Chapter 2

Doubt intuitionistic fuzzy ideals *

2.1 Introduction

The notion of uncertainty has drawn a havoc attention of the reasearchers during the last few decades and as a result its concept and definition have gone through a frequent modification during this time making it a much discussed term in science and mathematics having a vast sphere of utility. In its dazzling evolution, Zadeh [86], played the role of an avant-garde. Following him several sebsequent researchers worked in this field among them Atanassov's [2] name must be mentioned who propounded the idea of IFSs as an extension and generalisation of the notion of FSs.

Xi [84], in 1991, introduced concept of FS in BCK-algebras. Then in 1992, Huang gave another notion of fuzzy set in BCI-algebras. Following the same rout in 1994, Jun [43] established the definition of DFSA and DF-ideals in BCK/BCI-algebras to avoid the confusion created in [27] Huang's defination of fuzzy BCI-algebras by means of some effective results.

In 2000, Jun and Kim [42] explored the idea of IFSA and IF-ideals in BCK-algebras.

This chapter has inside the concept of DIFSAs and DIF-ideals in BCK/BCI-algebras. Findings of this chapter show that an IFS of BCK/BCI-algebra is DIFSA and DIF-ideal if and only if the complement of this IFS is an IFSA and an IF-ideal. And at the same time we are establishing some common properties related to them.

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*Part of the work presented in this chapter is published in *International Journal of Fuzzy Logic Systems*, **5**(1) 1-13 (2015).

2.2 DIFSA in BCK/BCI-algebras

In this section we have defined DIFSA and thereafter moved on investigating its several properties with the help of some certain examples.

Definition 2.2.1. *Let $M = (\alpha_M, \zeta_M)$ be an IFS in V, then M is called a* ***DIFSA*** *in V if*
(i) $\alpha_M(v_1 * v_2) \leq \alpha_M(v_1) \bigvee \alpha_M(v_2)$, *and*
(ii) $\zeta_M(v_1 * v_2) \geq \zeta_M(v_1) \bigwedge \zeta_M(v_2), \forall v_1, v_2 \in V$.

Theorem 2.2.1. *Let $M = (\alpha_M, \zeta_M)$ is a DIFSA in V, then*
(i) $\alpha_M(0) \leq \alpha_M(v_1)$, *and*
(ii) $\zeta_M(0) \geq \zeta_M(v_1), \forall v_1 \in V$.

Proof. We have that $\alpha_M(0) = \alpha_M(v_1 * v_1) \leq \alpha_M(v_1) \bigvee \alpha_M(v_1) \leq \alpha_M(v_1)$.
$\therefore \alpha_M(0) \leq \alpha_M(v_1), \forall v_1 \in V$.
Again we have $\zeta_M(0) = \zeta_M(v_1 * v_1) \geq \zeta_M(v_1) \bigwedge \zeta_M(v_1) \geq \zeta_M(v_1)$.
$\therefore \zeta_M(0) \geq \zeta_M(v_1), \forall v_1 \in V$. ☐

Theorem 2.2.2. *Let $M = (\alpha_M, \zeta_M)$ be a DIFSA in V. Then for any $v_1 \in V$, we have*

$$\alpha_M(v_1^m * v_1) \begin{cases} \leq \alpha_M(v_1), & \textit{if } m \textit{ is odd} \\ = \alpha_M(v_1), & \textit{if } m \textit{ is even.} \end{cases}$$

and

$$\zeta_M(v_1^m * v_1) \begin{cases} \geq \zeta_M(v_1), & \textit{if } m \textit{ is odd} \\ = \zeta_M(v_1), & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. Let $v_1 \in V$, then $\alpha_M(v_1 * v_1) = \alpha_M(0) \leq \alpha_M(v_1)$.
Let m is odd, and $m = 2q - 1$, where q is a positive integer.
Now assume that $\alpha_M(v_1^{2q-1} * v_1) \leq \alpha_M(v_1)$ for some positive integer q.

Then, $\alpha_M(v_1^{2q-1} * v_1) = \alpha_M(v_1^{2q+1} * v_1)$.

$$\begin{aligned}
\alpha_M(v_1^{2(q+1)-1} * v_1) &= \alpha_M(v_1^{2q+1} * v_1), \\
&= \alpha_M(v_1^{2q-1} * (v_1 * (v_1 * v_1))) \\
&= \alpha_M(v_1^{2q-1} * (v_1 * 0)) \\
&= \alpha_M(v_1^{2q-1} * v_1) \\
&\leq \alpha_M(v_1)
\end{aligned}$$

Hence, $\alpha_M(v_1^m * v_1) \leq \alpha_M(v_1)$, if m is odd.

