Chapter 4

Doubt intuitionistic fuzzy Sub-implicative ideals in BCI-algebras*

4.1 Introduction

Taking queue from Atanassov's thought, Palaniappan et al. [60] introduced the notions of IF SI-ideals and IF SC-ideals in BCI-algebras. Mostafa [59] established the idea of anti fuzzy SI-ideals in BCI-algebras. Jianming and Zhisong [40] described the notion of DFP-ideals in BCI-algebra.

Solairaju [79] investigated the idea of IF P-ideal including some related features in BCI-algebra.

A FS $M = \{\langle q', \zeta_M(q')\rangle : q' \in V\}$ in V is named as a DF SI-ideal [59] in V if
(i) $\zeta_M(0) \leq \zeta_M(q')$
(ii) $\zeta_M(r' * (r' * q')) \leq \zeta_M(((q' * (q' * r')) * (r' * q')) * s') \bigvee \zeta_M(s')$, for all $q', r', s' \in V$.

A FS $M = \{\langle q', \zeta_M(q')\rangle : q' \in V\}$ in V is identified as a DFP-ideal [40] in V if (i) $\zeta_M(0) \leq \zeta_M(q')$
(ii) $\zeta_M(q') \leq \zeta_M((q' * s') * (r' * s')) \bigvee \zeta_M(r') \forall q', r', s' \in V$.

The objective of this chapter is to define DIF SI-ideals and DIFP-ideals in BCI-algebras and to study its characteristics. The conditions for a DIF-ideal to be a DIF SI-ideals in BCI-algebras are also presented and relations among DIFP-ideals and

*A part of the work presented in this chapter is published in
International Journal of Computational Intelligence Systems, **8**(2) (2015), 240-249.

DIF SI-ideals are studied. Findings of the study concludes that every DIF-ideal in V is not a DIF SI-ideal in V. The conditions are presented for a DIF-ideal to be a DIF SI-ideals in BCI-algebras.

4.2 DIF SI-ideals in BCI-algebras

The current section introduces the concept of DIF SI-ideal in BCI-algebras and studies its properties.

Definition 4.2.1. *Let $M = (\alpha_M, \zeta_M)$ be an IFS of a BCI-algebra V, then M is recognized as* ***DIF SI-ideal*** *in V if*
(i) $\alpha_M(0) \leq \alpha_M(q^{'}), \zeta_M(0) \geq \zeta_M(q^{'})$
(ii) $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq \alpha_M(((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) * s^{'}) \bigvee \alpha_M(s^{'})$
(iii) $\zeta_M(r^{'} * (r^{'} * q^{'})) \geq \zeta_M(((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) * s^{'}) \bigwedge \zeta_M(s^{'})$, for all $q^{'}, r^{'}, s^{'} \in V$.

Theorem 4.2.1. *If a DIF SI-ideal in V meets the inequility $q^{'} \leq s^{'}$ then (i) $\alpha_M(q^{'}) \leq \alpha_M(s^{'})$ and (ii) $\zeta_M(q^{'}) \geq \zeta_M(s^{'})$.*

Proof. Let $q^{'}, r^{'}, s^{'} \in V$ be such that $q^{'} \leq s^{'}$ then $q^{'} * s^{'} = 0$ and since M is a DIF SI-ideal in V, so $\alpha_M(r^{'}*(r^{'}*q^{'})) \leq \alpha_M(((q^{'}*(q^{'}*r^{'}))*(r^{'}*q^{'}))*s^{'}) \bigvee \alpha_M(s^{'})$, when $r^{'} = q^{'}$, then using (A3) and (P5), we get $\alpha_M(q^{'}) \leq \alpha_M(q^{'} * s^{'}) \bigvee \alpha_M(s^{'}) = \alpha_M(0) \bigvee \alpha_M(s^{'}) = \alpha_M(s^{'})$. Therefore, $\alpha_M(q^{'}) \leq \alpha_M(s^{'})$.

Again, $\zeta_M(r^{'}*(r^{'}*q^{'})) \geq \zeta_M(((q^{'}*(q^{'}*r^{'}))*(r^{'}*q^{'}))*s^{'}) \bigwedge \zeta_M(s^{'})$, when $r^{'} = q^{'}$, then using (A3) and (P5), we get $\zeta_M(q^{'}) \geq \zeta_M(q^{'} * s^{'}) \bigwedge \zeta_M(s^{'}) = \zeta_M(0) \bigwedge \zeta_M(s^{'}) = \zeta_M(s^{'})$. Therefore, $\zeta_M(q^{'}) \geq \zeta_M(s^{'})$. Thus the proof ends. □

Proposition 4.2.2. *Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in a BCI-algebra V. Then $\alpha_M(0 * (0 * q^{'})) \leq \alpha_M(q^{'})$ and $\zeta_M(0 * (0 * q^{'})) \geq \zeta_M(q^{'})$, for all $q^{'} \in V$.*

Proof. $\alpha_M(0*(0*q^{'})) \leq \alpha_M(((q^{'}*(q^{'}*0))*(0*q^{'}))*s^{'}) \bigvee \alpha_M(s^{'}) = \alpha_M(((q^{'}*q^{'})*(0*q^{'}))*s^{'}) \bigvee \alpha_M(s^{'}) = \alpha_M((0 * (0 * q^{'})) * s^{'}) \bigvee \alpha_M(s^{'})$. When $s^{'} = q^{'}$ we get, $\alpha_M(0 * (0 * q^{'})) \leq \alpha_M((0 * (0 * q^{'})) * q^{'}) \bigvee \alpha_M(q^{'})$ or, $\alpha_M(0 * (0 * q^{'})) \leq \alpha_M(0) \bigvee \alpha_M(q^{'})$ [by using A2].

