C_U^\alpha)$ is a convex function of C_L^α and C_U^α. Henceforth, $D(C_L^\alpha, C_U^\alpha)$ ie., $d(\tilde{A}, C_d(\tilde{A}))$ has global minimum at C_L^*, C_U^* . Therefore, the nearest interval is given by

$$C_d(\tilde{A}) = \left[\int_0^1 A_L^\alpha d\alpha, \int_0^1 A_U^\alpha d\alpha\right]$$

so, with respect to metric d, this is nearest interval approximation of fuzzy number $\tilde{A}$. Similarly, if $\tilde{A} = (a_1, a_2, a_3, a_4)$ be a fuzzy number, the α-level interval of $\tilde{A}$ is defined as $(\tilde{A})^\alpha = [A_L^\alpha, A_U^\alpha]$. When $\tilde{A}$ is a TrFNs, then $A_L^\alpha = a_1 + \alpha(a_2 - a_1)$ and $A_U^\alpha = a_4 - \alpha(a_4 - a_3)$, for $0 < \alpha \leq 1$.

By nearest interval approximation method, lower and upper limits of the interval are given by respectively as

$$C_L^\alpha = \int_0^1 A_L^\alpha d\alpha = \int_0^1 [a_1 + \alpha(a_2 - a_1)]d\alpha] = \frac{1}{2}(a_2 + a_1)$$

$$\text{and} \quad C_U^\alpha = \int_0^1 A_U^\alpha d\alpha = \int_0^1 [a_4 - \alpha(a_4 - a_3)]d\alpha] = \frac{1}{2}(a_3 + a_4)$$

Using the above definition, the nearest crisp interval number considering $\tilde{A}$ as a trapezoidal fuzzy number is $[\frac{a_1+a_2}{2}, \frac{a_3+a_4}{2}]$.

Similarly, for a triangular fuzzy number $\widetilde{A}$, the nearest interval approximation is $[\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]$.

(ii) Signed Distance:

Following Yao and We [151], if two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ be represented by $\widetilde{A} = \bigcup_{\alpha\in[0,1]} [A_L^\alpha(\alpha), A_U^\alpha(\alpha)]$ and $\widetilde{B} = \bigcup_{\alpha\in[0,1]} [B_L^\alpha(\alpha), B_U^\alpha(\alpha)]$, then the signed distance of $\widetilde{A}$ and $\widetilde{B}$ is the distance between the mid points $M(A(\alpha)) = \frac{A_L(\alpha)+A_U(\alpha)}{2}$ of $[A_L(\alpha), A_U(\alpha)]$ and $M(B(\alpha)) = \frac{B_L(\alpha)+B_U(\alpha)}{2}$ of $[B_L(\alpha), B_U(\alpha)]$ over α in [0,1], where $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$ are the real intervals corresponding to $[A_L^\alpha(\alpha), A_U^\alpha(\alpha)]$ and $[B_L^\alpha(\alpha), B_U^\alpha(\alpha)]$ respectively, which is as follows

$$\begin{aligned} d(\widetilde{A}, \widetilde{B}) &= \frac{1}{1-0}\int_0^1 [M(A(\alpha)) - M(B(\alpha))]d\alpha, \\ &= \frac{1}{2}\int_0^1 [A_L(\alpha) + A_U(\alpha) - B_L(\alpha) - B_U(\alpha)]d\alpha. \end{aligned} \tag{2.5}$$

In particular, if $\widetilde{A}$ be a TFN and $\widetilde{0}$ be a TFN represented by (a_1, a_2, a_3) and $(0, 0, 0)$ respectively, then $d(\widetilde{A}, \widetilde{0}) = \frac{2a_2+a_1+a_3}{4}$.

(iii) Graded Mean and Modified Graded Mean Integration Representation:

In addition, Chen and Hsieh [23] introduced Graded Mean Integration Representation method based on the integral value of graded mean α-level of generalized fuzzy number. The graded mean α-level value of generalized fuzzy number $\widetilde{a} = (a_1, a_2, a_3, a_4)$ is $\alpha\left[\frac{a^-(\alpha)+a^+(\alpha)}{2}\right]$ for all $\alpha \in (0, 1]$. So, the graded mean integration representation of TrFN $\tilde{a}$ is

$$P(\tilde{a}) = \int_0^1 \alpha\left[\frac{a^-(\alpha)+a^+(\alpha)}{2}\right] d\alpha \Big/ \int_0^1 \alpha d\alpha = \frac{1}{6}[a_1 + 2a_2 + 2a_3 + a_4]. \tag{2.6}$$

Here, equal weightage has been given to the left and right parts of the membership function. But the weightage depends on th attitude or optimism of the decision maker. For this reason it has been modified to $\alpha\left[ka^-(\alpha) + (1-k)a^+(\alpha)\right]$, where $k \in [0, 1]$ is called the decision maker's attitude or optimism parameter. It is known as the modified graded mean α-level value of the fuzzy number $\widetilde{a}$. The value of k closer to 0 implies that the decision maker is more pessimistic while the value of k closer to 1 means that the decision maker is more optimistic. Therefore, the modified form of (2.6) is

$$\begin{aligned} P_k(\tilde{a}) &= \frac{\int_0^1 \alpha\left[ka^-(\alpha) + (1-k)a^+(\alpha)\right] d\alpha}{\int_0^1 \alpha d\alpha} \\ &= \frac{1}{3}[k(a_1 + 2a_2) + (1-k)(2a_3 + a_4)]. \end{aligned} \tag{2.7}$$

The method is also called as k-preference integration representation.

Defuzzification of fuzzy inference:

For calculation of transportation cost, the fuzzy inference module (for single input to single output) is given below-

Step1: Take transportation distance as input and transportation cost as output.

Step2: Calculate the membership values to the fuzzy sets High, Medium and Low for finding cost.

Step3: Evaluate the rules and find the rule strengths of each rule.

Step4: Calculates the membership functions of the fuzzy amount High, Medium, Low which are represented by the rules with non-zero strength.

Step5: Apply fuzzy union operator to find the fuzzy output.

Step6: Apply centroid method to find the defuzzified cost.

(iv) Centroid Method:

If $\widetilde{A}$ be a fuzzy number defined on $\Re$, then the centroid of $\widetilde{A}$ is defined by

$$C(\widetilde{A}) = \frac{\int_{-\infty}^{\infty} x\mu_{\widetilde{A}}(x)dx}{\int_{-\infty}^{\infty} \mu_{\widetilde{A}}(x)dx}. \tag{2.8}$$

For a triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ with membership function defined in (2.3), then $C(\widetilde{A}) = \frac{a_1+a_2+a_3}{3}$.

2.1.9 Possibility / Necessity / Credibility in Fuzzy Environment

Considering the degree of membership $\mu_{\widetilde{A}}(x)$ of an element x in a fuzzy set $\widetilde{A}$, defined on a referential U. Three interpretations of this degree (Dubois and Prade [34]) can be found in the literature as following

(i) Degree of similarity: On the basis of the degree of similarity, $\mu_{\widetilde{A}}(x)$ is the degree of proximity of x to prototype elements of $\widetilde{A}$. Basically, this is the oldest semantics of membership grades since Bellman *et al.* [12].

(ii) Degree of preference: On the basis of the degree of preference, $\widetilde{A}$ represents a set of more or less preferred objects (or values of a decision variable x) and $\mu_{\widetilde{A}}(u)$ represents an intensity of preference in favor of object u, or the feasibility of selecting u as a value of x. Fuzzy sets then exprese criteria or flexible constraints. This view is the one later put forward by Bellman and Zadeh [13]; it has given birth to an abundant literature on fuzzy optimization, especially decision analysis and fuzzy linear programming.

(iii) Degree of uncertainty: This interpretation was proposed by Zadeh [155] when he introduced the possibility theory and developed his theory of approximate reasoning

(Zadeh [153]). $\mu_{\tilde{A}}(u)$ is then the degree of possibility that a parameter x has value u, given that all that is known about it, is that 'x' is $\widetilde{A}$. Then the values encompasses by the support of the membership functions are mutually exclusive, and the membership degrees rank these values in terms of their respective possibility. Set functions called possibility and necessity measures can be derived so as to rank-order events in terms of unsurprising-ness and acceptance respectively.

Possibility, Necessity, Credibility and Expected Value of Fuzzy Parameter:

Let $\Re$ represents the set of real numbers. $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ respectively. Then taking degree of uncertainty as the semantics of fuzzy number, according to Dubois and Prade [32, 33], Liu and Iwamura [86, 87], Zadeh [155], possibility is defined by:

$$\text{Pos }(\tilde{A} \star \tilde{B}) = \sup\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), x, y \in \Re, x \star y\} \tag{2.9}$$

where the abbreviation Pos represent possibility and $\star$ is any one of the relations $>, <, =, \leq, \geq$. Analogously, if $\tilde{B}$ is a crisp number, say b, then

$$\text{Pos }(\tilde{A} \star b) = \sup\{\mu_{\tilde{A}}(x), x \in R, x \star b\} \tag{2.10}$$

On the other hand, necessity measure of an event $\tilde{A} \star \tilde{B}$ is a dual of possibility measure. The grade of necessity of an event is the grade of impossibility of the opposite event and is defined as:

$$\text{Nes }(\tilde{A} \star \tilde{B}) = 1 - \text{Pos }(\overline{\tilde{A} \star \tilde{B}}) \tag{2.11}$$

where the abbreviation Nes represents necessity measure and $\overline{\tilde{A} \star \tilde{B}}$ represents complement of the event $\tilde{A} \star \tilde{B}$.

On the basis of possibility and necessity measure, the credibility measure function Cr, analyzed by Liu and Liu [80], is defined as follows:

If we denote the support of $\tilde{a}$ by $R = \{r \in \Re | \mu_{\tilde{a}}(\gamma) > 0\}$, the credibility measure is given by

$$Cr(A) = \frac{1}{2}\Big[Pos(A) + Nes(A)\Big] \tag{2.12}$$

for any $A \in 2^{\Re}$, where $2^{\Re}$ is the power set of $\Re$ and Cr satisfies the following two conditions:

i) $Cr(\phi) = 0$ and $Cr(\Re)$ =1;

ii)$A \subset B$ implies $Cr(A) \leq Cr(B)$ for any $A, B \in 2^{\Re}$

Tthus, Cr is also a fuzzy measure defined on $(\Re, 2^{\Re})$. Besides, Cr is self dual, i.e., $Cr(A) = 1 - Cr(A^C)$ for any $A \in 2^{\Re}$.

In this thesis, based on the credibility measure (2.12) the following form is defined as

$$Cr(A) \; = \; \Big[\rho Pos(A) + (1-\rho)Nes(A)\Big], \tag{2.13}$$

for any $A \in 2^{\Re}$ and $0 < \rho < 1$, where ρ is the degree of pessimism. It also satisfies the above conditions.

For the triangular fuzzy number $\tilde{a} \; = \; (a_1, a_2, a_3))$ and the crisp number r, $Pos(\tilde{a} \geq r)$ and $Nes(\tilde{a} \geq r)$ are given by

$$Pos(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_2 \\ \dfrac{a_3 - r}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases} \tag{2.14}$$

$$Nes(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ 0 & \text{if} \quad r \geq a_2 \end{cases} \tag{2.15}$$

The credibility measure for of the events $\widetilde{a} \geq r$ and $\widetilde{a} \leq r$ for the TFN $\widetilde{a}$ can be defined as

$$Cr(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - \rho a_1}{a_2 - a_1} - \dfrac{(1-\rho)r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ \dfrac{\rho(a_3 - r)}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases} \tag{2.16}$$

$$Cr(\tilde{a} \leq r) = \begin{cases} 0 & \text{if} \quad r \leq a_1 \\ \dfrac{\rho(r - a_1)}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ \dfrac{\rho a_3 - a_2}{a_3 - a_2} - \dfrac{(1-\rho)r}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 1 & \text{if} \quad r \geq a_3 \end{cases} \tag{2.17}$$

Let X be a normalized fuzzy variable. Then expected value of the fuzzy variable X is defined by

$$E[X] = \int_0^\infty Cr(X \geq r)dr - \int_{-\infty}^0 Cr(X \leq r)dr \tag{2.18}$$

When the right hand side of (2.18) is of form $\infty - \infty$, the expected value is not defined. Also, the expected value operation has been proved to be linear for bounded fuzzy variables, i.e., for any two bounded fuzzy variables X and Y, we have $E[aX + bY] = aE[X] + bE[Y]$ for any real numbers a and b.

Lemma 2.1: Let $\tilde{a} = (a_1, a_2, a_3)$ be a triangular fuzzy number and r is a crisp number. The expected value of $\tilde{a}$ is given by

$$E[\tilde{a}] = \frac{1}{2}\Big[(1-\rho)a_1 + a_2 + \rho a_3\Big], \quad 0 < \rho < 1. \tag{2.19}$$

Proof. From (2.18), by using (2.16) and (2.17), we have

$$\begin{aligned} E[\tilde{a}] &= \int_0^{\infty} Cr(\tilde{a} \geq r)dr - \int_{-\infty}^{0} Cr(\tilde{a} \leq r)dr \\ &= \int_0^{a_1} Cr(\tilde{a} \geq r)dr + \int_{a_1}^{a_2} Cr(\tilde{a} \geq r)dr + \int_{a_2}^{a_3} Cr(\tilde{a} \geq r)dr + 0 \\ &= \int_0^{a_1} dr + \int_{a_1}^{a_2} \left[\frac{a_2 - \rho a_1}{a_2 - a_1} - \frac{(1-\rho)r}{a_2 - a_1}\right]dr + \int_{a_2}^{a_3} \left[\frac{\rho(a_3 - r)}{a_3 - a_2}\right]dr \\ &= \frac{1}{2}\Big[(1-\rho)a_1 + a_2 + \rho a_3\Big]. \end{aligned}$$

Lemma 2.2: For two TFN $\widetilde{a} = (a_1, a_2, a_3)$ and $\widetilde{b} = (b_1, b_2, b_3)$, Pos $(\tilde{a} \geq \tilde{b}) > \epsilon$ iff $\frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2} > \epsilon, \quad (a_2 < b_2,\ a_3 > b_1)$

Proof: Let us consider, Pos$(\tilde{a} \geq \tilde{b}) > \epsilon$.

From definition (2.14), if $\widetilde{a} = (a_1, a_2, a_3)$ and $\widetilde{b} = (b_1, b_2, b_3)$ be two TFNs then

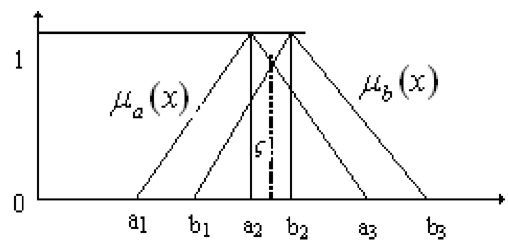

Figure 2.8: Pos $(\tilde{a} \geq \tilde{b})$

$$\text{Pos } (\tilde{a} \geq \tilde{b}) = \begin{cases} 1 & \text{for } a_2 \geq b_2 \\ \zeta_2 = \frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2} & \text{for } a_2 < b_2,\ a_3 > b_1 \\ 0 & \text{for } a_3 \leq b_1 \end{cases}$$

Hence, Pos $(\tilde{a} \geq \tilde{b}) > \epsilon$ iff $\zeta_2 = \frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2} > \epsilon, \quad (a_2 < b_2,\ a_3 > b_1)$.

Note: Pos $(a \geq \tilde{b}) > \epsilon$ iff $\zeta_2 = \frac{a-b_1}{b_2-b_1} > \epsilon, \quad (b_1 < a < b_2)$.

Lemma 2.3: For TrFN $\widetilde{a} = (a_1, a_2, a_3, a_4)$ and a real number b, then Pos $(\tilde{a} \leq b) > \epsilon$ iff $\frac{b-a_1}{a_2-a_1} > \epsilon, \quad (a_1 \leq b \leq a_2)$.