Again, let m is even, and $m = 2q$.

Now for $q = 1, \alpha_M(v_1^2 * v_1) = \alpha_M(v_1 * (v_1 * v_1)) = \alpha_M(v_1 * 0) = \alpha_M(v_1)$.

Also assume that, $\alpha_M(v_1^{2q} * v_1) = \alpha_M(v_1)$ for some positive integer q, then,

$$\begin{aligned}
\alpha_M(v_1^{2(q+1)} * v_1) &= \alpha_M(v_1^{2q} * (v_1 * (v_1 * v_1))), \\
&= \alpha_M(v_1^{2q} * v_1) \\
&= \alpha_M(v_1)
\end{aligned}$$

Hence, $\alpha_M(v_1^m * v_1) = \alpha_M(v_1)$, if m is even.

This proves the first part. similarly we can prove the second part. □

Theorem 2.2.3. *Let $M = (\alpha_M, \zeta_M)$ be a DIFSA in V. Then for any $v_1 \in V$, we have*
$\alpha_M(v_1 * v_1^m) = \alpha_M(0)$.
and $\zeta_M(v_1 * v_1^m) = \zeta_M(0)$.
for $m = 1, 2, 3, \ldots$

Proof: Straightforward

EXAMPLE 3. *Let a BCK-algebra $V = \{0, d, e, f\}$ be given by the table below:*

$*$	0	d	e	f
0	0	0	0	0
d	d	0	0	d
e	e	d	0	e
f	f	f	f	0

Let $M = (\alpha_M, \zeta_M)$ is an IFS in V defined by

V	0	d	e	f
α_M	0.5	0.5	0.6	0.5
ζ_M	0.5	0.5	0.3	0.5

Then $M = (\alpha_M, \zeta_M)$ is a DIFSA in V.

2.3 DIF-ideals in BCK/BCI-algebras

Current section we have defined DIF-ideal in BCK/BCI-algebras and thereafter moved on investigating its several properties taking help of some certain examples.

Definition 2.3.1. *Let V be a BCK/BCI-algebra. An IFS $M = (\alpha_M, \zeta_M)$ in V is said to be a* ***DIF-ideal*** *if*

(F1) $\alpha_M(0) \leq \alpha_M(v_1); \zeta_M(0) \geq \zeta_M(v_1)$,

(F2) $\alpha_M(v_1) \leq \alpha_M(v_1 * v_2) \bigvee \alpha_M(v_2)$,

(F3) $\zeta_M(v_1) \geq \zeta_M(v_1 * v_2) \bigwedge \zeta_M(v_2), \forall v_1, v_2 \in V$.

Theorem 2.3.1. *Let an IFS $M = (\alpha_M, \zeta_M)$ in V be a DIF-ideal in V. If the inequility $v_1 * p_1 \leq q_1$ holds in V, then*

(i) $\alpha_M(v_1) \leq max\{\alpha_M(p_1), \alpha_M(q_1)\}$

(ii) $\zeta_M(v_1) \geq min\{\zeta_M(p_1), \zeta_M(q_1)\}$

Proof: Let $v_1, p_1, q_1 \in V$, such that $v_1 * p_1 \leq q_1$. So $(v_1 * p_1) * q_1 = 0$ and thus,

$$\begin{aligned}
\alpha_M(v_1) &\leq max\{\alpha_M(v_1 * p_1), \alpha_M(p_1)\}, \\
&\leq max\{max\{\alpha_M((v_1 * p_1) * q_1), \alpha_M(q_1)\}, \alpha_M(p_1)\} \\
&= max\{max\{\alpha_M(0), \alpha_M(q_1)\}, \alpha_M(p_1)\} \\
&= max\{\alpha_M(q_1), \alpha_M(p_1)\}
\end{aligned}$$

$\therefore \alpha_M(v_1) \leq max\{\alpha_M(p_1), \alpha_M(q_1)\}.$

$$\begin{aligned} \text{Again, } \zeta_M(v_1) &\geq min\{\zeta_M(v_1 * p_1), \zeta_M(p_1)\}, \\ &\geq min\{min\{\zeta_M((v_1 * p_1) * q_1), \zeta_M(q_1)\}, \zeta_M(p_1)\} \\ &= min\{min\{\zeta_M(0), \zeta_M(q_1)\}, \zeta_M(p_1)\} \\ &= min\{\zeta_M(q_1), \zeta_M(p_1)\} \\ &= min\{\zeta_M(p_1), \zeta_M(q_1)\} \end{aligned}$$

Thus it is proved.