Therefore, $\alpha_M(0 * (0 * q^{'})) \leq \alpha_M(q^{'})$, for all $q^{'} \in V$.

Again, $\zeta_M(0*(0*q^{'})) \geq \zeta_M(((q^{'}*(q^{'}*0))*(0*q^{'}))*s^{'}) \bigwedge \zeta_M(s^{'}) = \zeta_M(((q^{'}*q^{'})*(0*q^{'}))*s^{'}) \bigwedge \zeta_M(s^{'}) = \zeta_M((0 * (0 * q^{'})) * s^{'}) \bigwedge \zeta_M(s^{'})$. When $s^{'} = q^{'}$ we get, $\zeta_M(0 * (0 * q^{'})) \leq \zeta_M((0 * (0 * q^{'})) * q^{'}) \bigwedge \zeta_M(q^{'})$ or, $\zeta_M(0 * (0 * q^{'})) \leq \zeta_M(0) \bigwedge \zeta_M(q^{'})$ [by using A2]. Therefore, $\zeta_M(0 * (0 * q^{'})) \geq \zeta_M(q^{'})$, for all $q^{'} \in V$. □

Example 16. *Consider a BCI-algebra $V = \{0, d, e, f\}$ as given in Example3 with table as follows:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS of V defined by

V	0	d	e	f
α_M	0.1	0.4	0.5	0.6
ζ_M	0.8	0.6	0.5	0.4

which is a DIF SI-ideal in V .

Theorem 4.2.3. *Every DIF SI-ideal in V is a DIFSA in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. If $r^{'} = q^{'}$, then from hypothesis(ii) and (iii) in Definition 4.2.1, $\alpha_M(q^{'}) \leq \alpha_M(q^{'} * s^{'}) \bigvee \alpha_M(s^{'})$ and $\zeta_M(q^{'}) \geq \zeta_M(q^{'} * s^{'}) \bigwedge \zeta_M(s^{'}), \forall q^{'}, s^{'} \in V$. Hence it also implies that, $\alpha_M(q^{'} * s^{'}) \leq \alpha_M((q^{'} * s^{'}) * s^{'}) \bigvee \alpha_M(s^{'})$ and $\zeta_M(q^{'} * s^{'}) \geq \zeta_M((q^{'} * s^{'}) * s^{'}) \bigwedge \zeta_M(s^{'})$, for all $q^{'}, r^{'}, s^{'} \in V$. Again, $((q^{'} * s^{'}) * s^{'}) \leq (q^{'} * s^{'}) * (s^{'} * s^{'}) = q^{'} * s^{'} \leq q^{'}$, [by using (P6), (A3), (P5), (P2)]. Hence by Theorem 4.2.1, $\alpha_M((q^{'} * s^{'}) * s^{'}) \leq \alpha_M(q^{'})$.

Thus, $\alpha_M(q^{'} * s^{'}) \leq \alpha_M(q^{'}) \bigvee \alpha_M(s^{'})$ and also, $\zeta_M((q^{'} * s^{'}) * s^{'}) \geq \zeta_M(q^{'})$. So, $\zeta_M(q^{'} * s^{'}) \leq \zeta_M(q^{'}) \bigwedge \zeta_M(s^{'})$. Hence, M is a DIFSA in V. □

Theorem 4.2.4. *Every DIF SI-ideal in V is a DIF-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. If $r^{'} = q^{'}$, then from hypothesis(ii) and (iii) in Definition 4.2.1, $\alpha_M(q^{'}) \leq \alpha_M(q^{'} * s^{'}) \bigvee \alpha_M(s^{'})$ and $\zeta_M(q^{'}) \geq \zeta_M(q^{'} * s^{'}) \bigwedge \zeta_M(s^{'}), \forall q^{'}, s^{'} \in V$.

Hence, M is a DIF-ideal in V. □

Reversly it may not hold. That is every DIF-ideal in V is not a DIF SI-ideal in V. It can be interpreted by the help of example below:

Example 17. *Let us consider the BCI-algebra V as defined in Example 16.*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS of V defined by

V	0	d	e	f
α_M	0	0.5	0.5	0.6
ζ_M	1	0.5	0.5	0.4

Here M is a DIF-ideal in V. But, M is not a DIF SI-ideal in V, as $\alpha_M(e*(e*d)) \nleq max\{\alpha_M(((d*(d*e))*(e*d))*0), \alpha_M(0)\}$. As it implies that, $\alpha_M(d) \leq \alpha_M(0)$, which is a contradiction.

Now a condition for a DIF-ideal in V to be a DIF SI-ideal in V is given here.