Proof: Let us consider, Pos$(\tilde{a} \leq b) > \epsilon$.
From definition (2.14), if $\widetilde{a} = (a_1, a_2, a_3, a_4)$ be TrFN and b be the crisp number then

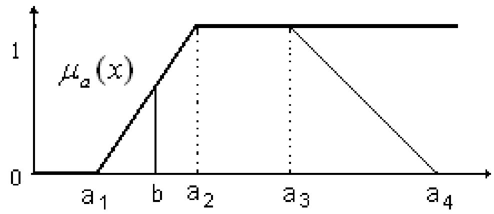

Figure 2.9: Pos $(\tilde{a} \leq b)$

$$\begin{aligned} \text{Pos } (\tilde{a} \leq b) &= \{sup(min(\mu_{\tilde{a}}(x), 1)), \ x, b \in R, x \leq b\} \\ &= \begin{cases} 0 & \text{for } -\infty < b < a_1 \\ \eta_1 = \frac{b-a_1}{a_2-a_1} & \text{for } a_1 \leq b \leq a_2 \\ 1 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.20}$$

which is depicted in the Figure 2.9. Therefore, it is clear that the event $(-\infty < b < a_1)$ is not acceptable (impossible event) with respect to the fuzzy event $\widetilde{a} \leq b$ as $\widetilde{a} < b$ which implies the value of $b >$ least value of $\widetilde{a}$. On the other hand, the event $b > a_2$ is certain case of the fuzzy event $\widetilde{a} \leq b$. Hence. we consider the case $a_1 \leq b \leq a_2$, which gives Pos $(\tilde{a} \leq b) = \frac{b-a_1}{a_2-a_1}$. Therefore, $Pos(\widetilde{a} \leq b) > \epsilon \Rightarrow \frac{b-a_1}{a_2-a_1} > \epsilon$.

Lemma-2.4: If $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be TrFNs with $0 < a_1$ and $0 < b_1$ then Nes$(\tilde{a} > \tilde{b}) > \alpha$ iff $\dfrac{b_4 - a_1}{a_2 - a_1 + b_4 - b_3} < 1 - \alpha$.
Proof: We have Nes$(\tilde{a} > \tilde{b}) > \alpha$

That is, $\{1 - Pos(\tilde{a} \leq b)\} > \alpha$

$\Rightarrow Pos(\tilde{a} \leq b) < 1 - \alpha$

So from Figurc 2.10 it is clcar that

$Pos(\tilde{a} \leq \tilde{b}) = \delta = \dfrac{b_4 - a_1}{a_2 - a_1 + b_4 - b_3}$

and hence the result follows.

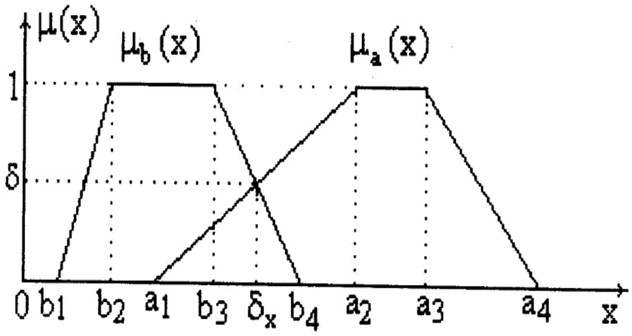

Figure 2.10: Measure of Pos($\tilde{a} \leq \tilde{b}$)

Lemma-2.5: If $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TrFN with $0 < a_1$ and b be a crisp number, then Nes($\tilde{a} > b$) $> \alpha$ iff $\dfrac{b - a_1}{a_2 - a_1} < 1 - \alpha$.

Proof: Proof follows from Lemma-2.4.

Note 2.1: As a TFN $\tilde{a} = (a_1, a_2, a_4)$, is a special case of a TrFN $\tilde{a} = (a_1, a_2, a_3, a_4)$ with $a_2 = a_3$, so results of Lemma-2.3,2.4,2.5 are remain valid for a TFN $\tilde{a} = (a_1, a_2, a_4)$, if we replace a_3 with a_2.

Lemma 2.6: When the membership functions of $\widetilde{a}$ and $\widetilde{b}$ in $[a - \Delta a, a, a + \Delta a]$, and $[b - \Delta b, b, b + \Delta b]$ are linear, then

$$\text{Pos } (a \simeq b) = \begin{cases} Pos(b \lesssim a) \\ Pos(b \gtrsim a) \end{cases}$$

Proof: For two TFN, $\widetilde{a}$ and $\widetilde{b}$, $a\widetilde{<}b$ means $\widetilde{a} < \widetilde{b}$. Hence, we have

$$\text{Pos } (a \simeq b) = \begin{cases} 1 & \text{for } a = b \\ \zeta_1 = \frac{a+\Delta a-b+\Delta b}{\Delta a+\Delta b} & \text{for } \; b-\Delta b < a < a+\Delta a \\ \zeta_3 = \frac{a-b+\Delta b}{\Delta b} & \text{for } \; b-\Delta b < a-\Delta a < a < b \\ 0 & \text{otherwise} \end{cases}$$

Pictorial representation is present in Figure 2.11. Similarly, for the second case we need

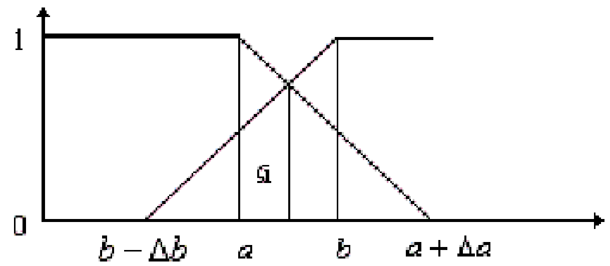

Figure 2.11: Measure of Pos($a \simeq b$)

to interchange only a and b.

If the attitude of a DM is toward optimistic, the equation (2.11) is the measure of best case and in pessimistic sense equation (2.9) gives the measure of worst case of that event. Now, if we consider ρ the optimistic and pessimistic index to determine the combined attitude of DM, then the measure of Weighted Possibility and Necessity (WPN) of $\widetilde{a} * \widetilde{b}$ is given by

$$WPN(\widetilde{a} * \widetilde{b}) = \rho Pos(\widetilde{a} * \widetilde{b}) + (1-\rho)Nes(\widetilde{a} * \widetilde{b}) \tag{2.21}$$

The optimistic (pessimistic) fill rate can be used to measure the maximum (minimum) chance to accomplish the target of the SC. The fill rate also determined as the weighted average of optimistic and pessimistic fill rates as shown in the following Figure 2.11

Note 2.2: In particular when $\rho = 0.5$, WPN is known as Credibility of that event, i.e.,

$$Cr(\widetilde{a} * \widetilde{b}) = \frac{1}{2}(Pos(\widetilde{a} * \widetilde{b}) + Nes(\widetilde{a} * \widetilde{b})) \tag{2.22}$$

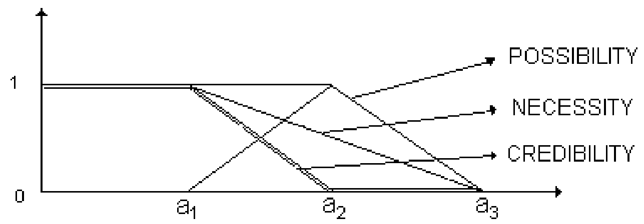

Figure 2.12: Possibility / Necessity / Credibility Weighted fill rate

Integration of a Fuzzy Function:

Let $\widetilde{f}(x)$ be a fuzzy function on $[a,b] \subseteq \Re$ to $\Re$ such that $\widetilde{f}(x)$ is a fuzzy number, i.e., a piecewise continuous convex normalized fuzzy set on $\Re$. Then integral of any continuous α-level curve of $\widetilde{f}(x)$ over $[a,b]$ always exists and the integral of $\widetilde{f}(x)$ over $[a,b]$ is then defined to be the fuzzy set

$$\widetilde{I}(a,b) = \left\{ \left(\int_a^b f^-(\alpha)(x)dx + \int_a^b f^+(\alpha)(x)dx, \alpha \right) \right\} \tag{2.23}$$

The determination of the integral $\widetilde{I}(a,b)$ becomes somewhat easier if the fuzzy function is assumed to be LR type. We shall therefore assume that $\widetilde{f}(x) = (f_1(x), f_2(x), f_3(x))_{LR}$ is a fuzzy number in LR representation for all $x \in [a,b]$, where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are assumed to be positive integrable functions on $[a,b]$. Dubois and Prade [31] have shown that under these conditions

$$\widetilde{I}(a,b) = \left(\int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \int_a^b f_3(x)dx \right)_{LR} \tag{2.24}$$

2.1.10 Rank of a fuzzy number

There are several methods available for ordering the fuzzy numbers. To compete the order of two fuzzy numbers, we introduce a rank value for each fuzzy number. We introduce a new ranking formula for fuzzy numbers. Let us consider a trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$. Ranking of $\widetilde{A}$ is calculated by following algorithm :

Step-1 : Calculate the mean position $[\overline{X}_{\widetilde{A}} = \frac{a_1+a_2+a_3+a_4}{4}]$of the fuzzy number along x-axis.

Step-2 : Calculate the spread $(S_{\widetilde{A}} = a_4 - a_1)$ of the fuzzy number.

Step-3 : Calculate the area $[A_{\widetilde{A}} = \frac{a_4+a_3-a_1-a_2}{2}]$ of the fuzzy number.

Step-4 : Calculate the ranking value $(R_{\widetilde{A}})$ of the fuzzy number by the following formula.

$$R_{\widetilde{A}} = \tfrac{1}{3}\{(2 \times \overline{X}_{\widetilde{A}} + S_{\widetilde{A}}) + A_{\widetilde{A}}\} \tag{2.25}$$

Property -1 : The rank value of any fuzzy number $(a_1, a_2, a_3, a_4)$is less or equal to a_4

Proof : Let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number where $a_1 \leq a_2 \leq a_3 \leq a_4$.
Then from the above definitions:
The ranking value is given by

$$\begin{aligned}
R_{\widetilde{A}} &= \frac{1}{3}\{(2 \times \overline{X}_{\widetilde{A}} + S_{\widetilde{A}}) + A_{\widetilde{A}}\} \\
&= \frac{1}{3}\{2 \times \frac{1}{4}(a_1 + a_2 + a_3 + a_4) + (a_4 - a_1) + \frac{1}{2}(a_4 + a_3 - a_1 - a_2)\} \\
&= \frac{1}{3}\{\frac{1}{2}(a_1 + a_2 + a_3 + a_4) + (a_4 - a_1) + \frac{1}{2}(a_4 + a_3 - a_1 - a_2)\} \qquad (2.26) \\
&= \frac{1}{3}\{2 * (a_4 + a_3 - a_1)\} \\
&\leq \frac{1}{3}(3 * a_4) \qquad [a_3 - a_1 \leq a_4 - a_1 \leq a_4] \\
&\leq a_4
\end{aligned}$$

Property -2 : The rank value of any fuzzy number $(a_1, a_2, a_3, a_4)$is positive when a_1, a_2, a_3, a_4 are all positive

Proof : Let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number where $a_1 \leq a_2 \leq a_3 \leq a_4$.

Then the ranking value is given by

$$\begin{aligned}
R_{\widetilde{A}} &= \frac{1}{3}\{(2 \times \overline{X}_{\widetilde{A}} + S_{\widetilde{A}}) + A_{\widetilde{A}}\} \\
&= \frac{1}{3}\{2 \times \frac{1}{4}(a_1 + a_2 + a_3 + a_4) + (a_4 - a_1) + \frac{1}{2}(a_4 + a_3 - a_1 - a_2)\} \\
&= \frac{1}{3}\{\frac{1}{2}(a_1 + a_2 + a_3 + a_4) + (a_4 - a_1) + \frac{1}{2}(a_4 + a_3 - a_1 - a_2)\} \qquad (2.27) \\
&= \frac{1}{3}(2 * a_4 + a_3 - a_1) \\
&\geq \frac{1}{3}(2 * a_4) \qquad [a_3 - a_1 \geq 0] \\
&> 0 \qquad [a_4 > 0]
\end{aligned}$$

Example-2.1: The order relations of three fuzzy numbers $\widetilde{A}_1 = (0.4, 0.5, 1)$, $\widetilde{A}_2 = (0.4, 0.7; 1)$, $\widetilde{A}_3 = (0.4, 0.9, 1)$, is represented in *Figure* 2.12. The rank value of three fuzzy numbers are $R_{\widetilde{A_1}} = 0.722$, $R_{\widetilde{A_2}} = 0.767$, $R_{\widetilde{A_3}} = 0.811$ (Followed 2.5). Hence $\widetilde{A_1} < \widetilde{A_2} < \widetilde{A_3}$.

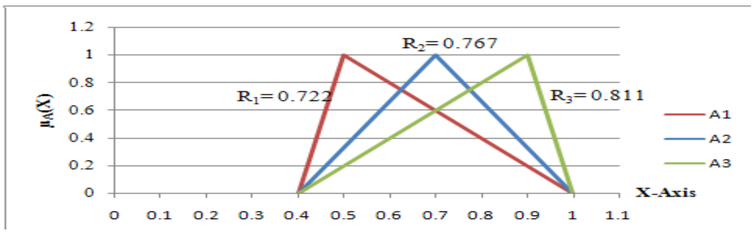

Figure 2.13: Order relation of three equal spread TFN

Example-2.2: The two fuzzy numbers in (represent in *Figure* 2.13)are respectively $\widetilde{A}_1(0.2, 0.5, 0.8)$, $\widetilde{A}_2(0.4, 0.5; 0.6)$, then we have: $R_{\widetilde{A_1}} = 0.633$, $R_{\widetilde{A_2}} = 0.433$. Hence $\widetilde{A_1} > \widetilde{A_2}$.

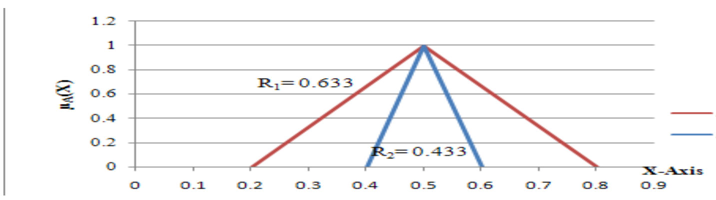

Figure 2.14: Order relation of two equal mean TFN

Example-2.3: The three fuzzy numbers in (represent in *Figure* 2.14) are respectively $\widetilde{A}_1(0.5, 0.7, 0.9)$, $\widetilde{A}_2(0.3, 0.7, 0.9)$, $\widetilde{A}_3(0.3, 0.4, 0.7, 0.9)$, then we have: $R_{\widetilde{A_1}} = 0.667$, $R_{\widetilde{A_2}} = 0.722$, $R_{\widetilde{A_3}} = 0.733$. Hence $\widetilde{A_1} < \widetilde{A_2} < \widetilde{A_3}$.

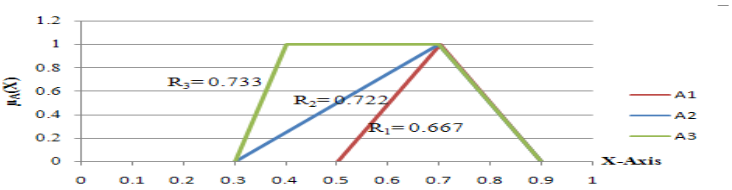

Figure 2.15: Order relation of three fuzzy number

Example-2.4: The three fuzzy numbers in (represent in *Figure* 2.15) are respectively $\widetilde{A}_1(0.3, 0.5, 0.8, 0.9)$, $\widetilde{A}_2(0.3, 0.5, 0.9)$, $\widetilde{A}_3(0.3, 0.5, 0.7)$, then we have: $R_{\widetilde{A_1}} = 0.767$, $R_{\widetilde{A_2}} = 0.678$, $R_{\widetilde{A_3}} = 0.533$, hence $\widetilde{A_3} < \widetilde{A_2} < \widetilde{A_1}$.