Corollary 2.3.1. *If $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V, then for any $v_1, p_1, p_2, p_3, \dots p_n \in V$ and $(\dots(v_1 * p_1) * p_2) * \dots) * p_n = 0$,*
$\alpha_M(v_1) \leq max\{\alpha_M(p_1), \alpha_M(p_2), \alpha_M(p_3), \dots \alpha_M(p_n)\}$
$\zeta_M(v_1) \geq min\{\zeta_M(p_1), \zeta_M(p_2), \zeta_M(p_3), \dots \zeta_M(p_n)\}$

Proposition 2.3.2. *In V let $M = (\alpha, \zeta)$ be a DIF-ideal. Then the followings hold for all $v_1, v_2, v_3 \in V$*
(a)if $v_1 \leq v_2$ then $\alpha(v_1) \leq \alpha(v_2), \zeta(v_1) \geq \zeta(v_2)$
(b) $\alpha(v_1 * v_2) \leq \alpha(v_1 * v_3) \bigvee \alpha(v_3 * v_2)$ and $\zeta(v_1 * v_2) \geq \zeta(v_1 * v_3) \bigwedge \zeta(v_3 * v_2)$

Proof. (a) If $v_1 \leq v_2$ then $v_1 * v_2 = 0$
Hence $\alpha(v_1) \leq \alpha(v_1 * v_2) \bigvee \alpha(v_2) = \alpha(0) \bigvee \alpha(v_2) = \alpha(v_2)$,
and $\zeta(v_1) \geq \zeta(v_1 * v_2) \bigwedge \zeta(v_2) = \zeta(0) \bigwedge \zeta(v_2) = \zeta(v_2)$.

(b) Since $(v_1 * v_2) * (v_1 * v_3) \leq (v_3 * v_2)$
It follows from (a) that
$\alpha\{(v_1 * v_2) * (v_1 * v_3)\} \leq \alpha(v_3 * v_2)$
Now $\alpha\{(v_1 * v_2) \leq \alpha\{(v_1 * v_2) * (v_1 * v_3)\} \bigvee \alpha(v_1 * v_3)$ [$\because M = (\alpha, \zeta)$ is a DIF-ideal.]
$\therefore \alpha(v_1 * v_2) \leq \alpha(v_3 * v_2) \bigvee \alpha(v_1 * v_3)$
Again, $\zeta(v_1 * v_2) \geq \zeta\{(v_1 * v_2) * (v_1 * v_3)\} \bigwedge \zeta(v_1 * v_3) \geq \zeta(v_3 * v_2) \bigwedge \zeta(v_1 * v_3)$
$\therefore \zeta(v_1 * v_2) \geq \zeta(v_1 * v_3) \bigwedge \zeta(v_3 * v_2)$ □

Theorem 2.3.3. *If an IFS $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V, it must be a DIFSA in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V.
Since $v_1 * v_2 \leq v_1, \forall v_1, v_2 \in V$.
Then $\alpha_M(v_1 * v_2) \leq \alpha_M(v_1)$
and $\zeta_M(v_1 * v_2) \geq \zeta_M(v_1)$
So, $\alpha_M(v_1 * v_2) \leq \alpha_M(v_1) \leq \alpha_M(v_1 * v_2) \bigvee \alpha_M(v_2) \leq \alpha_M(v_1) \bigvee \alpha_M(v_2), \forall v_1, v_2 \in V$.

$$[\because M = (\alpha_M, \zeta_M) \text{ is a DIF-ideal.}]$$

and $\zeta_M(v_1 * v_2) \geq \zeta_M(v_1) \geq \zeta_M(v_1 * v_2) \bigwedge \zeta_M(v_2) \geq \zeta_M(v_1) \bigwedge \zeta_M(v_2), \forall v_1, v_2 \in V$.
This shows that, $M = (\alpha_M, \zeta_M)$ is a DIFSA in V. □

Example 4. *Let us consider $V = \{0, d, f, g, h\}$ as a BCK-algebra in below tabulated form:*

$*$	0	d	f	g	h
0	0	0	0	0	0
d	d	0	0	0	0
f	f	f	0	f	0
g	g	g	g	0	0
h	h	f	d	f	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V defined by

V	0	d	f	g	h
α_M	0	0.4	0.6	0.6	0.6
ζ_M	1	0.5	0.4	0.4	0.4

Then $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V.
Thus M is a DIF-ideal as well as DIFSA in V.