Theorem 4.2.5. *If a DIF-ideal in V fulfills the inequalities, $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq \alpha_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))$, and $\zeta_M(r^{'} * (r^{'} * q^{'})) \geq \zeta_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))$, then it becomes a DIF SI-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V satisfying the inequalities, $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq \alpha_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))$, and $\zeta_M(r^{'} * (r^{'} * q^{'})) \geq \zeta_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))$. Now, $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq \alpha_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) \leq \alpha_M(((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) * s^{'}) \bigvee \alpha_M(s^{'})$, and $\zeta_M(r^{'} * (r^{'} * q^{'})) \geq \zeta_M((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) \geq \zeta_M(((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) * s^{'}) \bigwedge \zeta_M(s^{'})$, for all $q^{'}, r^{'}, s^{'} \in V$, [because M is a DIF-ideal]. Hence, M is a DIF SI-ideal in V. Hence the result follows. □

Lemma 4.2.1. *Every DIF-ideal in V becomes a DIF SI-ideal in V, when V is implicative BCI-algebra.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V, where V is an implicative BCI-algebra, then $\alpha_M(q^{'}) \leq max\{\alpha_M(q^{'} * s^{'}), \alpha_M(s^{'})\}$, for all $q^{'}, r^{'}, s^{'} \in V$. So, $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq max\{\alpha_M(r^{'} * ((r^{'} * q^{'})) * s^{'}), \alpha_M(s^{'})\}$, but V is implicative BCI-algebra, then $((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) = (r^{'} * (r^{'} * q^{'}))$. Hence $\alpha_M(r^{'} * (r^{'} * q^{'})) \leq max\{\alpha_M(((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) * s^{'}), \alpha_M(s^{'})\}$. Thus the proof ends. □

Theorem 4.2.6. *If V is implicative BCI-algebra, then an IFS M in V is a DIF-ideal in V if and only if it is an DIF SI-ideal in V.*

Proof. By using Lemma 4.2.1 and Theorem 4.2.3 we can prove it easily. □

Illustrate the Theorem 4.2.5, 4.2.6 and Lemma 4.2.1 by the help of example given below.

EXAMPLE 18. *Let us consider an implicative BCI-algebra $V = \{0, q, r\}$ with the table as follows:*

$*$	0	q	r
0	0	r	q
q	q	0	r
r	r	q	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V as defined by

V	0	q	r
α_M	0	0.8	0.8
ζ_M	1	0.2	0.2

Hence, M is a DIF-ideal as well as DIF SI-ideal in V.

Theorem 4.2.7. *Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. Then, so is $\bigoplus M = \{\langle q', \alpha_M(q'), \bar{\alpha}_M(q')\rangle / q' \in V\}$.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V, then $\alpha_M(0) \leq \alpha_M(q')$ and $\alpha_M(r' * (r' * q')) \leq max\{\alpha_M(((q' * (q' * r')) * (r' * q')) * s'), \alpha_M(s')\}$. Now, $\alpha_M(0) \leq \alpha_M(q')$, or $1 - \bar{\alpha}_M(0) \leq 1 - \bar{\alpha}_M(q')$, or $\bar{\alpha}_M(0) \geq \bar{\alpha}_M(q')$, for any $q' \in V$. Now for any $q', r', s' \in V$, $\alpha_M(r' * (r' * q')) \leq max\{\alpha_M(((q' * (q' * r')) * (r' * q')) * s'), \alpha_M(s')\}$. This gives, $1 - \bar{\alpha}_M(r' * (r' * q')) \leq max\{1 - \bar{\alpha}_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \bar{\alpha}_M(s')\}$ or, $\bar{\alpha}_M(r' * (r' * q')) \geq 1 - max\{1 - \bar{\alpha}_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \bar{\alpha}_M(s')\}$. Finally, $\bar{\alpha}_M(r' * (r' * q')) \geq min\{\bar{\alpha}_M(((q' * (q' * r')) * (r' * q')) * s'), \bar{\alpha}_M(s')\}$. Hence, $\bigoplus M = \{(q', \alpha_M(q'), \bar{\alpha}_M(q'))/q' \in V\}$ is a DIF SI-ideal in V. □

Theorem 4.2.8. *Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. Then so is $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in V\}$.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V, then $\zeta_M(0) \geq \zeta_M(q')$.
Also, $\zeta_M(r' * (r' * q')) \geq min\{\zeta_M(((q' * (q' * r')) * (r' * q')) * s'), \zeta_M(s')\}$.

Again, we have, $\zeta_M(0) \geq \zeta_M(q')$, or $1 - \bar{\zeta}_M(0) \geq 1 - \bar{\zeta}_M(q')$, or $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(x)$, for any $q' \in V$. Also for any $q', r', s' \in V$, $\zeta_M(r' * (r' * q')) \geq min\{\zeta_M(((q' * (q' * r')) * (r' * q')) * s'), \zeta_M(s')\}$.

This implies, $1 - \bar{\zeta}_M(r' * (r' * q') \geq min\{1 - \bar{\zeta}_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \bar{\zeta}_M(s')\}$. That is, $\bar{\zeta}_M(r' * (r' * q') \leq 1 - min\{1 - \bar{\zeta}_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \bar{\zeta}_M(s')\}$ or, $\bar{\zeta}_M(r' * (r' * q') \leq max\{\bar{\zeta}_M(((q' * (q' * r')) * (r' * q')) * s'), \bar{\zeta}_M(s')\}$. Hence, $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in V\}$ is a DIF SI-ideal in V. □

Theorem 4.2.9. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in V. Then $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V if and only if $\bigoplus M = \{\langle q', \alpha_M(q'), \bar{\alpha}_M(q')\rangle / q' \in V\}$ and $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in V\}$ are DIF SI-ideals in V.*

Proof. The proof follows the same route that was used in Theorem 4.2.7 and Theorem 4.2.8. □

The example provided below supports the Theorem 4.2.7, 4.2.8 and 4.2.9.