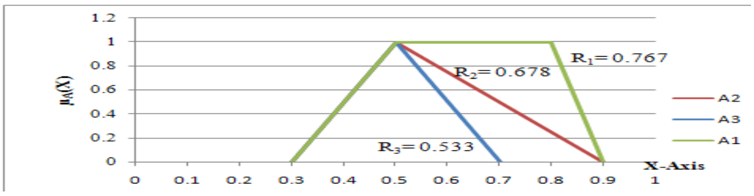

Figure 2.16: Order relation of three fuzzy number

Example-2.5: The two fuzzy numbers in (represent in *Figure* 2.16:) are respectively $\widetilde{A}_1(0.3, 0.3, 1)$, $\widetilde{A}_2(0.1, 0.7, 0.8)$, then we have: $R_{\widetilde{A_1}} = 0.706$, $R_{\widetilde{A_2}} = 0.706$, hence $\widetilde{A_1} = \widetilde{A_2}$.

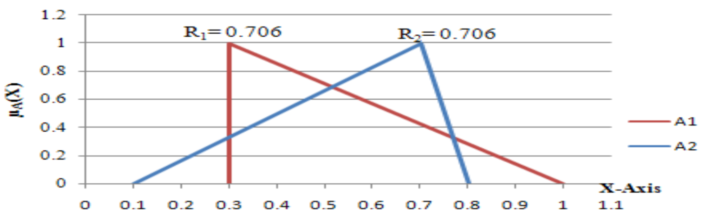

Figure 2.17: Order relation of two different spread TFN

2.1.11 Type-2 fuzzy set (T2FS)

In an universal set X, a type-2 fuzzy set $\tilde{\tilde{A}}$ is a fuzzy set in which the membership function of an ellement/point of $\tilde{\tilde{A}}$ is also fuzzy in nature, i.e., membership grade of each element/point is no longer a crisp value but a fuzzy set. This membership function is called type-2 membership function. A T2FS $\tilde{\tilde{A}}$ is defined as (Mendel and John [99])

$$\tilde{\tilde{A}} = \{((x, v), \mu_{\tilde{\tilde{A}}}(x, v)) : \forall x \in X, \forall v \in G_x \subseteq [0, 1]\}$$

where $0 \leq \mu_{\tilde{\tilde{A}}}(x, v) \leq 1$ is the type-2 membership function, is termed a secondary membership function having domain $G_x \subseteq [0, 1]$ is the primary membership of $x \in X$, so that all $v \in G_x$ are the primary membership grades of the point x. $\tilde{\tilde{A}}$ can also be written as (Mendel and John [100]) $\tilde{\tilde{A}} = \{(x, \tilde{\mu}_{\tilde{\tilde{A}}}(x)) \mid \forall x \in X\}$ or

$$\tilde{\tilde{A}} = \int_{x \in X} \tilde{\mu}_{\tilde{\tilde{A}}}(x) x = \int_{x \in X} \Big[\int_{v \in G_x} f_x(v)/v \Big] x$$

where $0 \leq f_x(v) \leq 1$ and for a particular $x = x'$ and $v = v' \in G_{x'}, f_{x'}(v') = \mu_{\tilde{\tilde{A}}}(x', v')$ is called secondary membership grade of (x', v'). $\int_x$ and $\int_v$ denote union over all admissible x and v. For discrete universes of discourse $\int$ is replaced by $\sum$.

When all the secondary membership grade values are 1 (i.e. $f_x(v) = 1, \forall(x, v)$) such T2FS is called interval type-2 fuzzy set (IT2FS) (Mendel et al. [101]). A IT2FS is characterized by the footprint of uncertainty (FOU) which is the union of all of the primary memberships function G_x, i.e. the mathematical form of FOU of a IT2FS $\tilde{\tilde{A}}$ is as

$$FOU(\tilde{\tilde{A}}) = \bigcup_{x \in X} G_x.$$

Regular fuzzy parameter (RFP) (Liu and Liu [84]): For a possibility space (τ, p, Pm), where τ is nonempty set of points, p is power set of τ and possibility measure (Pm) (Zadeh [153]) of a fuzzy event possibility space, a regular fuzzy parameter $\tilde{\eta}$ is defined as a measurable map from the set τ to [0, 1] in the sense that for every $t \in [0, 1]$, one has $\{\gamma \in \tau : \tilde{\eta}(\gamma) \leq t\} \in p$. A discrete RFP can be represented as follows

$\widetilde{\eta} \sim \begin{pmatrix} \rho_1 & \rho_2 & \dots & \rho_n \\ \mu_1 & \mu_2 & \dots & \mu_n \end{pmatrix}$, where $\rho_1 \in [0,1]$ and $\mu_i > 0, \forall i$ and $max\{\mu_i\} = 1$.

If $\widetilde{\eta} = (\rho_1, \rho_2, \rho_3, \rho_4)$ with $0 \le \rho_1 < \rho_2 < \rho_3 < \rho_4 \le 1$, then $\widetilde{\eta}$ is called a trapezoidal RFP.

If $\widetilde{\eta} = (\rho_1, \rho_2, \rho_3)$ with $0 \le \rho_1 < \rho_2 < \rho_3 \le 1$, then $\widetilde{\eta}$ is called a triangular RFP.

Example 2.6: Suppose $\widetilde{\widetilde{A}} = \{(x, \widetilde{\mu}_{\widetilde{\widetilde{A}}}(x)) : x \in X\}$ be a type-2 fuzzy parameter (T2FP) where $X = \{6, 7, 8\}$ and the primary memberships (possibilities) of the points of X are $J_6 = \{0.3, 0.4, 0.5\}$, $J_7 = \{0.6, 0.7, 0.9\}$, $J_8 = \{0.4, 0.6, 0.7, 0.9\}$. The secondary membership function of the point 6 is

$$\widetilde{\mu}_{\widetilde{\widetilde{A}}}(6) = \mu_{\widetilde{\widetilde{A}}}(6, v) = (0.5/0.3) + (1.0/0.4) + (0.8/0.5) \sim \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.5 & 1.0 & 0.8 \end{pmatrix},$$

i.e., $\mu_{\widetilde{\widetilde{A}}}(64, 0.3) = 0.5$, $\mu_{\widetilde{\widetilde{A}}}(6, 0.4) = 1$ and $\mu_{\widetilde{\widetilde{A}}}(6, 0.5) = 0.8$. Here $\mu_{\widetilde{\widetilde{A}}}(6, 0.3) = 0.5$ means secondary membership grade that the point 6 has the primary membership 0.3 is 0.6. So $\widetilde{\widetilde{A}}$ takes on the value 6 with membership $\begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.5 & 1.0 & 0.8 \end{pmatrix}$, which represents a RVP.

Similarly,

$$\widetilde{\mu}_{\widetilde{\widetilde{A}}}(7) = \mu_{\widetilde{\widetilde{A}}}(7, v) = (0.7/0.6) + (1.0/0.7) + (0.8/0.9) \sim \begin{pmatrix} 0.6 & 0.7 & 0.9 \\ 0.7 & 1.0 & 0.8 \end{pmatrix}$$

$$\widetilde{\mu}_{\widetilde{\widetilde{A}}}(8) = \mu_{\widetilde{\widetilde{A}}}(8, v) = (0.5/0.4) + (0.6/0.6) + (1.0/0.7) + (0.7/0.9)$$
$$\sim \begin{pmatrix} 0.4 & 0.6 & 0.7 & 0.9 \\ 0.5 & 0.6 & 1.0 & 0.7 \end{pmatrix}$$

So discrete T2FP $\widetilde{\widetilde{A}}$ is given by $\widetilde{\widetilde{A}} = (0.5/0.3)/6 + (1.0/0.4)/6 + (0.8/0.5)/6 + (0.7/0.6)/7 + (1.0/0.7)/7 + (0.8/0.9)/7 + (0.5/0.4)/8 + (0.6/0.6)/8 + (1.0/0.7)/8 + (0.7/0.9)/8$.

$\tilde{\tilde{A}}$ is also written as

$$\tilde{\tilde{A}} = \begin{cases} 6, \text{ with } \tilde{\mu}_{\tilde{\tilde{A}}}(6); \\ 7, \text{ with } \tilde{\mu}_{\tilde{\tilde{A}}}(7); \\ 8, \text{ with } \tilde{\mu}_{\tilde{\tilde{A}}}(8); \end{cases}$$

Example 2.7: A type-2 triangular fuzzy parameter $\tilde{\tilde{\eta}} = (\rho_1, \rho_2, \rho_3; \theta_l, \theta_r)$, where $\rho_1, \rho_2, \rho_3 \in \Re$ and $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\tilde{\eta}}$

takes a value x and the secondary possibility distribution function $\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)$ of $\tilde{\tilde{\eta}}$ is defined as follows

$$\tilde{\mu}_{\tilde{\tilde{\eta}}}(x) = \Big(\frac{x-\rho_1}{\rho_2-\rho_1} - \theta_l \min\{\frac{x-\rho_1}{\rho_2-\rho_1}, \frac{\rho_2-x}{\rho_2-\rho_1}\}, \frac{x-\rho_1}{\rho_2-\rho_1}, \frac{x-\rho_1}{\rho_2-\rho_1} + \theta_r \min\{\frac{x-\rho_1}{\rho_2-\rho_1}, \frac{\rho_2-x}{\rho_2-\rho_1}\}\Big) \tag{2.28}$$

for any $x \in [\rho_1, \rho_2]$, and

$$\tilde{\mu}_{\tilde{\tilde{\eta}}}(x) = \Big(\frac{\rho_3-x}{\rho_3-\rho_2} - \theta_l \min\{\frac{\rho_3-x}{\rho_3-\rho_2}, \frac{x-\rho_2}{\rho_3-\rho_2}\}, \frac{\rho_3-x}{\rho_3-\rho_2}, \frac{\rho_3-x}{\rho_3-\rho_2} + \theta_r \min\{\frac{\rho_3-x}{\rho_3-\rho_2}, \frac{x-\rho_2}{\rho_3-\rho_2}\}\Big) \tag{2.29}$$

for any $x \in (\rho_2, \rho_3]$,

A type-2 triangular fuzzy parameter is an extension of a triangular fuzzy parameter (TFP). In TFP (ρ_1, ρ_2, ρ_3), the membership grade of every point is a fixed number in [0,1]. However in a type-2 triangular fuzzy parameter $\tilde{\tilde{\eta}} = (\rho_1, \rho_2, \rho_3; \theta_l, \theta_r)$, the primary memberships of the points are no longer fixed values, instead they have a range between 0 and 1. Here θ_l and θ_r are used to represent the spreads of primary memberships of type-2 TFP. Obviously if $\theta_l = \theta_r = 0$, then type-2 TFP $\tilde{\tilde{\eta}}$ becomes a TFP and the equations (2.28) and (2.29) together become the membership function of a type-1 TFV. Now from equations (2.28) and (2.29), $\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)$ can be expressed as

$$\tilde{\mu}_{\tilde{\tilde{\eta}}}(x) = \begin{cases} \Big(\frac{x-\rho_1}{\rho_2-\rho_1} - \theta_l \frac{x-\rho_1}{\rho_2-\rho_1}, \frac{x-\rho_1}{\rho_2-\rho_1}, \frac{x-\rho_1}{\rho_2-\rho_1} + \theta_r \frac{x-\rho_1}{\rho_2-\rho_1}\Big), & \text{if } x \in [\rho_1, \frac{\rho_1+\rho_2}{2}]; \\ \Big(\frac{x-\rho_1}{\rho_2-\rho_1} - \theta_l \frac{\rho_2-x}{\rho_2-\rho_1}, \frac{x-\rho_1}{\rho_2-\rho_1}, \frac{x-\rho_1}{\rho_2-\rho_1} + \theta_r \frac{\rho_2-x}{\rho_2-\rho_1}\Big), & \text{if } x \in (\frac{\rho_1+\rho_2}{2}, \rho_2]; \\ \Big(\frac{\rho_3-x}{\rho_3-\rho_2} - \theta_l \frac{x-\rho_2}{\rho_3-\rho_2}, \frac{\rho_3-x}{\rho_3-\rho_2}, \frac{\rho_3-x}{\rho_3-\rho_2} + \theta_r \frac{x-\rho_2}{\rho_3-\rho_2}\Big), & \text{if } x \in (\rho_2, \frac{\rho_2+\rho_3}{2}]; \\ \Big(\frac{\rho_3-x}{\rho_3-\rho_2} - \theta_l \frac{\rho_3-x}{\rho_3-\rho_2}, \frac{\rho_3-x}{\rho_3-\rho_2}, \frac{\rho_3-x}{\rho_3-\rho_2} + \theta_r \frac{\rho_3-x}{\rho_3-\rho_2}\Big), & \text{if } x \in (\frac{\rho_2+\rho_3}{2}, \rho_3]; \end{cases} \tag{2.30}$$

Now we illustrate numerically the Example 2.7. Consider the type-2 TFP $\tilde{\tilde{\eta}} = (2, 4, 6; 0.4, 0.8)$. Then the secondary possibility distribution of $\tilde{\tilde{\eta}}$ is given by

$$\tilde{\mu}_{\tilde{\tilde{\eta}}}(x) = \begin{cases} (0.3(x-2), 0.5(x-2), 0.9(x-2)), & \text{if } x \in [2, 3]; \\ (0.5(x-2) - 0.2(4-x), 0.5(x-2), 0.5(x-2) + 0.4(4-x)), & \text{if } x \in (3, 4]; \\ (0.5(6-x) - 0.2(x-4), 0.5(6-x), 0.5(6-x) + 0.4(x-4)), & \text{if } x \in (4, 5]; \\ (0.3(6-x), 0.5(6-x), 0.9(6-x)), & \text{if } x \in (5, 6]; \end{cases}$$

Here secondary possibility degree of every value of x is a triangular RVP, e.g., $\tilde{\mu}_{\tilde{\tilde{\eta}}}(2.5) = (0.15, 0.25, 0.45)$, $\tilde{\mu}_{\tilde{\tilde{\eta}}}(5.6) = (0.12, 0.20, 0.36)$, etc. The FOU of $\tilde{\tilde{\eta}}$ is depicted in Figure 2.18.