Converse of Theorem 2.3.3 may not true. Which can be consolidated through Example 3 .
As $\zeta_M(e) = 0.3$
$\zeta_M(e) = 0.3 \ngeq 0.5 = \zeta_M(e * d) \bigwedge \zeta_M(d)$, therefore, $M = (\alpha_M, \zeta_M)$ is not a DIF-ideal.

We now provide a condition for an IFS $M = (\alpha_M, \zeta_M)$ which is a DIFSA in V to be a DIF-ideal in V.

Theorem 2.3.4. *Let an IFS* $M = (\alpha_M, \zeta_M)$ *be a DIFSA in* V*. If the inequiality* $v_1 * v_2 \leq v_3$ *holds in* V*, then* M *would be a DIF-ideal in* V*.*

Proof. Let $M = (\alpha_M, \zeta_M)$ be the DIFSA in V. Then from Theorem 2.2.1, $\alpha_M(0) \leq \alpha_M(v_1) \ldots (1)$, and $\zeta_M(0) \geq \zeta_M(v_1) \ldots (2), \forall v_1 \in V$.

As $v_1 * v_2 \leq v_3$ holds in V, then from Theorem 2.3.1,

We get, $\alpha_M(v_1) \leq max\{\alpha_M(v_2), \alpha_M(v_3)\}$, $\zeta_M(v_1) \geq min\{\zeta_M(v_2), \zeta_M(v_3)\}, \forall v_1, v_2, v_3 \in V$.

Again since, $v_1 * (v_1 * v_2) \leq v_2$, Then, $\alpha_M(v_1) \leq max\{\alpha_M(v_1 * v_2), \alpha_M(v_2)\}$,

and $\zeta_M(v_1) \geq min\{\zeta_M(v_1 * v_2), \zeta_M(v_2)\}$.

Hence, $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V. □

Theorem 2.3.5. *Let an IFS* $M = (\alpha_M, \zeta_M)$ *be a DIF-ideal in* V*. Then*

$$\alpha_M(0 * (0 * v_1)) \leq \alpha_M(v_1).$$
$$\zeta_M(0 * (0 * v_1)) \geq \zeta_M(v_1), \forall v_1 \in V.$$

Proof. $\alpha_M(0 * (0 * v_1)) \leq \alpha_M\{(0 * (0 * v_1)) * v_1\} \vee \alpha_M(v_1) \leq \alpha_M(0) \vee \alpha_M(v_1) = \alpha_M(v_1), \forall v_1 \in V$.

Therefore, $\alpha_M(0 * (0 * v_1)) \leq \alpha_M(v_1), \forall v_1 \in V$.

Again,

$\zeta_M(0*(0*v_1)) \geq \zeta_M\{(0*(0*v_1))*v_1\} \wedge \zeta_M(v_1) \geq \zeta_M(0) \wedge \zeta_M(v_1) = \zeta_M(v_1), \forall v_1 \in V$.

Therefore, $\zeta_M(0 * (0 * v_1)) \geq \zeta_M(v_1), \forall v_1 \in V$. □

Theorem 2.3.6. *Let* $M = (\alpha_M, \zeta_M)$ *be a DIF-ideal in* V*. Then so is* $\bigoplus M = \{(v_1, \alpha_M(v_1), \bar{\alpha}_M(v_1))/v_1 \in V\}$.

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V, then

$\alpha_M(0) \leq \alpha_M(v_1)$, and $\alpha_M(v_1) \leq \alpha_M(v_1 * v_2) \bigvee \alpha_M(v_2)$.

Now, $\alpha_M(0) \leq \alpha_M(v_1) \Rightarrow 1 - \bar{\alpha}_M(0) \leq 1 - \bar{\alpha}_M(v_1) \Rightarrow \bar{\alpha}_M(0) \geq \bar{\alpha}_M(v_1)$, for any $v_1 \in V$.