Example 19. *Let us consider a BCI-algebra $V = \{0, s, t, u\}$ as given by below tabulated form:*

$*$	0	s	t	u
0	0	0	0	u
s	s	0	0	u
t	t	t	0	u
u	u	u	u	0

Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V defined by

V	0	s	t	u
α_M	0	0.3	0.5	0.6
ζ_M	0.8	0.6	0.5	0.4

Then $\bigoplus M = \{\langle q', \alpha_M(q'), \bar{\alpha}_M(q')\rangle / q' \in V\}$, where $\alpha_M(q')$ and $\bar{\alpha}_M(q')$ are defined as follows:

V	0	s	t	u
α_M	0	0.3	0.5	0.6
$\bar{\alpha}_M$	1	0.7	0.5	0.4

Also $\bigotimes M = \{\langle q', \bar{\zeta}_M(q'), \zeta_M(q')\rangle / q' \in V\}$, *whose* $\zeta_M(q')$ *and* $\bar{\zeta}_M(q')$ *are defined by*

V	0	s	t	u
$\bar{\zeta}_M$	0.2	0.4	0.5	0.6
ζ_M	0.8	0.6	0.5	0.4

So, it can be verified that $\bigoplus M$ *and* $\bigotimes M$ *are DIF SI-ideals 0f* V*.*

Theorem 4.2.10. *An IFS* $M = (\alpha_M, \zeta_M)$ *is a DIF SI-ideal in a BCI-algebra* V *if and only if the DIVFs* α_M *and* $\bar{\zeta}_M$ *are DF SI-ideals in* V*.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. Then it is obvious that α_M is a DF SI-ideals in V, and from Theorem 4.2.8, we can prove that $\bar{\zeta}_M$ is a DF SI-ideals in V.

Conversely, let α_M be a DF SI-ideals in V. Therefore $\alpha_M(0) \leq \alpha_M(q')$ and $\alpha_M(r' * (r' * q')) \leq max\{\alpha_M(((q' * (q' * r')) * (r' * q')) * s'), \alpha_M(s')\}$, for all $q', r', s' \in V$. Again, let $\bar{\zeta}_M$ is a DF SI-ideals in V, so, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(q')$, gives $1 - \zeta_M(0) \leq 1 - \zeta_M(q')$, implies $\zeta_M(0) \geq \zeta_M(q')$.

Also, $\bar{\zeta}_M(r' * (r' * q')) \leq max\{\bar{\zeta}_M(((q' * (q' * r')) * (r' * q')) * s'), \bar{\zeta}_M(s')\}$ or, $1 - \zeta_M(r' * (r' * q')) \leq max\{1 - \zeta_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \zeta_M(s')\}$ or, $\zeta_M(r' * (r' * q')) \geq 1 - max\{1 - \zeta_M(((q' * (q' * r')) * (r' * q')) * s'), 1 - \zeta_M(s')\}$. Finally, $\zeta_M(r' * (r' * q')) \geq min\{\zeta_M(((q' * (q' * r')) * (r' * q')) * s'), \zeta_M(s')\}$, for all $q', r', s' \in V$. Hence, $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V. □

Corollary 4.2.1. *The sets,* $D_{\alpha_M} = \{q' \in V / \alpha_M(q') = \alpha_M(0)\}$ *and* $D_{\zeta_M} = \{q' \in V / \zeta_M(q') = \zeta_M(0)\}$ *are SI-ideals in* V*, when* $M = (\alpha_M, \zeta_M)$ *is a DIF SI-ideal in* V*.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V. Obviously, $0 \in D_{\alpha_M}$ and D_{ζ_M}. Now, let $q', r', s' \in V$, such that $(((q' * (q' * r')) * (r' * q')) * s') \in D_{\alpha_M}$, $s' \in D_{\alpha_M}$. Then $\alpha_M(((q' * (q' * r')) * (r' * q')) * s') = \alpha_M(0) = \alpha_M(s')$. Now, $\alpha_M(r' * (r' * q')) \leq max\{\alpha_M(((q' * (q' * r')) * (r' * q')) * s'), \alpha_M(s')\} = \alpha_M(0)$.

Again, since α_M is a DF SI-ideals in V, $\alpha_M(0) \leq \alpha_M(r' * (r' * q'))$. Therefore, $\alpha_M(0) = \alpha_M(r' * (r' * q'))$. which shows that, $(r' * (r' * q')) \in D_{\alpha_M}$, for all $q', r' \in V$. Therefore, D_{α_M} is a SI-ideal in V.

Also, let $q', r', s' \in V$, such that $(((q' * (q' * r')) * (r' * q')) * s') \in D_{\zeta_M}$, $s' \in D_{\zeta_M}$. Then $\zeta_M(((q' * (q' * r')) * (r' * q')) * s') = \zeta_M(0) = \zeta_M(s')$. Now, $\zeta_M(r' * (r' * q')) \geq min\{\zeta_M(((q' * (q' * r')) * (r' * q')) * s'), \zeta_M(s')\} = \zeta_M(0)$.