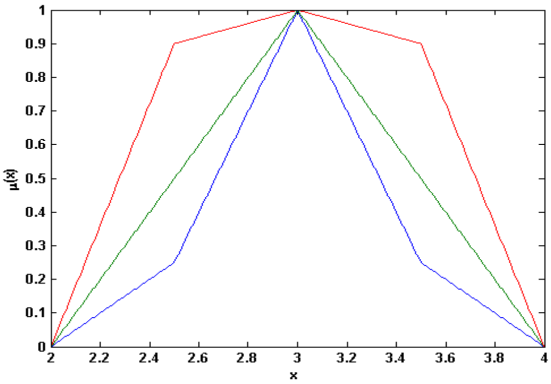

Figure 2.18: FOU of $\tilde{\eta}$

2.1.12 Critical values (CVs) for RVPs

Three kinds of critical values (CVs) of a RVP $\tilde{\eta}$ has been introduced by Qin et al. [122] and these are defined by:

(i) the optimistic CV of $\tilde{\tilde{\eta}}$, denoted by $CV*[\tilde{\tilde{\eta}}]$, is defined as

$$CV*[\tilde{\tilde{\eta}}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Pm\{\tilde{\eta} \geq \alpha\}]$$

(ii) the pessimistic CV of $\tilde{\eta}$, denoted by $CV_*[\tilde{\tilde{\eta}}]$, is defined as

$$CV_*[\tilde{\tilde{\eta}}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Nm\{\tilde{\tilde{\eta}} \geq \alpha\}]$$

(iii) the CV of $\tilde{\tilde{\eta}}$, denoted by $CV[\tilde{\eta}]$, is defined as

$$CV[\tilde{\tilde{\eta}}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Cr\{\tilde{\tilde{\eta}} \geq \alpha\}]$$

In perticular, if $\tilde{\tilde{\eta}} = (\rho_1, \rho_2, \rho_3, \rho_4)$ be a trapezoidal RVP. Then we have

(i) the optimistic $CV^*[\tilde{\tilde{\eta}}] = \rho_4/(1 + \rho_4 - \rho_3)$

(ii) the pessimistic $CV_*[\tilde{\tilde{\eta}}] = \rho_2/(1 + \rho_2 - \rho_1)$

$$\text{(iii) the } CV[\tilde{\tilde{\eta}}] = \begin{cases} \frac{2\rho_2-\rho_1}{1+2(\rho_2-\rho_1)}, & \text{if } \rho_2 > \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \rho_2 \leq \frac{1}{2} < \rho_3; \\ \frac{\rho_4}{1+2(\rho_4-\rho_3)}, & \text{if } \rho_3 \leq \frac{1}{2}; \end{cases}$$

Example 2.8: Let $\tilde{\tilde{\eta}}$ be a discrete RVP defined by $\tilde{\tilde{\eta}} = \begin{pmatrix} 0.3 & 0.4 & 0.5 & 0.8 \\ 0.4 & 0.7 & 1.0 & 0.6 \end{pmatrix}$

Then for $\alpha \in [0, 1]$,

$$Pm\{\tilde{\tilde{\eta}} \geq \alpha\} =^{sup}_{r\geq\alpha} \mu_{\tilde{\tilde{\eta}}}(r) = \begin{cases} 1, & \text{if } \alpha \leq 0.5; \\ 0.6, & \text{if } 0.5 < \alpha \leq 0.8; \\ 0, & \text{if } 0.8 < \alpha \leq 1; \end{cases}$$

$$Nm\{\tilde{\tilde{\eta}} \geq \alpha\} = 1 -^{sup}_{r\leq\alpha} \mu_{\tilde{\tilde{\eta}}}(r) = \begin{cases} 1, & \text{if } \alpha \leq 0.3; \\ 0.6, & \text{if } 0.3 < \alpha \leq 0.4; \\ 0.3, & \text{if } 0.4 < \alpha \leq 0.5; \\ 0, & \text{if } 0.5 < \alpha \leq 1; \end{cases} \tag{2.31}$$

$$Cr\{\tilde{\tilde{\eta}} \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq 0.3; \\ 0.80, & \text{if } 0.3 < \alpha \leq 0.4; \\ 0.65, & \text{if } 0.4 < \alpha \leq 0.5; \\ 0.3, & \text{if } 0.5 < \alpha \leq 0.8; \\ 0, & \text{if } 0.8 < \alpha \leq 1; \end{cases}$$

therefore we have

$$\begin{aligned} CV^*[\tilde{\tilde{\eta}}] &= \sup_{\alpha\in[0,1]}[\alpha \wedge Pm\{\tilde{\tilde{\eta}} \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.5]}[\alpha \wedge 1] \vee \sup_{\alpha\in[0.5,0.8]}[\alpha \wedge 0.6] \vee \sup_{\alpha\in[0.8,1]}[\alpha \wedge 0] \\ &= 0.5 \vee 0.6 \vee 0 = 0.6 \\ CV_*[\tilde{\tilde{\eta}}] &= \sup_{\alpha\in[0,1]}[\alpha \wedge Nm\{\tilde{\tilde{\eta}} \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.3]}[\alpha \wedge 1] \vee \sup_{\alpha\in[0.3,0.4]}[\alpha \wedge 0.6] \vee \sup_{\alpha\in[0.4,0.5]}[\alpha \wedge 0.3] \vee \sup_{\alpha\in[0.5,1]}[\alpha \wedge 0] \\ &= 0.2 \vee 0.4 \vee 0.3 \vee 0 = 0.4 \end{aligned} \tag{2.32}$$

$$\begin{aligned} CV[\tilde{\tilde{\eta}}] &= \sup_{\alpha\in[0,1]}[\alpha \wedge Pm\{\tilde{\tilde{\eta}} \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.3]}[\alpha \wedge 1] \vee \sup_{\alpha\in[0.3,0.4]}[\alpha \wedge 0.80] \vee \sup_{\alpha\in[0.4,0.5]}[\alpha \wedge 0.65] \vee \sup_{\alpha\in[0.5,0.8]}[\alpha \wedge 0.3] \\ &\quad \vee \sup_{\alpha\in[0.8,0.1]}[\alpha \wedge 0] \\ &= 0.2 \vee 0.4 \vee 0.5 \vee 0.3 \vee 0 = 0.5 \end{aligned} \tag{2.33}$$

The following theorems introduce the critical values (CVs) of triangular and trapezoidal RVPs.

2.1.13 CV-based reduction method for type-2 fuzzy parameters

To reduce the complexity of T2FS, a common idea is to convert a T2FS into a T1FS so that the methodologies to deal with T1FSs can also be applied to T2FSs. Qin et al. [122] proposed a CV-based reduction method which reduces a type-2 fuzzy parameter to a type-1 fuzzy parameter (may or may not be normal). Let $\tilde{\tilde{\eta}}$ be a T2FP with secondary possibility distribution function $\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)$ (which represents a RVP). The method is to introduce the critical values (CVs) as representing values for RFP $\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)$, i.e., $CV^*[\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)], CV_*[\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)]$ or $CV[\tilde{\mu}_{\tilde{\tilde{\eta}}}(x)]$ and so corresponding type-1 fuzzy parameters (T1FPs) are derived using these CVs of the secondary possibilities. Then these methods are respectively called optimistic CV reduction, pessimistic CV reduction and CV reduction method.

Example 2.6 (continued). For the T2FP A. in Ex. 2.6, $\tilde{\mu}_{\tilde{A}}(6), \tilde{\mu}_{\tilde{A}}(7)$ and $\tilde{\mu}_{\tilde{A}}(8)$ are discrete RFPs. So the CVs of these RFPs can be obtained by using ($2.1.12), we have $CV^*[\tilde{\mu}_{\tilde{A}}(6)] = sup_{\alpha\in[0,1]}[\alpha \wedge Pm\{\tilde{\mu}_{\tilde{A}}(6) \geq \alpha\}]$, where

$$Pm\{\tilde{\mu}_{\tilde{A}}(6) \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq 0.4; \\ 0.7, & \text{if } 0.4 < \alpha \leq 0.6; \\ 0, & \text{if } 0.6 < \alpha \leq 1; \end{cases}$$

$$\begin{aligned} CV^*[\tilde{\mu}_{\tilde{A}}(6)] &= \sup_{\alpha\in[0,0.4]}[\alpha \wedge 1] \vee \sup_{\alpha\in[0.4,0.6]}[\alpha \wedge 0.7] \vee \sup_{\alpha\in[0.6,1]}[\alpha \wedge 0] \\ &= 0.4 \vee 0.6 \vee 0 = 0.6 \end{aligned}$$

In this way, we obtain

$$CV^*[\tilde{\mu}_{\tilde{A}}(6)] = 0.6, CV^*[\tilde{\mu}_{\tilde{A}}(7)] = 0.8, CV^*[\tilde{\mu}_{\tilde{A}}(8)] = 0.6$$

$$CV_*[\tilde{\mu}_{\tilde{A}}(6)] = 0.4, CV_*[\tilde{\mu}_{\tilde{A}}(7)] = 0.6, CV_*[\tilde{\mu}_{\tilde{A}}(8)] = 0.6$$

$$CV[\tilde{\mu}_{\tilde{A}}(6)] = 0.4, CV[\tilde{\mu}_{\tilde{A}}(7)] = 0.65, CV[\tilde{\mu}_{\tilde{A}}(8)] = 0.6$$

Then applying optimistic CV, pessimistic CV and CV reduction methods, the T2 FV $\tilde{A}$ is reduced respectively to the following T1FPs

$$\begin{pmatrix} 6 & 7 & 8 \\ 0.6 & 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 6 & 7 & 8 \\ 0.4 & 0.6 & 0.6 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 7 & 8 \\ 0.4 & 0.65 & 0.6 \end{pmatrix}$$

Theorem 2.1 (Qin et al. [122]). Let $\tilde{\tilde{\eta}}$ be a type-2 triangular fuzzy parameter defined as $\tilde{\tilde{\eta}} = (\rho_1, \rho_2, \rho_3; \theta_l, \theta_r)$. Then we have:

(i) Using the optimistic CV reduction method, the reduction η_1 of $\tilde{\tilde{\eta}}$ has the following possibility distribution

$$\mu_{\eta_1}(x) = \begin{cases} \frac{(1+\theta_r)(x-\rho_1)}{\rho_2-\rho_1+\theta_r(x-\rho_1)}, & \text{if } x \in [\rho_1, \frac{\rho_1+\rho_2}{2}]; \\ \frac{(1-\theta_r)x+\theta_r\rho_2-\rho_1}{\rho_2-\rho_1+\theta_r(\rho_2-x)}, & \text{if } x \in [\frac{\rho_1+\rho_2}{2}, \rho_2]; \\ \frac{(-1+\theta_r)x-\theta_r\rho_2+\rho_3}{\rho_3-\rho_2+\theta_r(x-\rho_2)}, & \text{if } x \in [\rho_2, \frac{\rho_2+\rho_3}{2}]; \\ \frac{(1+\theta_r)(\rho_3-x)}{\rho_3-\rho_2+\theta_r(\rho_3-x)}, & \text{if } x \in [\frac{\rho_2+\rho_3}{2}, \rho_3]. \end{cases} \tag{2.34}$$

(ii) Using the pessimistic CV reduction method, the reduction η_2 of $\tilde{\tilde{\eta}}$ has the following possibility distribution

$$\mu_{\eta_2}(x) = \begin{cases} \frac{x-\rho_1}{\rho_2-\rho_1+\theta_1(x-\rho_1)}, & \text{if } x \in [\rho_1, \frac{\rho_1+\rho_2}{2}]; \\ \frac{x-\rho_1}{\rho_2-\rho_1+\theta_1(\rho_2-x)}, & \text{if } x \in (\frac{\rho_1+\rho_2}{2}, \rho_2]; \\ \frac{\rho_3-x}{\rho_3-\rho_2+\theta_1(x-\rho_2)}, & \text{if } x \in (\rho_2, \frac{\rho_2+\rho_3}{2}]; \\ \frac{\rho_3-x}{\rho_3-\rho_2+\theta_1(\rho_3-x)}, & \text{if } x \in (\frac{\rho_2+\rho_3}{2}, \rho_3]. \end{cases} \tag{2.35}$$

(iii) Using the CV reduction method, the reduction η_3 of $\tilde{\tilde{\eta}}$ has the following possibility distribution

$$\mu_{\eta_3}(x) = \begin{cases} \frac{(1+\theta_r)(x-\rho_1)}{\rho_2-\rho_1+2\theta_r(x-\rho_1)}, & \text{if } x \in [\rho_1, \frac{\rho_1+\rho_2}{2}]; \\ \frac{(1-\theta_l)x+\theta_l\rho_2-\rho_1}{\rho_2-\rho_1+2\theta_l(\rho_2-x)}, & \text{if } x \in (\frac{\rho_1+\rho_2}{2}, \rho_2]; \\ \frac{(-1+\theta_l)x-\theta_l\rho_2+\rho_3}{\rho_3-\rho_2+2\theta_l(x-\rho_2)}, & \text{if } x \in (\rho_2, \frac{\rho_2+\rho_3}{2}]; \\ \frac{(1+\theta_r)(\rho_3-x)}{\rho_3-\rho_2+2\theta_r(\rho_3-x)}, & \text{if } x \in (\frac{\rho_2+\rho_3}{2}, \rho_3]. \end{cases} \tag{2.36}$$

From the above examples it is observed that reduced type-1 fuzzy parameters as obtained by CV-based reduction methods are not always normalized, i.e. are general fuzzy parameters. For such cases, generalized credibility measure $\tilde{Cr}$ is used instead of the credibility measure. The following theorem approaches to find crisp equivalent forms of constraints involving type-2 triangular fuzzy parameters. This theorem is established using generalized credibility measure for the reduced fuzzy parameter from type-2 triangular fuzzy parameter by CV reduction method.

Theorem 2.2 (Qin et al. [122]). Let $\tilde{\eta}_i$ be the reduction of the type-2 triangular fuzzy parameter $\tilde{\tilde{\eta}}_i = (\rho_1^i, \rho_2^i, \rho_3^i; \theta_{l,i}\theta_{r,i})$ obtained by the CV reduction method for $i = 1, 2, ..., n$. Suppose $\tilde{\tilde{\eta}}_1, \tilde{\tilde{\eta}}_2, ..., \tilde{\tilde{\eta}}_n$ are mutually independent, and $k_i \geq 0$ for $i = 1, 2, ..., n$.

(i) Given the generalized credibility level $\alpha \in (0, 0.5]$,

if $\alpha \in (0, 0.5]$, then $\tilde{Cr}\{\sum_{i=1}^{n} k_i\eta_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(1 - 2\alpha + (1 - 4\alpha)\theta_{r,i})k_i\rho_1^i + 2\alpha k_i\rho_2^i}{1 + (1 - 4\alpha)\theta_{r,i}} \leq t,$$

and if $\alpha \in (0.25, 0.5]$, then $\tilde{Cr}\{\sum_{i=1}^{n} k_i\eta_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(1 - 2\alpha)k_i\rho_1^i + (2\alpha + (4\alpha - 1)\theta_{l,i})k_i\rho_2^i}{1 + (4\alpha - 1)\theta_{l,i}} \leq t,$$

(ii) Given the generalized credibility level $\alpha \in (0.5, 1]$,

if $\alpha \in (0.5, 0.75]$, then $\tilde{Cr}\{\sum_{i=1}^{n} k_i\eta_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(2\alpha - 1)k_i\rho_3^i + (2(1 - \alpha) + (3 - 4\alpha)\theta_{l,i})k_i\rho_2^i}{1 + (3 - 4\alpha)\theta_{l,i}} \leq t,$$

and if $\alpha \in (0.75, 1]$, then $\tilde{Cr}\{\sum_{i=1}^{n} k_i\eta_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(2\alpha - 1 + (4\alpha - 3)\theta_{r,i})k_i\rho_3^i + 2(1 - \alpha)k_i\rho_2^i}{1 + (4\alpha - 3)\theta_{r,i}} \leq t$$

2.1.14 Nearest interval approximation of continuous type-2 fuzzy parameters

We search the CV-based reductions of the type-2 fuzzy parameter for approximation of continuous T2FP by crisp interval. Then we obtain the corresponding $\alpha - cuts$ of these CV-based reductions. Now we illustrate the above method with type-2 triangular fuzzy parameter. Let.$\tilde{\tilde{\eta}}$ be a type-2 triangular fuzzy parameter defined as. $\alpha - cuts = (\rho_1, \rho_2, \rho_3; \theta_l, \theta_r)$. Then from Theorem 1, we have the optimistic CV reduction, pessimistic CV reduction and CV reduction of.$\tilde{\tilde{\eta}}$ as η_1, η_2, η_3, respectively with the possibility distributions given. Now using the definition of $\alpha - cuts$ of a fuzzy parameter [145] we find $\alpha - cuts$ of the reductions of η

$\alpha - cut$ **of the optimistic CV reduction** η_1 **of** η **:** Applying the definition of $\alpha - cuts$ of a fuzzy parameter we find $\alpha - cuts$ of the reductions of η_1 as $[\eta_{1L}(\alpha), \eta_{1U}(\alpha)]$ where

$$\eta_{1L}(\alpha) = \begin{cases} \frac{(1+\theta_r)\rho_1 + (\rho_2 - \rho_1 - \theta_r \rho_1)\alpha}{(1+\theta_r) - \theta_r \alpha}, & 0 \leq \alpha \leq 0.5; \\ \frac{(\rho_1 - \theta_r \rho_2) + (\rho_2 - \rho_1 + \theta_r \rho_2)\alpha}{(1-\theta_r) + \theta_r \alpha}, & 0.5 \leq \alpha \leq 1. \end{cases}$$

$$\eta_{1U}(\alpha) = \begin{cases} \frac{(\rho_3 - \theta_r \rho_2) - (\rho_3 - \rho_2 - \theta_r \rho_2)\alpha}{(1-\theta_r) + \theta_r \alpha}, & 0.5 \leq \alpha \leq 1; \\ \frac{(1+\theta_r)\rho_3 - (\rho_3 - \rho_2 + \theta_r \rho_3)\alpha}{(1+\theta_r) - \theta_r \alpha}, & 0 le \alpha \leq 0.5. \end{cases}$$