Now for arbitrary $v_1, v_2 \in V$,

$$\alpha_M(v_1) \leq max\{\alpha_M(v_1 * v_2), \alpha_M(v_2)\}$$
$$\Rightarrow 1 - \bar{\alpha}_M(v_1) \leq max\{1 - \bar{\alpha}_M(v_1 * v_2), 1 - \bar{\alpha}_M(v_2)\}$$
$$\Rightarrow \bar{\alpha}_M(v_1) \geq 1 - max\{1 - \bar{\alpha}_M(v_1 * v_2), 1 - \bar{\alpha}_M(v_2)\}$$
$$\Rightarrow \bar{\alpha}_M(v_1) \geq min\{\bar{\alpha}_M(v_1 * v_2), \bar{\alpha}_M(v_2)\}.$$

So, $\bigoplus M = \{(v_1, \alpha_M(v_1), \bar{\alpha}_M(v_1))/v_1 \in V\}$ is a DIF-ideal in V. $\square$

Theorem 2.3.7. *Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V. Then so is $\bigotimes M = \{(v_1, \bar{\zeta}_M(v_1), \zeta_M(v_1))/v_1 \in V\}$.*

Proof. Since $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V, then $\zeta_M(0) \geq \zeta_M(v_1)$. And $\zeta_M(v_1) \geq \zeta_M(v_1 * v_2) \bigwedge \zeta_M(v_2)$. Also we have, $\zeta_M(0) \geq \zeta_M(v_1) \Rightarrow 1 - \bar{\zeta}_M(0) \geq 1 - \bar{\zeta}_M(v_1) \Rightarrow \bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1)$, for any $v_1 \in V$.
Consider, for any $v_1, v_2 \in V$,

$$\begin{aligned}
\zeta_M(v_1) &\geq min\{\zeta_M(v_1 * v_2), \zeta_M(v_2)\} \\
\Rightarrow 1 - \bar{\zeta}_M(v_1) &\geq min\{1 - \bar{\zeta}_M(v_1 * v_2), 1 - \bar{\zeta}_M(v_2)\} \\
\Rightarrow \bar{\zeta}_M(v_1) &\leq 1 - min\{1 - \bar{\zeta}_M(v_1 * v_2), 1 - \bar{\zeta}_M(v_2)\} \\
\Rightarrow \bar{\zeta}_M(v_1) &\leq max\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}.
\end{aligned}$$

Hence, $\bigotimes M = \{(v_1, \bar{\zeta}_M(v_1), \zeta_M(v_1))/v_1 \in V\}$ is a DIF-ideal in V. $\square$

Theorem 2.3.8. *An IFS $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V iff $\bigoplus M = \{(v_1, \alpha_M(v_1), \bar{\alpha}_M(v_1))/v_1 \in V\}$ and $\bigotimes M = \{(v_1, \bar{\zeta}_M(v_1), \zeta_M(v_1))/v_1 \in V\}$ are DIF-ideals in V.*

Now Theorem 2.3.8 is illustrated by using the example provided below.

Example 5. *Let consider $V = \{0, s, t, u, v\}$ as a BCK-algebra in below tabulated form:*

$*$	0	s	t	u	v
0	0	0	0	0	0
s	s	0	s	0	s
t	t	t	0	t	0
u	u	s	u	0	u
v	v	v	t	v	0

Let $M = \{(v_1, \bar{\zeta}_M(v_1)) : v_1 \in V\}$ be an IVFS defined as

V	0	s	t	u	v
α_M	0.4	0.5	0.6	0.5	0.6
ζ_M	0.5	0.4	0.3	0.4	0.3

Then $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V.
Then $\bigoplus M = \{(v_1, \alpha_M(v_1), \bar{\alpha}_M(v_1))/v_1 \in V\}$ be an IFS defined by

V	0	s	t	u	v
α_M	0.4	0.5	0.6	0.5	0.6
$\bar{\alpha}_M$	0.6	0.5	0.4	0.5	0.4

And $\bigotimes M = \{(v_1, \bar{\zeta}_M(v_1), \zeta_M(v_1))/v_1 \in V\}$ *be an IFS defined by*

V	0	s	t	u	v
$\bar{\zeta}_M$	0.5	0.6	0.7	0.6	0.7
ζ_M	0.5	0.4	0.3	0.4	0.3

So, it is clear that $\bigoplus M = \{(v_1, \alpha_M(v_1), \bar{\alpha}_M)/v_1 \in V\}$ *and* $\bigotimes M = \{(v_1, \bar{\zeta}_M(v_1), \zeta_M(v_1))/v_1 \in V\}$ *are DIF-ideal.*

Theorem 2.3.9. *An IFS* $M = (\alpha_M, \zeta_M)$ *is a DIF-ideal in* V *iff the FSs* α_M *and* $\bar{\zeta}_M$ *are DF-ideals in* V.