Again, since $\bar{\zeta}_M$ is a DF SI-ideals in V, $\zeta_M(0) \geq \zeta_M(r' * (r' * q'))$. Therefore, $\zeta_M(0) = \zeta_M(r' * (r' * q'))$. So, $(r' * (r' * q')) \in D_{\zeta_M}$, for all $q', r' \in V$. Therefore, D_{ζ_M} is a SI-ideal in V. □

Definition 4.2.2. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in V, and $c, d \in [0, 1]$, then UC of level c and LC of level d of M, is as followes:*

$$\alpha^{\leq}_{M,c} = \{q' \in V / \alpha_M(q') \leq c\}$$
$$and \;\; \zeta^{\geq}_{M,d} = \{q' \in V / \zeta_M(q') \geq d\}.$$

Theorem 4.2.11. *If $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V, then $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are SI-ideals in V for any $c, d \in [0, 1]$.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V, and let $c \in [0, 1]$ with $\alpha_M(0) \leq c$. Also we have, $\alpha_M(0) \leq \alpha_M(q')$, for all $q' \in V$, but $\alpha_M(q') \leq c$, for all $q' \in \alpha^{\leq}_{M,c}$. So, $0 \in \alpha^{\leq}_{M,cc}$. Let $q', r', s' \in V$ with $(((q' * (q' * r')) * (r' * q')) * s') \in \alpha^{\leq}_{M,c}$ and $s' \in \alpha^{\leq}_{M,c}$, then, $\alpha_M(((q' * (q' * r')) * (r' * q')) * s') \in \alpha^{\leq}_{M,c}$ and $\alpha_M(s') \in \alpha^{\leq}_{M,c}$. Therefore, $\alpha_M(((q' * (q' * r')) * (r' * q')) * s') \leq c$ and $\alpha_M(s') \leq c$. Since α_M is a DF SI-ideals in V, it follows that, $\alpha_M(r' * (r' * q')) \leq \alpha_M(((q' * (q' * r')) * (r' * q')) * s') \bigvee \alpha_M(s') \leq c$ and hence $(r' * (r' * q')) \in \alpha^{\leq}_{M,c}$, for all $q', r', s' \in V$. Therefore, $\alpha^{\leq}_{M,c}$ is a SI-ideal in V for $c \in [0, 1]$. In such way, it also proved that $\zeta^{\geq}_{M,d}$ is a SI-ideal in V for $d \in [0, 1]$. □

Theorem 4.2.12. *If $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are either empty or SI-ideals in V for $c, d \in [0, 1]$, then $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V.*

Proof. Let $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ be either empty or SI-ideals in V for $c, d \in [0, 1]$. For any $q' \in V$, let $\alpha_M(q') = c$ and $\zeta_M(q') = d$. Then $q' \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$, so $\alpha^{\leq}_{M,c} \neq \phi \neq \zeta^{\geq}_{M,d}$. Since $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are SI-ideals of V, therefore $0 \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$. Hence, $\alpha_M(0) \leq c = \alpha_M(q')$ and $\zeta_M(0) \geq d = \zeta_M(q')$, where $q' \in V$. If there exist $k_1, k_2, k_3 \in V$ such that $\alpha_M(k_2 * (k_2 * k_1)) > max\{\alpha_M(((k_1 * (k_1 * k_2)) * (k_2 * k_1)) * k_3), \alpha_M(k_3)\}$, then by taking,

$c_0 = \frac{1}{2}(\alpha_M(k_2*(k_2*k_1))+max\{\alpha_M(((k_1*(k_1*k_2))*(k_2*k_1))*k_3), \alpha_M(k_3)\})$. We have, $\alpha_M(k_2 * (k_2 * k_1) > c_0 > max\{\alpha_M(((k_1 * (k_1 * k_2)) * (k_2 * k_1)) * k_3), \alpha_M(k_3)\}$. Hence, $k_2 * (k_2 * k_1) \notin \alpha^{\leq}_{M,c_0}, (((k_1 * (k_1 * k_2)) * (k_2 * k_1)) * k_3) \in \alpha^{\leq}_{M,c_0}$ and $k_3 \in \alpha^{\leq}_{M,c_0}$, that is $\alpha^{\leq}_{M,c_0}$ is not a SI-ideal in V, which is a contradiction. Therefore, $\alpha_M(r' * (r' * q')) \leq \alpha_M(((q' * (q' * r')) * (r' * q')) * s') \bigvee \alpha_M(s')$, for some $q', r', s' \in V$.

$\zeta_M(r' * (r' * q')) \geq \zeta_M(((q' * (q' * r')) * (r' * q')) * s') \bigwedge \zeta_M(s')$, for some $q', r', s' \in V$.

Hence, $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V. □

4.3 DIFP-ideal in BCI-algebras

In this section, we define DIF P-ideal in BCI-algebras and investigate its properties.