$\alpha - cut$ **of the pessimistic CV reduction** η_2 **of** η **:** $\alpha - cuts$ of the reductions of η_2 as $[\eta_{2L}(\alpha), \eta_{2U}(\alpha)]$ where

$$\eta_{2L}(\alpha) = \begin{cases} \frac{\rho_1 + (\rho_2 - \rho_1 - \theta_l \rho_1)\alpha}{1 - \theta_l \alpha}, & 0 \leq \alpha \leq 0.5; \\ \frac{\rho_1 + (\rho_2 - \rho_1 + \theta_l \rho_2)\alpha}{1 + \theta_l \alpha}, & 0.5 \leq \alpha \leq 1. \end{cases}$$

$$\eta_{2U}(\alpha) = \begin{cases} \frac{\rho_3 - (\rho_3 - \rho_2 - \theta_l \rho_2)\alpha}{1 + \theta_l \alpha}, & 0.5 \leq \alpha \leq 1; \\ \frac{\rho_3 - (\rho_3 - \rho_2 + \theta_l \rho_3)\alpha}{1 - \theta_l \alpha}, & 0 \leq \alpha \leq 0.5. \end{cases}$$

$\alpha-cut$ **of the CV reduction** η_3 **of** η **:** $\alpha-cuts$ of the reductions of η_3 as $[\eta_{3L}(\alpha), \eta_{3U}(\alpha)]$ where

$$\eta_{3L}(\alpha) = \begin{cases} \frac{(1+\theta_r)\rho_1+(\rho_2-\rho_1-2\theta_r\rho_1)\alpha}{(1+\theta_r)-2\theta_r\alpha}, & 0 \leq \alpha \leq 0.5; \\ \frac{(\rho_1-\theta_l\rho_2)+(\rho_2-\rho_1+2\theta_l\rho_2)\alpha}{(1-\theta_l)+2\theta_l\alpha}, & 0.5 \leq \alpha \leq 1. \end{cases}$$

$$\eta_{3U}(\alpha) = \begin{cases} \frac{(\rho_3-\theta_l\rho_2)-(\rho_3-\rho_2-2\theta_l\rho_2)\alpha}{(1-\theta_l)+2\theta_l\alpha}, & 0.5 \leq \alpha \leq 1; \\ \frac{(1+\theta_r)\rho_3-(\rho_3-\rho_2+2\theta_r\rho_3)\alpha}{(1+\theta_r)-2\theta_r\alpha}, & 0 le\alpha \leq 0.5. \end{cases}$$

Now we know that nearest interval approximation of a fuzzy number (Grzegorzewski [47]). $\widetilde{A}$ with distance metric d is given by $C_d(\widetilde{A}) = [C_L, C_U]$, where $C_L = \int_0^1 A_L(\alpha)d\alpha$ and $C_U = \int_0^1 A_U(\alpha)d\alpha$, where distance metric d to measure distance of $\widetilde{A}$ from $C_d(\widetilde{A})$ is given by

$$d(\widetilde{A}, C_d(\widetilde{A})) = \sqrt{\int_0^1 \{A_L(\alpha) - C_L\}^2 d\alpha + \int_0^1 \{A_U(\alpha) - C_U\}^2 d\alpha}$$

We can find out the nearest interval approximation of η for the $\alpha - cuts$ of optimistic CV, pessimistic CV or CV reduction of $\tilde{\tilde{\eta}}$ using above method.

Nearest interval approximation of $\tilde{\tilde{\eta}}$ **using** $\alpha - cut$ **of the optimistic CV reduction** η_1 **of** $\tilde{\tilde{\eta}}$: In this case the nearest interval approximation of $\tilde{\tilde{\eta}}$ is obtained as $[C_L, C_U]$ where

$$\begin{aligned} C_L &= \int_0^1 \eta_{1L} d\alpha \\ &= \int_0^{0.5} \frac{(1+\theta_r)\rho_1 + (\rho_2 - \rho_1 - \theta_r\rho_1)\alpha}{(1+\theta_r) - \theta_r\alpha} d\alpha + \int_{0.5}^1 \frac{(\rho_1 - \theta_r\rho_2) + (\rho_2 - \rho_1 + \theta_r\rho_2)\alpha}{(1-\theta_r) + \theta_r\alpha} d\alpha \\ &= \frac{(1+\theta_r)\rho_1}{\theta_r} ln(\frac{1+\theta_r}{1+0.5\theta_r}) - \frac{\rho_2 - \rho_1 - \theta_r\rho_1}{\theta_r^2}[0.5\theta_r - (1+\theta_r)ln(\frac{1+\theta_r}{1+0.5\theta_r})] \\ &\quad - \frac{\rho_1 - \theta_r\rho_2}{\theta_r} ln(1 - 0.5\theta_r) + \frac{\rho_2 - \rho_1 + \theta_r\rho_2}{\theta_r^2}[0.5\theta_r + (1-\theta_r)ln(1-0.5\theta_r)] \end{aligned} \quad (2.37)$$

$$\begin{aligned} C_U &= \int_0^1 \eta_{1U} d\alpha \\ &= \int_0^{0.5} \frac{(1+\theta_r)\rho_3 - (\rho_3 - \rho_2 + \theta_r\rho_3)\alpha}{(1+\theta_r) - \theta_r\alpha} d\alpha + \int_{0.5}^1 \frac{(\rho_3 - \theta_r\rho_2) - (\rho_3 - \rho_2 - \theta_r\rho_2)\alpha}{(1-\theta_r) + \theta_r\alpha} d\alpha \\ &= \frac{(1+\theta_r)\rho_3}{\theta_r} ln(\frac{1+\theta_r}{1+0.5\theta_r}) + \frac{\rho_3 - \rho_2 + \theta_r\rho_3}{\theta_r^2}[0.5\theta_r - (1+\theta_r)ln(\frac{1+\theta_r}{1+0.5\theta_r})] \\ &\quad - \frac{\rho_3 - \theta_r\rho_2}{\theta_r} ln(1 - 0.5\theta_r) - \frac{\rho_3 - \rho_2 + \theta_r\rho_2}{\theta_r^2}[0.5\theta_r + (1-\theta_r)ln(1-0.5\theta_r)] \end{aligned} \quad (2.38)$$

Nearest interval approximation of $\tilde{\tilde{\eta}}$ **using** $\alpha - cut$ **of the pessimistic CV reduction** η_2 **of** $\tilde{\tilde{\eta}}$: In this case the nearest interval approximation of $\tilde{\tilde{\eta}}$ is obtained as

$[C_L, C_U]$ where

$$\begin{aligned}
C_L &= \int_0^1 \eta_{1L} d\alpha \\
&= \int_0^{0.5} \frac{\rho_1 + (\rho_2 - \rho_1 - \theta_l \rho_1)\alpha}{1 - \theta_l \alpha} d\alpha + \int_{0.5}^1 \frac{\rho_1 + (\rho_2 - \rho_1 + \theta_l \rho_2)\alpha}{1 + \theta_l \alpha} d\alpha \\
&= -\frac{\rho_1}{\theta_l} ln\big(1 + 0.5\theta_l\big) - \frac{\rho_2 - \rho_1 - \theta_l \rho_1}{\theta_l^2}[0.5\theta_l + ln(1 - 0.5\theta_l)] \\
&\quad + \frac{\rho_1}{\theta_l} ln\big(\frac{1 + \theta_l}{1 + 0.5\theta_l}\big) + \frac{\rho_2 - \rho_1 + \theta_l \rho_2}{\theta_l^2}[0.5\theta_l - ln(\frac{1 + \theta_l}{1 + 0.5\theta_l})] \qquad (2.39) \\
C_U &= \int_0^1 \eta_{1U} d\alpha \\
&= \int_0^{0.5} \frac{\rho_3 - (\rho_3 - \rho_2 - \theta_l \rho_3)\alpha}{1 - \theta_l \alpha} d\alpha + \int_{0.5}^1 \frac{\rho_3 - (\rho_3 - \rho_2 + \theta_l \rho_2)\alpha}{1 + \theta_l \alpha} d\alpha \\
&= -\frac{\rho_3}{\theta_l} ln\big(1 - 0.5\theta_l\big) + \frac{\rho_3 - \rho_2 - \theta_l \rho_3}{\theta_l^2}[0.5\theta_l + ln(1 - 0.5\theta_l)] \\
&\quad + \frac{\rho_3}{\theta_l} ln\big(\frac{1 + \theta_l}{1 + 0.5\theta_l}\big) - \frac{\rho_3 - \rho_2 - \theta_l \rho_3}{\theta_l^2}[0.5\theta_l - ln(\frac{1 + \theta_l}{1 + 0.5\theta_l})] \qquad (2.40)
\end{aligned}$$

Nearest interval approximation of $\tilde{\tilde{\eta}}$ using $\alpha - cut$ of the CV reduction η_1 of $\tilde{\tilde{\eta}}$: In this case the nearest interval approximation of $\tilde{\tilde{\eta}}$ is obtained as $[C_L, C_U]$ where

$$\begin{aligned}
C_L &= \int_0^1 \eta_{1L} d\alpha \\
&= \int_0^{0.5} \frac{(1+\theta_r)\rho_1 + (\rho_2 - \rho_1 - 2\theta_r \rho_1)\alpha}{(1+\theta_r) - 2\theta_r \alpha} d\alpha + \int_{0.5}^1 \frac{(\rho_1 - \theta_l \rho_2) + (\rho_2 - \rho_1 + 2\theta_l \rho_2)\alpha}{(1 - \theta_l) + 2\theta_l \alpha} d\alpha \\
&= \frac{(1+\theta_r)\rho_1}{2\theta_r} ln(1 + \theta_r) - \frac{\rho_2 - \rho_1 - 2\theta_r \rho_1}{4\theta_r^2}[\theta_r - (1+\theta_r) ln(1+\theta_r)] \\
&\quad + \frac{\rho_1 - \theta_l \rho_2}{2\theta_l} ln(1 + \theta_l) + \frac{\rho_2 - \rho_1 + 2\theta_l \rho_2}{4\theta_l^2}[\theta_l + (1 - \theta_l) ln(1 + \theta_l)] \qquad (2.41) \\
C_U &= \int_0^1 \eta_{1U} d\alpha \\
&= \int_0^{0.5} \frac{(1+\theta_r)\rho_3 - (\rho_3 - \rho_2 + 2\theta_r \rho_3)\alpha}{(1+\theta_r) - 2\theta_r \alpha} d\alpha + \int_{0.5}^1 \frac{(\rho_3 - \theta_l \rho_2) - (\rho_3 - \rho_2 - 2\theta_l \rho_2)\alpha}{(1 - \theta_r) + 2\theta_l \alpha} d\alpha \\
&= \frac{(1+\theta_r)\rho_3}{2\theta_r} ln(1 + \theta_r) + \frac{\rho_3 - \rho_2 + 2\theta_r \rho_3}{4\theta_r^2}[\theta_r - (1+\theta_r) ln(1+\theta_r)] \\
&\quad + \frac{\rho_3 - \theta_l \rho_2}{2\theta_l} ln(1 + \theta_l) - \frac{\rho_3 - \rho_2 - 2\theta_l \rho_2}{4\theta_l^2}[\theta_l + (1 - \theta_l) ln(1 + \theta_l)] \qquad (2.42)
\end{aligned}$$

Now we find nearest interval approximation of type-2 triangular fuzzy parameter $\tilde{\tilde{\eta}} = (2, 3, 4; 0.5, 0.8)$, the credibilistic, pessimistic and optimistic interval approximations of $\tilde{\tilde{\eta}}$ are obtained as [2.4925, 3.5074], [2.5567, 3.4432] and [2.4086, 3.5913] respectively.

2.2 Optimization

under given circumstances, Optimization is the process to obtain the best return(s)/result(s) of a syustem. In construction, design, maintenance and production of any engineering or management system, engineers or decision makers have to accept many managerial and technological decisions at different stages. The final terget of all such decisions is either to maximize the longing profit or to minimize the effort cost/required. Therefore, the problem of optimization is connected with the maximization/minimization of a transcendental or an algebraic function of one or more variables which ere known as objective function under some available resources that are described as constraints. Optimization problem in Crisp Environment is mainly classified into two types

i **Single-Objective Linear Programming (SOLP)/ Non-Linear Programming (SONLP) Problem in Crisp Environment**

A single-objective mathematical programming (SOMP) problem is an optimization problem consisting of only one objective function. We can formulate the minimization of a SOMP problem as the following:

$$\left.\begin{array}{ll} \text{Find} & x=(x_1,x_2,....,x_n)^T \\ \text{which minimizes} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j=1,2,....,m \\ x_i \geq 0, & i=1,2,....,n \end{array} \right\} \end{array}\right\} \tag{2.43}$$

where, $f(x)$ and $g_j(x)$, $j=1,2,....,m$ are functions defined in $\Re^n$.

It is stated that, if both the constraints and the objective function are linear, the above SOMP problem turns out a single-objective linear programming problem (SOLPP). In another way, it is known as a single-objective non-linear programming problem (SONLPP).

A decision variable vector $x=(x_1,x_2,....,x_n)^T$ that gratifies all the constraints is knonwn as a feasible solution to the problem. The gathering of all such solutions shapes the feasible region. The SONLPP (2.43) is to search out a feasible solution x^* such that for each feasible point $f(x) \geq f(x^*)$ for minimization problem and $f(x) \leq f(x^*)$ for maximization problem . Here, x^* is named an solution or optimal solution to the problem.

ii **Multi-Objective Linear Programming Problem (MOLPP)/ Non-Linear Programming Problem (MONLPP) in Crisp Environment**

As the world has turned out to be more complex and nearly every important problem of real-world entangles more than one objective. In such situations, according to multiple criteria, the decision makers search imperative to assess best possible approximate alternatives solution. A common minimization type multi-objective programming problem is of the following structure:

$$\left.\begin{array}{ll} \text{Find} & x = (x_1, x_2, ..., x_n)^T \\ \text{which minimizes} & F(x) = (f_1(x), f_2(x), ..., f_k(x))^T \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j = 1, 2, ..., m \\ x_i \geq 0, & i = 1, 2, ..., n \end{array} \right\} \end{array}\right\} \tag{2.44}$$

where $f_1(x), f_2(x), ..., f_k(x)$ are $k\,(\geq 2)$ objectives. It is stated that, if the objectives $f_l(x)$, for $l = 1, 2, ..., k_0$ of the original problem are minimized and the objectibe $f_l(x)$ for $l = k_0+1, k_0+2, ..., k$, are maximized, then the objective in the mathematical formulation will be

$$\text{Min } F(x) = (f_1(x), f_2(x)...., f_{k_0}(x), -f_{k_0+1}(x), -f_{k_0+2}(x), ..., -f_k(x))^T.$$

subject to the same constraints as in (2.44).

If $f_l(x)$ $(l = 1, 2, ..., k)$, and $g_j(x)$ $(j = 1, 2, ..., m)$ are linear, the corresponding problem is called MOLPP. When all or any one of the above functions is non-linear, it is referred as a MONLPP. Here, the problem is often referred to as a Vector Minimum Problem.

Convex and Non-convex of Multi-objective Optimization Problem (MOOP): The multi-objective optimization problem (2.44) is said to be convex if all the objective functions and the feasible region are convex, otherwise it is called non-convex.