Definition 4.3.1. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in a BCI-algebra V, then M is identified as **DIFP-ideal** in V if (i) $\alpha_M(0) \leq \alpha_M(q'), \zeta_M(0) \geq \zeta_M(q')$*
(ii) $\alpha_M(q') \leq max\{\alpha_M((q' * s') * (r' * s')), \alpha_M(r')\}$
(iii) $\zeta_M(q') \geq min\{\zeta_M((q' * s') * (r' * s')), \zeta_M(r')\}$, for all $q', r', s' \in V$

EXAMPLE 20. *Let us consider a BCI-algebra $V = \{0, r, s, t\}$ as presented in the table below:*

$*$	0	r	s	t
0	0	0	t	s
r	r	0	t	s
s	s	s	0	t
t	t	t	s	0

Now let consider a DIFS $M = (\alpha_M, \zeta_M)$ in V as follows:

V	0	r	s	t
α_M	0	0.5	0.6	0.6
ζ_M	1	0.5	0.4	0.4

Then $M = (\alpha_M, \zeta_M)$ be a DIFP-ideal in V.

Theorem 4.3.1. *Every DIFP-ideal in V is a DIF-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFP-ideal in V, then (i) $\alpha_M(0) \leq \alpha_M(q')$; $\zeta_M(0) \geq \zeta_M(q')$, (ii) $\alpha_M(q') \leq max\{\alpha_M((q' * s') * (r' * s')), \alpha_M(r')\}$
and (iii) $\zeta_M(q') \geq min\{\zeta_M((q' * s') * (r' * s')), \zeta_M(r')\}, \forall q', r', s' \in V$. If we put $s' = 0$,

then from hypothesis(ii) and (iii), we get, $\alpha_M(q^{'}) \leq \alpha_M((q^{'} * 0) * (r^{'} * 0)) \bigvee \alpha_M(r^{'})$ and $\zeta_M(q^{'}) \geq \zeta_M((q^{'} * 0) * (r^{'} * 0)) \bigwedge \zeta_M(r^{'}), \forall q^{'}, r^{'} \in V$. Hence, every DIFP-ideal in V satisfies the inequalities: $\alpha_M(q^{'}) \leq \alpha_M(q^{'} * r^{'}) \bigvee \alpha_M(r^{'})$ and $\zeta_M(q^{'}) \geq \zeta_M(q^{'} * r^{'}) \bigwedge \zeta_M(r^{'})$, for all $q^{'}, r^{'} \in V$. Hence, M is a DIF-ideal in V. □

Theorem 4.3.1 may not hold in reverse direction in general, the below given example proves this fact.

EXAMPLE 21. *Consider the BCI-algebra V that was taken in Example 19:*

$*$	0	s	t	u
0	0	0	0	u
s	s	0	0	u
t	t	t	0	u
u	u	u	u	0

Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V defined by

V	0	s	t	u
α_M	0	0.3	0.4	0.5
ζ_M	1	0.7	0.6	0.5

But $M = (\alpha_M, \zeta_M)$ is not a DIFP-ideal in V, since $\alpha_M(t) = 0.4$ and $max(\alpha_M((t * u) * (s * u)), \alpha_M(s)) = \alpha_M(s) = 0.3$, that implies $\alpha_M(t) \nleq max(\alpha_M((t * u) * (s * u)), \alpha_M(s))$

Now let us uphold a new condition for the IFS $M = (\alpha_M, \zeta_M)$, which is a DIF-ideal in V to be a DIFP-ideal in V.

Proposition 4.3.2. *A DIF-ideal in a BCI-algebra V becomes a DIFP-ideal if the below stated postulates meet.*

(i) $\alpha_M(q^{'} * r^{'}) \leq \alpha_A((q^{'} * s^{'}) * (r^{'} * s^{'}))$ and (ii) $\zeta_M(q^{'} * r^{'}) \geq \alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \forall q^{'}, r^{'}, s^{'} \in V$.

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V satisfying (i) $\alpha_M(q^{'} * r^{'}) \leq \alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'}))$ and (ii) $\zeta_M(q^{'} * r^{'}) \geq \alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \forall q^{'}, r^{'}, s^{'} \in V$. Then $\alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})) \bigvee \alpha_M(r^{'}) \geq \alpha_M(q^{'} * r^{'}) \bigvee \alpha_M(r^{'}) \geq \alpha_M(q^{'})$. Again, $\zeta_M((q^{'} * s^{'}) * (r^{'} * s^{'})) \bigwedge \zeta_M(r^{'}) \leq \zeta_M(q^{'} * r^{'}) \bigwedge \zeta_M(r^{'}) \leq \zeta_M(q^{'})$. In this way, the proof ends. □

Proposition 4.3.3. *For a DIFP-ideal $M = (\alpha_M, \zeta_M)$ in a BCI-algebra V, $\alpha_M(q^{'}) \leq \alpha_M(0 * (0 * q^{'}))$ and $\zeta_M(q^{'}) \geq \zeta_M(0 * (0 * q^{'}))$, for all $q^{'} \in V$.*

Proof. It is straightforward □

Corollary 4.3.1. *The sets, $D_{\alpha_M} = \{q^{'} \in V/\alpha_M(q^{'}) = \alpha_M(0)\}$ and $D_{\zeta_M} = \{q^{'} \in V/\zeta_M(q^{'}) = \zeta_M(0)\}$ are P-ideals in V when $M = (\alpha_M, \zeta_M)$ is a DIFP-ideal of that BCI-algebra V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFP-ideal in V. Obviously, $0 \in D_{\alpha_M}$ and D_{ζ_M}. Now, assume $q^{'}, r^{'}, s^{'} \in V$, so that $(q^{'} * s^{'}) * (r^{'} * s^{'}) \in D_{\alpha_M}$, and $r^{'} \in D_{\alpha_M}$. Then $\alpha_M(q^{'}) \leq max\{\alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \alpha_M(r^{'})\} = \alpha_M(0)$. But $\alpha_M(0) \leq \alpha_M(q^{'})$, for all $q^{'} \in V$. Therefore, $\alpha_M(0) = \alpha_M(q^{'})$. So, $q^{'} \in D_{\alpha_M}$, for all $q^{'}, r^{'}, s^{'} \in V$. Therefore, D_{α_M} is a P-ideal in V.