2.3 Single-Objective Optimization

The single objective optimization problem based on the following basic concepts

Local Minimum: A point $x^* \in X$ is said to be a local minimum of (2.43) if there exists an $\epsilon > 0$ such that $f(x) \geq f(x^*), \;\; \forall x \in X \; : \; ||x - x^*|| < \epsilon,$.

Convex Function: A function $f(x_1, x_2, ..., x_n)$ becomes convex if the corresponding Hessian Matrix $H(x_1, x_2, ..., x_n) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{n \times n}$ is positive definite/ positive semi-definite. If it is negetive definite/ negetive semi-definite for the function $f(x_1, x_2, ..., x_n)$, then $f(x)$ is called concave function.

Global Minimum: $x^* \in X$ is said to be a global minima of (2.43) if $f(x) \geq f(x^*), \;\; \forall x \in X$. In another way, if the function $f(x)$ is convex then the local minimum solution $x \in X$ is global minimum.

Convex Programming Problem: The problem narrated in (2.43) is to be called convex programming problem if the constraint functions $g_j(x_1, x_2, ..., x_n), j = 1, 2, ..., m$ and the objective function $f(x_1, x_2, ..., x_n)$ are convex.

Lagrange Function: The Lagrange function corresponding to the constrained optimization problem (2.43) is that

$$L(x, y, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j [g_j(x) + y_j^2]. \tag{2.45}$$

where Y_j^2's are slack variables and λ_j's are Lagrange multipliers. It can be shown that in case of minimization problem, the values of λ_j's will be positive and for maximization problem it will be negative.

2.3.1 Solution Techniques for Single-Objective Linear/ Non-Linear Problem in Crisp Environment

Necessary Condition for Optimality: If a function $f(x)$ is defined for all $x \in X$ and has a relative minimum at $x = x^*$, where $x^* \in X$ and all the partial derivatives $\frac{\partial f(x)}{\partial x_r}$ for $r = 1, 2, ..., n$ are exists at $x = x^*$, then $\frac{\partial f(x^*)}{\partial x_r} = 0$.

Sufficient Condition for Optimality: The sufficient condition for a stationary point x^* to be an extreme point is that the matrix of second partial derivatives (Hessian Matrix) of $f(x)$ evaluated at $x = x^*$ is (i) positive definite when x^* is a relative minimum point, and (ii) negative definite when x^* is a relatively maximum point.

2.3.2 Generalized Reduced Gradient (GRG) Technique:

The GRG technique is a process for working out NLP problems used for both equality and inequality constraints. Consider the NLP problem as

$$\left.\begin{array}{ll} \text{Find} & x = (x_1, x_2, ...x_n)^T \\ \text{which maximizes} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j = 1, 2, ..., m \\ h_r(x) = 0, & r = 1, 2, ..., p \\ x_i \geq 0, & i = 1, 2, ..., n \end{array} \right\} \end{array}\right\} \tag{2.46}$$

By adding a non-negative slack variable $s_j\,(\geq 0),\ \ j = 1, 2, ..., m$ to each of the above inequality constraints, the problem (2.46) can be stated as,

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_n)^T & \\ g_j(x) + s_j = 0, & j = 1, 2, ..., m \\ h_r(x) = 0, & r = 1, 2, ..., p \\ x_i \geq 0 & i = 1, 2, ...n \\ s_j \geq 0, & j = 1, 2, ...m \end{array} \right\} \end{array}\right\} \tag{2.47}$$

where the lower and upper bounds on the slack variables, $s_j,\ j = 1, 2, ..., m$ are taken as a zero and a large number (infinity) respectively.

Denoting s_j by x_{j+n}, $g_j(x) + s_j$ by ξ_j, $h_r(x)$ by ξ_{m+r}, the above problem can be rewritten as,

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_{n+m})^T & \\ \xi_j(x) = 0, & j = 1, 2, ...m + p \\ x_i \geq 0 & i = 1, 2, ...n + m \end{array} \right\} \end{array}\right\} \tag{2.48}$$

This GRG technique has been developed eliminating variables using the equality constraints. Theoretically, dependent $(m+p)$ variables can be revealed in terms of remaining $(n - p)$ independent variables. Therefore, $(n + m)$ decision variables can be divided arbitrarily into two sets as

$$x = (y, z)^T$$

where y is $(n - p)$ independent or design variables and z is $(m + p)$ state or dependent variables and

$$\begin{aligned} y &= (y_1, y_2, ..., y_{n-p})^T \\ z &= (z_1, z_2, ..., z_{m+p})^T \end{aligned}$$

Here, the design variables are entirely independent and the state variables are dependent on the design variables used to gratify the constraints $\xi_j(x) = 0,\ (j = 1, 2, ..., m + p)$. Consider the first variations of the objective and constraint functions, it is defined as:

$$df(x) = \sum_{i=1}^{n-p} \frac{\partial f}{\partial y_i} dy_i + \sum_{i=1}^{m+p} \frac{\partial f}{\partial z_i} dz_i = \nabla_y^T f\, dy + \nabla_z^T f\, dz \tag{2.49}$$

$$d\xi_j(x) = \sum_{i=1}^{n-p} \frac{\partial \xi_j}{\partial y_i} dy_i + \sum_{i=1}^{m+p} \frac{\partial \xi_j}{\partial z_i} dz_i$$

$$\text{or} \quad d\xi = C\, dy + D\, dz \tag{2.50}$$

$$\text{where} \quad \nabla_y^T f = \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, ..., \frac{\partial f}{\partial y_{n-p}} \right)$$

$$\text{and} \quad \nabla_z^T f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, ..., \frac{\partial f}{\partial z_{m+p}} \right)$$

$$C = \begin{bmatrix} \frac{\partial \xi_1}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_1}{\partial y_{n-p}} \\ \frac{\partial \xi_2}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_2}{\partial y_{n-p}} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \frac{\partial \xi_{m+p}}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_{l+m}}{\partial y_{n-p}} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{\partial \xi_1}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_1}{\partial z_{m+p}} \\ \frac{\partial \xi_2}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_2}{\partial z_{m+p}} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \frac{\partial \xi_{m+p}}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_{m+p}}{\partial z_{m+p}} \end{bmatrix},$$

$$dy = (dy_1,\, dy_2, ..., dy_{n-p})^T$$

$$\text{and} \quad dz = (dz_1, dz_2, ..., dz_{m+p})^T$$

Assuming that the constraints are originally satisfied at the vector x $(\xi(x) = 0)$, any change in the vector dx must correspond to $d\xi = 0$ to maintain feasibility at $x + dx$. Thus, equation (2.50) can be solved as

$$Cdy + Ddz = 0$$

$$\text{or} \quad dz = -D^{-1}Cdy \tag{2.51}$$

The change in the objective function due to the change in x is given by the equation (2.49), which can be expressed, using equation (2.51) as

$$df(x) = (\nabla_y^T f - \nabla_z^T f D^{-1} C) dy$$

$$\text{or} \quad \frac{df(x)}{dy} = G_R \tag{2.52}$$

$$\text{where} \quad G_R = \nabla_y^T f - \nabla_z^T f D^{-1} C \tag{2.53}$$

Hence GR is called the generalized reduced gradient. Geometrically, the reduced gradient can be narrated as a projection of the original $n-$dimensional gradient into the $(n-m)$ dimensional feasible region described by the design variables.

For the existence of minimum of an unconstrained function a necessary condition is that the components of the gradient vanish. Similarly, a constrained function assumes its minimum value when the appropriate components of the reduced gradient are zero.

2.3.3 Genetic Algorithm(GA):

Genetic Algorithm is a thorough search algorithms to be made on the basis of on the mechanics of genesis (crossover, mutation etc.) and natural selection. It has been advanced by Holland (cf. Holland [58]), his colleagues and his students at the University of Michigan (cf. Goldberg [46]). For its several advantages over conventional optimization methods, one can effectively apply it to many optimization problems. Holland was stimulated by Darwin's theory about evolution and constructed GAs depending upon the fundamental principle of the theory: 'Survival of the fittest'. The theoretical basis for the GA is the Schema Theorem which narrates that individual chromosomes with short, low-order, highly fit schemata or building blocks receive an exponentially increasing number of trials in consecutive generations. It is known to that in natural genesis, chromosomes are the main carriers carrying the hereditary information from parent to offspring and hereditary factors which are presented by genes, are lined up on chromosomes. In this process, hereditary factors of parents are mixed-up and these are carried to their offsprings. Again according to Darwinian principle, none but the fittest animals can endure in nature. In this process, a better offspring is regenerated by a pair of fittest parent normally.

For an optimization problem, we can follow the same phenomenon to build up a genetic algorithm . Here potential solutions of the problem are analogous with the chromosomes and chromosome of better offspring with the better solution of the problem. Crossover and mutation take place among a set of potential solutions to get a new set of solutions and it goes on until terminating conditions are faced with. For a particular problem a GA consists of following six components.

(a) Representation of genetic for potential solutions(**chromosomes**) to the problem

(b) Initialization to create an **initial population** of potential solutions(chromosomes).

(c) Evalution to be used to determine **fitness** of each solution.

(d) An evolution function that plays the environmental role, rating solutions in term of their fitness, i.e., **selection process** for mating pool.

(e) **crossover** and **mutation** which change the composition of children.

(f) Different parameters to be used in the genetic algorithm.

Procedures for different GA components

(a) Representation of Chromosome : To find a feasible solution to the problem, we can basically use the concept of chromosome in the GA. A chromosome has the structure of a string of genes from where some value can be taken from a specified search space. Normally, chromosome representations are of two types as the following - (i) the binary vector representation based on bits and (ii) the real number representation. In this research work, the real number representation scheme has been used.

(b) Initialization: A population is a set of chromosomes ie solutions. There are N chromosomes ie solutions such as X_1, X_2, X_3, ... X_N which are generated randomly from search space of the problem. Here, each X_i $i = 1, 2 \ldots N$ satisfies the constraints of the problem. We take this solution set as initial population and it is the beginning point for a GA to advance to longing solutions. At this pace, probability of mutation p_m and probability of crossover p_c are also initialized. These two parameters are applied to select chromosomes from the mating pool for genetic operations- mutation and crossover respectively.

(c) Fitness value: In the population all chromosomes are assessed using a fitness function. Whether a chromosome convenient or not for the environment under consideration, that is measured by the fitness value. The higher fitness chromosomes will receive larger probabilities of inheritance in subsequent generations, while lower fitness chromosomes will more likely be extruded. The selection of a good and accurate fitness function is thus a pointer to the success of working out any problem in a quick way. The value of a objective function is taken as fitness of $f(X)$ for the solution X in the thesis.

(d) Selection process to create mating pool: Selection in a GA is a method by which some solutions be selected from the population for mating pool. From this mating pool, pairs of solutions in the current generation are choosed as parents to regenerate offspring. There are different selection processes, such as ranking selection, roulette wheel selection, sampling selection, stochastic universal local selection, tournament selection, truncation selection etc. Here, Roulette wheel selection process has been used in several cases. This procedure is made of following steps:

(i) Calculate total fitness of the population $F = \sum_{i=1}^{N} f(X_i)$

(ii) Find the probability of selection p_i of each solution X_i by the formula

$$p_i = \frac{f(X_i)}{F}$$

.

(iii) Find the cumulative probability q_i for each solution X_i by the following formula

$$q_i = \sum_{j=0}^{i} p_j \qquad \text{where} \quad q_N = 1$$

(iv) Create a random number 'r' from the range [0,1].

(v) If $r < q_1$ then select X_1, otherwise select X_i $(2 \leq i \leq N)$ where $q_{i-1} \leq r \leq q_i$.

(vi) To select N solutions from current population repeat step (iv) and (v) N times . Clearly one solution may be selected more than once.

(vii) This selected solution set is represented by $P^1(T)$.

(e) Crossover: Crossover is an important operator in the GA. It works to interchange the main characteristics of parent chromosomes and carries them on the offspring. It consists of two steps:

(i) Selection for crossover: At first, generates a random number r from the range [0..1] for each solution of $P^1(T)$. If $r < p_c$, then we can take the solution for crossover. Here p_c is the probability of crossover.

(ii) Crossover process: Crossover are taken place on the selected chromosomes ie, solutions. For each pair of coupled solutions X_1 and X_2 a random number λ is generated from the range $[0, 1]$. Then X_1 and X_2 are replaced by their offspring's X_{11} and X_{21} respectively as follows

$$X_{11} = \lambda X_1 + (1 - \lambda)X_2, \qquad X_{21} = \lambda X_2 + (1 - \lambda)X_1$$

, if X_{11}, X_{21} satisfied the constraints of the problem.

(f) Mutation: After the crossover operation, the mutation operation is applied to maintain the diversity of the population and it recovers possible loss of some good characteristics. It consists of two steps:

(i) Selection for mutation: Generates a random number r from the range $[0, 1]$ for each solution of $P^1(T)$. If $r < p_m$, the solution is taken for mutation. Here p_m is the probability of mutation.

(ii) Mutation schem: Select a random integer r in the range $[1, K]$ to mutate a solution $X = (x_1, x_2, ., x_K)$. Then the variables x_r is replaced by randomly generated value within the boundary of r^{th} component of X.

After the operations of selection, crossover and mutation, the new population is ready for its next generation, i.e., $P^1(T)$ is considered as population of new generation. Therefore, the genetic algorithm is written as follows in which T is iteration counter and $P(T)$ is the population of potential solutions for iteration T.

(i). Set iteration counter T=0.

(ii). Initialize the probability of crossover p_c and the probability of mutation p_m.

(iii). Initialize P(T).

(iv). Evaluate P(T).

(v). Repeat

- (a). Select N solutions from P(T), for mating pool using Roulette-wheel selection process. Let this set be $P(T)^1$.
- (b). Select solutions from $P(T)^1$, for crossover depending on p_c .
- (c). Made crossover on selected solutions for crossover to get population $P(T)^2$.
- (d). Select solutions from $P(T)^2$, for mutation depending on p_m.
- (e). Made mutation on selected solutions for mutation to get population $P(T+1)$.
- (f). $T \leftarrow T + 1$.
- (g). Evaluate $P(T)$.

(vi). Until(Termination condition does not hold).

(vii). Output: Fittest solution(chromosome) of P(T).

Handling of Constraints in GA

The basic idea of handling constraints is to design solutions carefully by genetic operators to keep all these within the feasible solution set. To ensure that the chromosomes are feasible, we have to check all new chromosomes (x) generated by genetic operators. To check the feasibility of a solution, a function is designed for each target optimization problem, the output value 1 means that the chromosome is feasible, 0 for infeasible. The algorithm for finding the feasibility of an individual (solution) (x) for the optimization problem (2.43) is given below:

```
for i = 1 to l do
  if(g_i(x) ≤ 0)
    continue;
   else
    return 0;
  endif
endfor
for j = 1 to m do
  if(h_j(x) = 0)
    continue;
   else
    return 0;
  endif
endfor
 return 1
```

2.3.4 Single-Objective Problem in Fuzzy Environment

The main aim of fuzzy optimization is to search the "best "solution (decision alternative) under imprecise information and / or vague resources limits. There are many forms of imprecision when dealing with fuzzy optimization. Normally, in fuzzy optimization fuzzy sets are utilized in two different ways .

i. To express uncertainty in the goals and the constraints (objective functions).

ii. To express flexibility in the goals and the constraints.