Also, let $q^{'}, r^{'}, s^{'} \in V$, such that $(q^{'} * s^{'}) * (r^{'} * s^{'}) \in D_{\zeta_M}$, and $r^{'} \in D_{\zeta_M}$. Then $\zeta_M(q^{'}) \geq max\{\zeta_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \zeta_M(r^{'})\} = \zeta_M(0)$. But, $\zeta_M(0) \geq \zeta_M(q^{'})$ for all $q^{'} \in V$. Therefore, $\zeta_M(0) = \zeta_M(q^{'})$. It follows that, $q^{'} \in D_{\zeta_M}$, for all $q^{'}, r^{'}, s^{'} \in V$. Therefore, D_{ζ_M} is a P-ideal in V. □

Theorem 4.3.4. *Every DIFP-ideal in V is a DIF SI-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIFP-ideal in V.

Now,

$$
\begin{aligned}
&(0 * (0 * (r^{'} * (r^{'} * q^{'})))) * ((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'})) \\
=&(0 * ((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))) * (0 * (r^{'} * (r^{'} * q^{'})))[\text{by P1}] \\
=&((0 * (q^{'} * (q^{'} * r^{'}))) * (0 * (r^{'} * q^{'}))) * ((0 * r^{'}) * (0 * (r^{'} * q^{'})[\text{ by P6 }] \\
=&(((0 * q^{'}) * (0 * (q^{'} * r^{'}))) * (0 * (r^{'} * q^{'}))) * ((0 * r^{'}) * (0 * (r^{'} * q^{'}))) \\
\leq&((0 * q^{'}) * (0 * (q^{'} * r^{'}))) * (0 * r^{'})[\text{ by P3}] \\
=&((0 * q^{'}) * (0 * r^{'})) * (0 * (q^{'} * r^{'}))[\text{ by P1}] \\
=&(0 * (q^{'} * r^{'})) * (0 * (q^{'} * r^{'}))[\text{ by P6}] \\
=&0[\text{ by A3}]
\end{aligned}
$$

Hence, $(0 * (0 * (r^{'} * (r^{'} * q^{'})))) \leq ((q^{'} * (q^{'} * r^{'})) * (r^{'} * q^{'}))$.

Since, M is a DIF-ideal, then, $\alpha_M(0*(0*(r'*(r'*q')))) \leq \alpha_M((q'*(q'*r'))*(r'*q'))$ and $\zeta_M(0*(0*(r'*(r'*q')))) \geq \zeta_M((q'*(q'*r'))*(r'*q'))$.
But by Proposition 4.3.3, we have $\alpha_M(r'*(r'*q')) \leq \alpha_M(0*(0*(r'*(r'*q'))))$ and $\zeta_M(r'*(r'*q')) \geq \zeta_M(0*(0*(r'*(r'*q'))))$. Hence, $\alpha_M(r'*(r'*q')) \leq \alpha_M((q'*(q'*r'))*(r'*q'))$ and $\zeta_M(r'*(r'*q')) \geq \zeta_M((q'*(q'*r'))*(r'*q'))$. By Theorem 4.2.5, we see that $M = (\alpha_M, \zeta_M)$ is a DIF SI-ideal in V.

But, the reverse of Theorem 4.3.4 may not be true, which is illustrated by Example 19. As, $\alpha_M(t) \nleq \alpha_M((t*u)*(s*u)) \bigvee \alpha_M(s)$. □

Theorem 4.3.5. *Union of any two DIFP-ideals in V, is also a DIFP-ideal in V if one is contained in another.*

Proof. Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two DIFP-ideals in V. Again let, $C = M \cup N = (\alpha_C, \zeta_C)$, where $\alpha_C = \alpha_M \vee \alpha_N$ and $\zeta_C = \zeta_M \wedge \zeta_N$. Let $q', r', s' \in V$, then, $\alpha_C(0) = max\{\alpha_M(0), \alpha_N(0)\} \leq max\{\alpha_M(q'), \alpha_N(q')\} = \alpha_C(q')$ and $\zeta_C(0) = min\{\zeta_M(0), \zeta_N(0)\} \geq min\{\zeta_M(q'), \zeta_N(q')\} = \zeta_C(q')$, for all $q' \in V$.