In the first case, fuzzy sets express the generalised formulations of intervals that are manipulated according to the rules which are extensions of the interval calculus by using λ-cuts of fuzzy sets. In the second case, fuzzy sets express the degree of the aspiration levels of the goals or of satisfaction of the constraints, given the flexibility in the formulation. Hence, the constraints (and goals) that are essentially crisp one assume to have some flexibility that can be exploited for improving the optimization objective. The general formulation of fuzzy optimization in the presence of flexible constraints and goals is given by

$$\begin{aligned} \text{Max} \quad & \widetilde{Z} = \widetilde{f}(x) \\ \text{subject to} \quad & g_i(x) \widetilde{\leq} 0, \quad i = 1, 2, ..., m \\ & x \in X \end{aligned} \tag{2.54}$$

The sign $\widetilde{\leq}$ denotes that $g_i(x) \leq 0$ can be satisfied to a degree smaller than 1. The fuzzy maximization corresponds to achieving the highest possible aspiration level for the goal f(x), given the constraints to the problem. The concepts of fuzzy goal and fuzzy constraints were first introduced by Bellman and Zadeh [13]. According to Bellman and Zadeh [13] decision making model, the fuzzy decision $\mu_D(x)$ defined as

$$\mu_D(x) = \mu_Z(x)_o \mu_1(x)_o \mu_2(x)_o \ldots\ldots\ldots_o \mu_m(x). \tag{2.55}$$

where 'o' denotes an operator of aggregation for fuzzy sets, μ_Z indicates the degree of satisfaction of the goal by x $\in$ X and $\mu_i(x)$ (i=1,2,...,m) denote the degree of satisfaction for the fuzzy constraints by the decision alternative x $\in$ X. The decision function (2.55) is complicated enough and a numerical method has to used to search for the optimism. Two commonly used types for the proposed fuzzy objective are discussed in the following.

(i). **Imprecise Parameters in Objective Function and Constraints**

In reality, a decision maker is not able to define exectly different parameters of the optimization problem under consideration. In such cases, the specifications (parameters) are either defined as fuzzy numbers ie, as non-stochastic sense with feasible membership functions or as random numbers ie, as stochastic sense with feasible probability distributions. In case of non-stochastic sense, the problems belong to the class of 'Single-Objective Programming Problem in Fuzzy Environment '.

A crisp non-linear programming problem may be stated as follows:

$$\left.\begin{array}{lll} \text{Min} & f(x,a) & \\ \text{subject to} & g_j(x,a) \leq b_j & j=1,2,...,m \\ & x_i \geq 0 & i=1,2,...,n \end{array}\right\} \quad (2.56)$$

where, $x = (x_1, x_2, ..., x_n)^T$ is crisp decision vector, $a = (a_1, a_2, ..., a_k)^T$ is crisp parameter vector, $b = (b_1, b_2, ..., b_m)^T$ is crisp requirement vector.

When the vectors a and b are fuzzy in nature, i.e., $\tilde{a}$ and $\tilde{b}$, the above problem (2.56) is reduced to a fuzzy non-linear programming problem as follows

$$\left.\begin{array}{lll} \text{Min} & \tilde{f}(x,\tilde{a}) & \\ \text{subject to} & \tilde{g}_j(x,\tilde{a}) \leq \tilde{b}_j & j=1,2,...,m \\ & x_i \geq 0 & i=1,2,...,n \end{array}\right\} \quad (2.57)$$

where, $x = (x_1, x_2, ..., x_n)^T$ is crisp decision vector, $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_k)^T$ is fuzzy parameter vector, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)^T$ is fuzzy requirement vector (where '$\sim$' represents the fuzziness of the parameters).

(ii). **Single-objective Programming Problem Under Fuzzy Expected Value**

A general single-objective mathematical programming problem with fuzzy parameters should have the following form:

$$\text{Max } f(u, \widetilde{\xi}) \quad (2.58)$$

$$\text{subject to} \quad g_j(u, \widetilde{\xi}) \leq 0, j = 1, 2, ...m.$$

where $\widetilde{\xi}$ and u are fuzzy and decision vectors respectively. To transform the fuzzy objective and constraints to their crisp equivalents, Liu and Liu [80] proposed a new approch to transform the problem into an equivalent crisp model which is as follows:

$$\begin{aligned} & \text{Max } E[f(u,\widetilde{\xi})] \\ \text{subject to} \quad & E[g_j(u,\widetilde{\xi})] \leq 0, j = 1, 2, ...m. \end{aligned} \tag{2.59}$$

2.3.5 Solution Techniques for Single Objective Optimization Problem in Fuzzy Environment

Under Possibility and / or Necessity and / or Credibilty Measure

When the ojective function $f(x,\tilde{a})$ becomes imprecise in nature, in that case the statement maximize $f(x,\tilde{a})$ is not well defined. In that case, one can maximize the optimistic (pessimistic) return z, corresponding to the objective function using possibility (necessity) measure of the fuzzy event $\{\tilde{a}|f(x,\tilde{a}) \geq z\}$ as suggested by Liu and Iwamura [86,87]. Therefore, when $\widetilde{a}$ is a fuzzy vector, one can convert the above problem (2.57) to the following equivalent possibility/necessity constrained programming problem (analogous to the chance constrained programming problem).

$$\begin{aligned} \max \quad & z \\ \text{subject to} \quad & pos/nes\{\tilde{a}|f(x,\tilde{a}) \geq z\} \geq \beta \\ & x \in X \end{aligned} \tag{2.60}$$

where β is the predetermined confidence level for fuzzy objective, pos$\{.\}$ (nes$\{.\}$) denotes the possibility (necessity) of the event in $\{.\}$. Here, the objective value z should be the maximum that the objective function $f(x,\tilde{a})$ achieves with at least possibility (necessity) β, in optimistic (pessimistic) sense.

2.4 Multi-Objective Optimization

The multi-objective optimization problem is based on the following basic concepts.

Ideal Objective Vector: An objective vector minimizing each of the objective functions is called an ideal (or perfect) objective vector.

Complete Optimal Solution: x^* is said to be a complete optimal solution to the MONLPP in (2.44) iff there exists $x^* \in X$ such that $f_i(x^*) \leq f_i(x),\ i = 1, 2, ..., k$ for all $x \in X$. In general, the target functions of the MONLPP dispute with each other, a complete optimal solution does not always exist and so non dominated (Pareto) optimality concept has been instituted.

Pareto Optimal Solution: A solution is said to be pareto optimal of MOLPP or MONLPP if none of the objective functions can be upgraded/ improved with out degrading some of the objective functions. Unless an optimization problem is convex, only locally optimal solution is assured using standard mathematical programming techniques. Hence, the idia of Pareto-optimality needs to be modified to institute the notion of a locally Pareto-optimal solution for a non-convex problem as defined by Geoffrion [49].

Locally Pareto Optimal Solution: $x^* \in X$ is called a locally Pareto optimal solution to the MONLPP if and only if there exists $r > 0$ such that x^* is Pareto optimal in $X \cap N(x^*, r)$, there does not exist another $x \in X \cap N(x^*, r)$ such that $f_i(x) \leq f_i(x^*)$, where $N(x^*, r)$ is a r-neighborhood of x^*

Concept of Domination: The concept of domination is very useful in evolutionary multi-objective optimization algorithms. In these algorithms, two solutions are compared on the basis of whether one dominates the other solution or not. Let us consider the operator $\sqsupseteq$ between two solutions i and j as $i \sqsupseteq j$. It denotes that solution i is better than solution j on a particular objective. Similarly $i \sqsubseteq j$ for a particular objective implies that solution i is worse than solution j on this objective. With this assumption, a solution i is said to dominate the other solution j, if both the following conditions hold.

- The solution i is not worse than j in all the objectives.
- The solution i is strictly better than j in at least one objective, i.e., $f_k(i) \sqsupset f_k(j)$ for at least one $k \in \{1, 2, ..K\}$

2.4.1 Solution Techniques for Multi-Objective Linear/ Non-Linear Problem in Crisp Environment

Weighted Sum Method: The weighted sum method scalarizes a set of objectives into a single objective by multiplying each objective with user's supplied weights. The weights of an objective are usually selected in proportion to the objective's relative importance in the problem. However setting up a properly weight vector depends on the scaling of each objective function. It is likely that different objectives take different orders of magnitude. When such objectives are weighted to form a composite objective function, it would be better to scale them properlyly so that each objective possesses more or less the same order of magnitude. This procedure is called normalization of objectives. After the objectives are normalized, a composite objective function $F(x)$ can be made by summing the weighted normalized objectives and the MONLPP given in equation (2.44) is then transformted into a single-objective optimization problem as follows:

$$\text{Minimize } F(x) = \sum_{i=1}^{k} w_i f_i(x), w_i \in [0, 1], \; x \in X \tag{2.61}$$

Here, w_i is the weight of the i-th objective function. Since the minimum of the above problem does not change if all the weights are multiplied by a constant, it is the usual practice to choose weights such that their sum is one, i.e., $\sum_{i=1}^{k} w_i = 1$.

2.4.2 Multi-Objective Genetic Algorithm (MOGA)

Genetic algorithm approach was first suggested by Holland [58]. Because of its generalization, it has been strongly applied to many optimization problems, for its diverse advantages over conventional optimization methods. There are many approaches to deal with the multi-objective optimization problems using genetic algorithms. These algorithms can be classified into two types-(i) Non-Elitist MOGA and (ii) Elitist MOGA. Among Non-Elitist MOGA, Srinivas and Deb's NSGA [137] enjoyed more attention. Two common features of all these algorithms are- (i) assigning fitness to population members based on non-dominated sorting and (ii) preserving diversity among solutions of the same non-dominated front. Diversity is maintained using a sharing function depending on the problem. Among Elitist MOGAs one can refers Rudolph's Elitist Multi-objective evolutionary algorithm (Rudolph [129]), Deb *et al.* 's [29] Elitist Non-dominated Shorting Multi-objective Genetic Algorithm. These algorithms normally select solution from parent population for cross-over and mutation randomly. After these operations, parent and child population are combined together and among them better solutions are choosed for next iteration. A fast and elitist MOGA is developed following Deb *et al.* [29] and it is used to solve few transportation models. This algorithm is named **Fast and Elitist Multi-objective Genetic Algorithm (FEMOGA)**. This multi-objective genetic algorithm possesses the following two important components.

(a) **Dividing of a population of solutions into subsets with non-dominated solutions:** A a problem is considered having M objectives and a population P of feasible solutions of the problem of size N is taken. The population is partitioned P into subsets $F_1, F_2, , F_k$, such that every subset contains non-dominated solutions, but every solution of F_i is not dominated by any solution of F_{i+1}, for $i = 1, 2, ..k-1$. To make this propertydo this for each solution (x) of P, the following two entitie are calculated:

   (i) Number of solutions of P which dominate x, denoted by n_x.

(ii) Set of solutions of P that are dominated by x. denoted by S_x.

To compute above two entities steps $O(MN^2)$ computations are required. Clearly F_1 holds every solution x having $n_x = 0$. Now, for each solution $x \in F_1$, visit every member y of S_x and decrease n_y by 1. In doing so for any member y if $n_y = 0$, then $y \in F_2$. In this way F_2 is established. The above procedure is continued to every member of F_2 and thus F_3 is obtained. This process is continued until all subsets are identified. For each solution x in the second or higher level of non-dominated subsets, n_x can be at most $(N-1)$. So, each solution x be visited at most $(N-1)$ times before n_x becomes zero. At this point, the solution is attributed a subset and it will never be visited again. Since, there is at most $(N-1)$ such solutions, the total complexity is $O(N^2)$. So, total complexity of this component is $O(MN^2)$.

(b) Determination of distance of a solution from other solutions in a subset: To calculate the distance of a solution from other solutions in a subset, the following steps are followed:

(i) First categorize the subset according to each objective function in ascending order of magnitude.

(ii) For each objective function, the boundary solutions are imposed an infinite distance value (a large value).

(iii) All other intermediate solutions are assigned a distance value for the objective, equal to the absolute normalized difference in the objective values of two adjjoining solutions.

(iv) This computation is carried on with other objective functions.

(v) The final distance of a solution from others is computed as the sum of individual distance values corresponding to each objective.

Since M independent sorting of at most N solutions (in case the subset contains all the solutions of the population) are involved, the above algorithm has $O(MNlogN)$ computational complexity. Appling the above two operations, the proposed multi-objective genetic algorithm have the following form:

(i). Stand iteration counter $T = 1$.

(ii). Create an initial population set of solution $P(T)$ of size N.

(iii). Set probability of mutation p_m and probability of crossover p_c.

(iv). Select solution from $P(T)$ for mutation and crossover.

(v). Make crossover as well as mutation on selected solution to get the child set $C(T)$.

(vi). Stand $P_1 = P(T) \bigcup C(T)$, here $\bigcup$ stands for union operation.

(vii). Divide P_1 into some disjoint subsets having non-dominated solutions. Let these sets be $F_1, F_2, .., F_k$.

(viii). Select maximum integer n such that order of $P_2(= F_1 \bigcup F_2 \bigcup \; ... \; \bigcup F_n)$less or equal to$N$.

(ix). If $O(P_2) < N$, sort solutions of F_{n+1} in descending order of their distance from other solutions of the subset. Then, select first $N - O(P_2)$ solutions from F_{n+1} and add with P_2, where $O(P_2)$ represents order of P_2.

(x). Stand $T = T + 1$ and $P(T) = P_2$.

(xi). If termination condition does not hold, go to step-4.

(xii). Output: P(T)

(xiii). End algorithm.

MOGAs with non-dominated sorting and sharing are mainly censured for their

- $O(MN^3)$ computational complexity,
- non-elitism approach,
- the need for specifying a sharing parameter to keep up diversity of solutions in the population.

In the above algorithm, these drawbacks are get over. Since in the above algorithm computational complexity of $step - vii$ is $O(MN^2)$, $step - ix$ is $O(MNlogN)$ and other steps are $\leq O(N)$, so overall time complexity of the algorithm is $O(MN^2)$. Here selection of new population after crossover and mutation on old population, is done by constucting a mating pool by combining the parent and offspring population and among them, best

N solutions are taken as solutions of new population. By this way, elitism is initiated in the algorithm. When some solutions from a non-dominated set F_j (i.e., a subset of F_j) are selected for new population, those are accepted whose distance compared to others (which are not selected) are much i.e., isolated solutions are accepted. In this way taking some isolated solutions in the new population, diversity among the solutions is introduced in the algorithm, without using any sharing function. Since computational complexity of this algorithm $< O(MN^3)$ and elitism is introduced, this algorithm is named as FEMOGA. Different procedures of the above FEMOGA are discussed in the following section.

Procedures of the proposed FEMOGA

(a) Representation: To represent a solution a 'K dimensional real vector' X=(x_1, x_2, ... x_K) is used , where x_1, x_2, ... x_K denote different decision variables of the problem such that constraints of the problem are satisfied.

(b) Initialization: From the search space, N such solutions $X_1, X_2, X_3, ..., X_N$ are randomly generated by random number generator such that each X_i satisfies the constraints of the problem. This solution set is taken as initial population P(1). Also set $p_c \in [0, 1]$, $p_m \in [0, 1]$, T=1.

(c) Crossover:

(i) **Selection for crossover:** Generate a random number r from the range [0,1] for each solution of $P(T)$. If $r < p_c$, then the solution is taken for crossover.

(ii) **Crossover process:** On the selected solutions crossover is taken place. For each pair of coupled solutions X_1 and X_2 a random number λ is generated from the range [0,1] and offsprings X_{11} and X_{21} are calculated by

$$X_{11} = \lambda X_1 + (1 - \lambda)X_2, \quad X_{21} = \lambda X_2 + (1 - \lambda)X_1$$

.

(d) Mutation:

(i) **Selection for mutation:** Generate a random number r from the range $[0, 1]$ for each solution of $P(T)$. If $r < p_m$, then the solution is taken for mutation.

(ii) **Mutation process:** To mutate a solution $X = (x_1, x_2, x_3, \ ... \ x_K)$, choose a random integer r in the range $[1, K]$. Then x_r is replaced by randomly generated value within the boundary of r^{th} component of X.

(e) Division of $P(T)$ into disjoint subsets possessing non-dominated solutions: The following algorithm is developed to divide $P(T)$ into some disjoint subsets having non-dominated solutions:

for every $x \in P(T)$ do
  set $S_x = \Phi$,where Φ expresses null set
  $n_x = 0$
  for every $y \in P(T)$ do
    if x dominates y then
      $S_x = S_x \bigcup \{y\}$
    else if y dominates x then
      $n_x = n_x + 1$
    end if
  ed For
  if $n_x = 0$ then
    $F_1 = F_1 \bigcup \{x\}$
  end If
end For
set i=1
while $F_i \neq \Phi$ do
  $F_{i+1} = \Phi$
  for every $x \in F_i$ do
    for every $y \in S_x$ do
      $n_y = n_y - 1$

if $n_y = 0$ then

$F_{i+1} = F_{i+1} \bigcup \{y\}$

end If

end For

end For

i=i+1

end while

Output:$F_1, F_2, ..F_{i-1}$.

(f) Determination of distance of a solution of a subset F from other solutions:

For this purpose Following algorithm has been used:

set n=number of solutions in F

for every $x \in F$ do

$x_{\text{distance}} = 0$

end For

for every objective m do

sort F, in ascending order of magnitude of m^{th} objective.

$F[1] = F[n] = M$, where M is a big quantity.

for i=2 to n-1 do

$F[i]_{\text{distance}} = F[i]_{\text{distance}} + (F[i+1].\text{objm} - F[i-1].\text{objm})/(f_m^{\max} - f_m^{\min})$

end for

end for

In the algorithm $F[i]$ represents $i - th$ solution of F, $F[i]$.objm represent $m - th$ objective value of $F[i]$. Also $f_m^{\min}$ and $f_m^{\max}$ represent the minimum and maximum values of $m - th$ objective function.

2.4.3 Fuzzy Multi-Objective Optimization Problem (FMOOP)

Let us consider, a fuzzy non-linear multi-objective maximizing problem where objective goal has some imprecise or fuzzy parameters as below :

$$\left.\begin{array}{ll}\text{Find} & x=(x_1,x_2,....,x_n)^T \\ \text{which optimizes} & F(x)=(\widetilde{f}_1(x),\widetilde{f}_2(x),....,\widetilde{f}_k(x))^T \\ \text{subject to} & x\in X \\ \text{where } X= & \left\{x: \begin{array}{ll} g_j(x)\leq 0, & j=1,2,....,l \\ h_r(x)=b_r, & r=1,2,....,m \\ x_i\geq 0, & i=1,2,....,n \end{array}\right\}\end{array}\right\} \quad (2.62)$$

Here, $\widetilde{f}_1(x),\widetilde{f}_2(x),...$ and $\widetilde{f}_k(x)$ $(k\geq 2)$ are k objectives. It is noted that, if the objectives $\widetilde{f}_i(x)$, for $i=1,2,....,k_0$ of the original problem are minimize and $\widetilde{f}_i(x)$ for $i=k_0+1,k_0+2,....,k$, are maximized, then the objective be converted for minimization as follows:

$$\text{Min } F(x)=(\widetilde{f}_1(x),\widetilde{f}_2(x)....,\widetilde{f}_{k_0}(x),-\widetilde{f}_{k_0+1}(x),-\widetilde{f}_{k_0+2}(x),....,-\widetilde{f}_k(x))^T.$$

subject to the same constraints as in (2.62).
Here, objectives are imprecise in nature.

2.4.4 Interactive Fuzzy Decision Making Method (for Fuzzy Multi-Objective Optimization Problems (FMOOP))

Now taking the imprecise nature of decision maker's (DM) judgment, for each of the objective functions DM may have different fuzzy or imprecise goals. In this case, an interactive approach is used for the man-machine interaction.

Pay-off matrix: Here, DM first bring out the membership functions for each objective functions f_j, (j=1,2,...k) respectively from DM's outlook with the help of individual minimum and individual maximum by non-linear optimization method (GRG).

Membership function: On the basis of individual minimum and maximum, a DM can formulate and select any one from among the following three types of membership functions.

(i) Linear membership functions,

(ii) Quadratic membership functions,

(iii) Exponential membership functions.

The membership functions for the corresponding objective functions f_j, (j=1,2,...k) may be written as

Type-1 : Linear membership function

For each objective function, the corresponding linear membership function is as follows:

$$\mu_{f_j}(x) = \begin{cases} 0 & \text{for } f_j^0 > f_j(x) \\ 1 - \dfrac{f_j^1 - f_j(x)}{f_j^1 - f_j^0} & \text{for } f_j^0 \leq f_j(x) \leq f_j^1 \\ 1 & \text{for } f_j(x) > f_j^1 \end{cases} \tag{2.63}$$

Type-2 : Quadratic membership function

For each objective function, the corresponding quadratic membership function is as follows:

$$\mu_{f_j}(x) = \begin{cases} 0 & \text{for } f_j^0 > f_j(x) \\ 1 - \left(\dfrac{f_j^1 - f_j(x)}{f_j^1 - f_j^0}\right)^2 & \text{for } f_j^0 \leq f_j(x) \leq f_j^1 \\ 1 & \text{for } f_j(x) > f_j^1 \end{cases} \tag{2.64}$$

Type-3 : Exponential membership function

For each objective function, the corresponding exponential membership function is as follows:

$$\mu_{f_j}(x) = \begin{cases} 0 & \text{for } f_j^0 > f_j(x) \\ \alpha_r\left[1 - e^{-\beta_K\left(\frac{(f_j^1 - f_j(x))}{(f_j^1 - f_j^0)}\right)}\right] & \text{for } f_j^0 \leq f_j(x) \leq f_j^1 \\ 1 & \text{for } f_j(x) > f_j^1 \end{cases} \tag{2.65}$$

Where the constants and the tolerance of j-th objective function f_j can be determined by asking the DM.

Parametric values: The goal parametric values are determined by DM for the membership function which to be determined following Sakawa et al. [132] as:

$$\begin{aligned} f_j^1 &= f_j(x_i^{max}) \\ f_j^0 &= \operatorname*{Min}_{l \neq j}\{f_l(x_i^{max})\} \end{aligned}$$

Level of significant: After considering the different non-linear/linear membership functions (MF) for each of the objective functions to create a candidate for the saticficing solution following Bellman and Zadeh [13] and Zimmermann [157], the DM is asked to specify his / her reference level of achievement for the membership values. Let $\overline{\mu}_{f_j}$ is the reference membership level of the objective function. The better reference membership levels are attainable for the better requirement that can be formulated as

$$\underset{x_i \epsilon X}{\text{Min}} \ \underset{1 \leq l \leq j}{\text{Max}} (\overline{\mu}_{f_l} - \mu_{f_l}) \tag{2.66}$$

which is equivalent to

$$\begin{aligned} &\text{Max} \quad \beta \\ &\text{where} \quad \beta \leq (\overline{\mu}_{f_j} - \mu_{f_j}) \ , \end{aligned}$$

Preferential Objective: Let us considered that objective f_T is more important than f_S (S, T=1,2,...k. and $S \neq T$) which is represented as $f_S \prec f_T$. It is rational for us to expect that objective with higher priorities also have higher level of satisfaction, this means that the solution is obtained from finding the maximum level of significant. Then condition of priority can be stated as:

$$\mu_{f_S}(x_i^*) \leq \mu_{f_T}(x_i^*)$$

Now, after getting β^* , if the DM chooses Z_T as the most important objective function from among all objective functions f_j (j=1,2,...k), then the problem becomes (for $\beta = \beta^*$)

$$\begin{aligned} &\text{Max} \quad f_T(x_i) \\ &\text{subject to} \quad \beta^* \leq (\overline{\mu}_{f_j} - \mu_{f_j}) \\ &\text{where} \quad 0 \leq \beta \leq 1 \end{aligned}$$

2.4.5 Implementation of MOGA with FMOOP

Genetic Algorithms are general purpose stochastic search algorithms based on the mechanics of natural selection and natural genetics. It has been developed by Prof. John Holland [58], his colleagues and his students at the University of Michigan and later it has been made comfortably accepted by Prof. David Goldberg [46] at the University of Illinois. The original Genetic Algorithm and its many variants, collectively are known as genetic algorithms. **These are computational processes that mimic the natural process of evolution i.e.,it copies the phenomena of biological evolution**. An important observations in the Darwinian evolutionary systems are as follows:

(i) one or more populations of individuals competing for limited resources,

(ii) the notion of dynamically changing populations due to the birth and death of individuals,

(iii) a concept of fitness which reflects the ability of an individuals to survive and reproduce, and

(iv) a concept of variational inheritance: offspring closely resemble their parents, but are not identical.

GAs use two basic procedures from evolution:(i) inheritance, or the flow of features from one generation to the next, and (ii) competition, or survival of the fittest, which results in weeding out the bad features from individuals in the population. All genetic algorithms work on a population, or a collection of several alternative solutions to the given problem. In the population each individual is called a chromosome or string, in analogy to chromosomes in natural systems. In each iteration of GA, a new generation is evolved from the exiting population in an attempt to get solutions. **One of the causes of the success of GAs is their population based strategy which stops them from getting trapped in a local optimal solution and as a result, it increases their**

probability of finding a global optimal solution. Five components of a Genetic Algorithm is as follows:

(i) a genetic representation for potential solutions to the problem,

(ii) a way to generate an initial population of potential solutions,

(iii) an evaluation function that plays in rating of solutions in terms of their fitness,

(iv) genetic operators (crossover, mutation, selection) that change the composition of children,

(v) values for various parameters which are used by the genetic algorithm (probabilities of applying genetic operators, population size, etc.)

Genetic Algorithms are different from more normal optimization and search procedures in four ways:

(i) GAs work with a coding of the parameter set, not the parametrs themselves.

(ii) GAs search from a population of points, not a single point.

(iii) GAs utilize payoff (objective function) information, not derivatives or other auxiliary knowledge.

(iv) GAs operate probabilistic transition rules, not deterministic rules.

The procedure of a generic GA (Goldberg [46], Michalewicz [103] and Srinivas [137]) is described

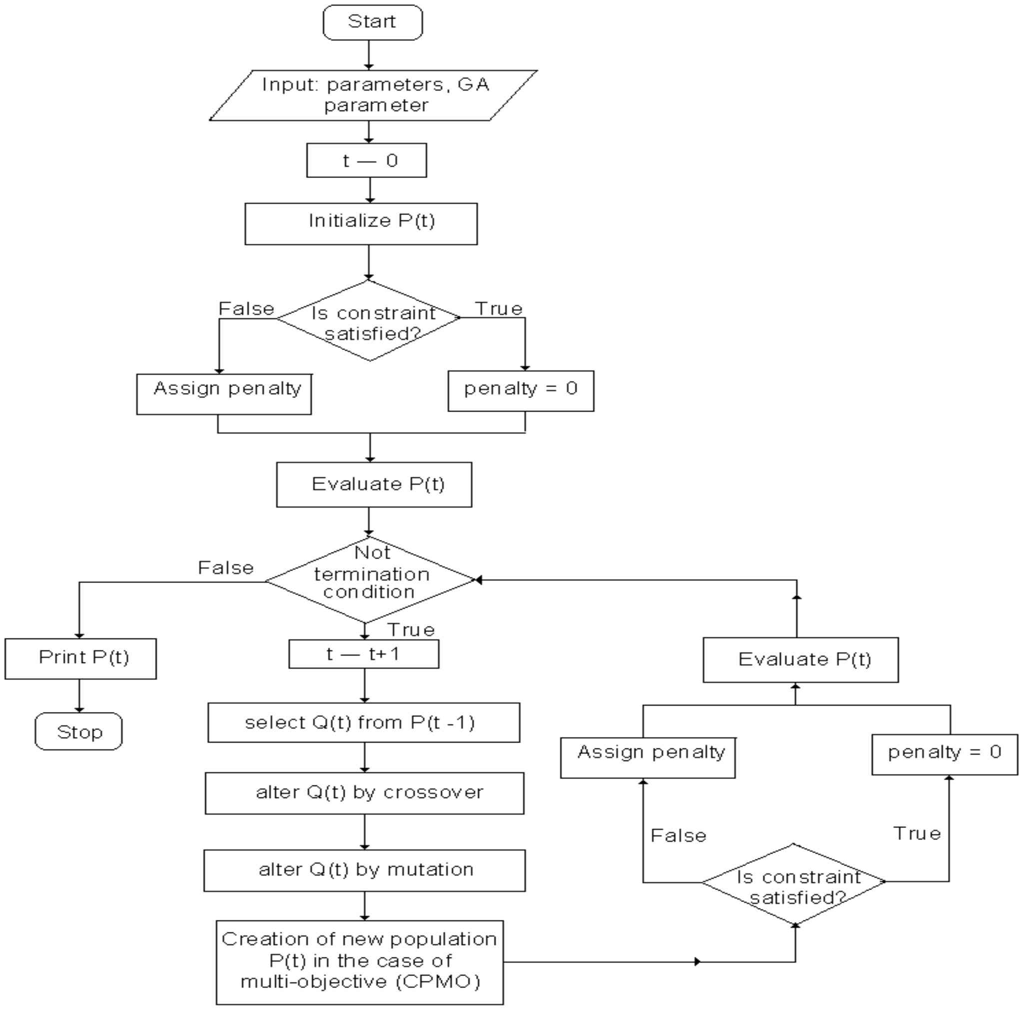

Figure 2.19: Representation of flowchart of MOGA

For the present GA, an overall procedure has been given in Figure 2.19

Using this algorithm,GA optimizes the proposed model. The algorithm of the GAs (cf Michalewicz [103]) is given bellow.

begin

$t \leftarrow 0$

initializc Population(t)

evaluate Population(t)

while(not terminate-condition)

{

$t \leftarrow t + 1$

select Population(t) from Population(t-1)

alter(crossover and mutate) Population(t)

evaluate Population(t)

}

Print Optimum Rcsult

end.

The GA process for fuzzy inferance model is in the following normal form

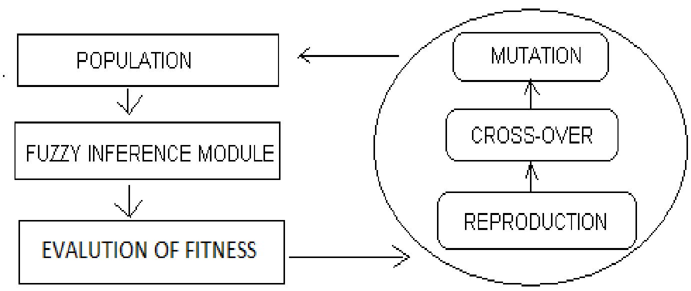

Figure 2.20: GA process for fuzzy inferance model

The basic GA for the fuzzy transportation model is given below

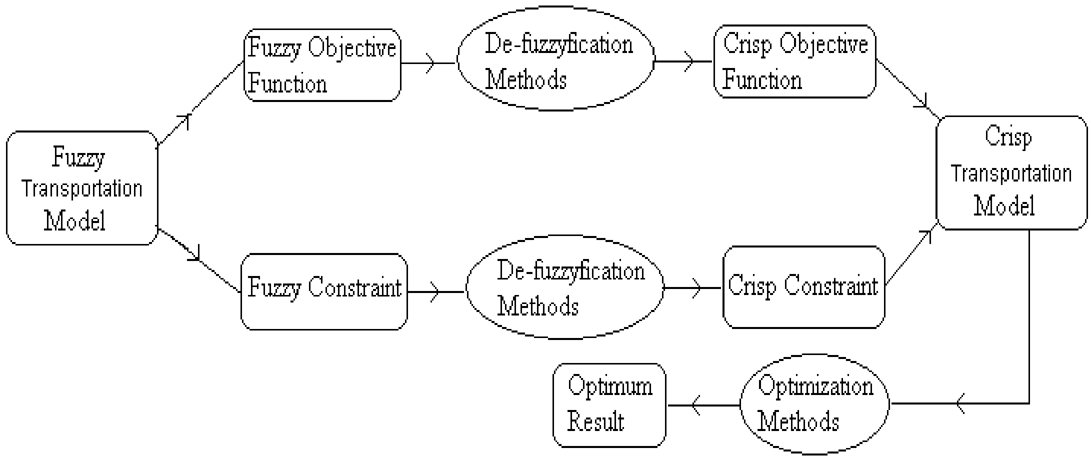

Figure 2.21: Graphical representation of fuzzy transportation model → crisp transportation model