Also,

$$\begin{aligned}
\alpha_C(q') &= max\{\alpha_M(q'), \alpha_N(q')\} \\
&\leq max\{max[\alpha_M((q'*s')*(r'*s')), \alpha_M(r')], max[\alpha_N((q'*s')*(r'*s')), \alpha_N(r')]\} \\
&= max\{max[\alpha_M((q'*s')*(r'*s')), \alpha_N((q'*s')*(r'*s'))], max[\alpha_M(r'), \alpha_N(r')]\} \\
&= max[\alpha_C((q'*s')*(r'*s')), \alpha_C(r')].
\end{aligned}$$

Similarly, it can verify that, $\zeta_C(q') \geq min[\zeta_C((q'*s')*(r'*s')), \zeta_C(r')]$.
In this way, proof ends. □

Theorem 4.3.6. *Let M and N be two IFSs in V, such that one is subset of other. Also M and N are two DIFP-ideals in V. Then $M \cap N$ is also a DIFP-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two DIFP-ideals in V. Again let, $D = M \cap N = (\alpha_D, \zeta_D)$, where $\alpha_D = min\{\alpha_M, \alpha_N\}$ and $\zeta_D = max\{\zeta_M, \zeta_N\}$. Let $q' \in V$, then $\alpha_D(0) = min\{\alpha_M(0), \alpha_N(0)\} \leq min\{\alpha_M(q'), \alpha_N(q')\} = \alpha_D(q')$ and $\zeta_D(0) = max\{\zeta_M(0), \zeta_N(0)\} \geq max\{\zeta_M(q'), \zeta_N(q')\} = \zeta_D(q')$.

Also, for $q^{'}, r^{'}, s^{'} \in V$

$$\begin{aligned}
\alpha_D(q^{'}) &= min\{\alpha_M(q^{'}), \alpha_N(q^{'})\} \\
&\leq min[max\{\alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \alpha_M(r^{'})\}, max\{\alpha_N((q^{'} * s^{'}) * (r^{'} * s^{'})), \alpha_N(r^{'})\}] \\
&= max[\min\{\alpha_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \alpha_N((q^{'} * s^{'}) * (r^{'} * s^{'}))\}, min\{\alpha_M(r^{'}), \alpha_N(r^{'})\}], \\
&\quad \text{[because one is contained in another]} \\
&= max[\alpha_D((q^{'} * s^{'}) * (r^{'} * s^{'})), \alpha_D(r^{'})].
\end{aligned}$$

Again,

$$\begin{aligned}
\zeta_D(q^{'}) &= max\{\zeta_M(q^{'}), \zeta_N(q^{'})\} \\
&\geq max[min\{\zeta_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \zeta_M(r^{'})\}, min\{\zeta_N((q^{'} * s^{'}) * (r^{'} * s^{'})), \zeta_N(r^{'})\}] \\
&= min[\max\{\zeta_M((q^{'} * s^{'}) * (r^{'} * s^{'})), \zeta_N((q^{'} * s^{'}) * (r^{'} * s^{'}))\}, max\{\zeta_M(r^{'}), \zeta_N(r^{'})\}], \\
&\quad \text{[because one is contained in another]} \\
&= min[\zeta_D((q^{'} * s^{'}) * (r^{'} * s^{'})), \zeta_D(r^{'})].
\end{aligned}$$

Thus the proof ends. □

Now the Theorem 4.3.5 and Theorem 4.3.6 are verified by the following example.

EXAMPLE 22. *Consider a BCI-algebra that was given in Example 19 as follows:*

$*$	0	s	t	u
0	0	0	0	u
s	s	0	0	u
t	t	t	0	u
u	u	u	u	0

Let $M = (\alpha_M, \zeta_M)$ be a DIF SI-ideal in V defined by

V	0	s	t	u
α_M	0	0.7	0.7	0.8
ζ_M	1	0.3	0.3	0.2

Then $M = (\alpha_M, \zeta_M)$ is a DIFP-ideal in V.

Also, let $N = (\alpha_N, \zeta_N)$ be an IFS in V as defined by

V	0	s	t	u
α_N	0	0.4	0.4	0.5
ζ_N	1	0.6	0.6	0.5

Then $N = (\alpha_N, \zeta_N)$ *is a DIFP-ideal in* V*.*

Again assume that $P = M \cup N = (\alpha_P, \zeta_P)$*, where* $\alpha_P = \alpha_M \vee \alpha_N$ *and* $\zeta_P = \zeta_M \wedge \zeta_N$ *and* P *is interpreted as:*

V	0	s	t	u
α_P	0	0.7	0.7	0.8
ζ_P	1	0.3	0.3	0.2

Then $P = (\alpha_P, \zeta_P)$ *is a DIFP-ideal in* V*.*

Now let, $Q = M \cap N = (\alpha_Q, \zeta_Q)$ *where* $\alpha_Q = \alpha_M \wedge \alpha_N$ *and* $\zeta_Q = \zeta_M \vee \zeta_N$*.*

Then the IFS Q *is represented by:*

V	0	s	t	u
α_Q	0	0.4	0.4	0.5
ζ_Q	1	0.6	0.6	0.5

Then $Q = (\alpha_Q, \zeta_Q)$ *is also a DIFP-ideal in* V*.*

4.4 Summary

The notion of DIF SI-ideals and DIFP-ideals in BCI-algebras are introduced in current chapter. Here it is shown that any DIFP-ideal is always a DIF SI-ideal. We also examplifies that a DIF SI-ideal may not always be DIFP-ideal. Besides, the chapter also contains some other properties about DIF SI-ideals and DIFP-ideals in BCI-algebras.