Proof. In V let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal. Then it is obvious that α_M is a DF-ideal in V. Further it can conclude from Theorem 2.3.7 that $\bar{\zeta}_M$ is a DF-ideal in V.

From reverse angle if, α_M is a DF-ideal in V, then

$\alpha_M(0) \leq \alpha_M(v_1)$ And $\alpha_M(v_1) \leq max\{\alpha_M(v_1 * v_2), \alpha_M(v_2)\}, \forall v_1, v_2 \in V$.

Again since $\bar{\zeta}_M$ is a DIF-ideal in V, so, $\bar{\zeta}_M(0) \leq \bar{\zeta}_M(v_1) \Rightarrow 1-\zeta_M(0) \leq 1-\zeta_M(v_1) \Rightarrow \zeta_M(0) \geq \zeta_M(v_1)$.

And

$$\begin{aligned} &\bar{\zeta}_M(v_1) \leq max\{\bar{\zeta}_M(v_1 * v_2), \bar{\zeta}_M(v_2)\}. \\ &\Rightarrow 1-\zeta_M(v_1) \leq max\{1-\zeta_M(v_1 * v_2), 1-\zeta_M(v_2)\} \\ &\Rightarrow \zeta_M(v_1) \geq 1 - max\{1-\zeta_M(v_1 * v_2), 1-\zeta_M(v_2)\} \\ &\Rightarrow \zeta_M(v_1) \geq min\{\zeta_M(v_1 * v_2), \zeta_M(v_2)\}, \forall v_1, v_2 \in V. \end{aligned}$$

Hence, $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V. □

EXAMPLE 6. *Let us consider a* BCK*-algebra* $V = \{0, d_1, e_1, f_1\}$ *given by the table* *below:*

$*$	0	d_1	e_1	f_1
0	0	0	0	0
d_1	d_1	0	d_1	d_1
e_1	e_1	d_1	0	0
f_1	f_1	d_1	f_1	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V defined by

V	0	d_1	e_1	f_1
α_M	0.1	0.5	0.7	0.6
ζ_M	0.8	0.4	0.2	0.4

Since $\alpha_M(e_1) = 0.7 \nleq 0.6 = max\{\alpha_M(e_1 * f_1), \alpha_M(f_1)\}$,
and $\zeta_M(e_1) = 0.2 \ngeq 0.4 = min\{\zeta_M(e_1 * f_1), \zeta_M(f_1)\}$.
Therefore $M = (\alpha_M, \zeta_M)$ is not a DIF-ideal in V.

Corollary 2.3.2. *For any DIF-ideal $M = (\alpha_M, \zeta_M)$ in a BCK/BCI-algebra V. The sets,*

$$D_{\alpha_M} = \{v_1 \in V/\alpha_M(v_1) = \alpha_M(0)\}, \text{ and } D_{\zeta_M} = \{v_1 \in V/\zeta_M(v_1) = \zeta_M(0)\} \text{ are ideals in } V.$$

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal of V. Obviously, $0 \in D_{\alpha_M}, D_{\zeta_M}$.
Now let $v_1, v_2 \in V$, such that $v_1 * v_2, v_2 \in D_{\alpha_M}$.
Then $\alpha_M(v_1 * v_2) = \alpha_M(0) = \alpha_M(v_2)$.
And so, $\alpha_M(v_1) \leq max\{\alpha_M(v_1 * v_2), \alpha_M(v_2)\} = \alpha_M(0)$
Again, since α_M is a DF-ideal in V, $\alpha_M(0) \leq \alpha_M(v_1)$, $\therefore$ $\alpha_M(0) = \alpha_M(v_1)$. Hence, $v_1 \in D_{\alpha_M}$. Therefore, D_{α_M} is an ideal in V.
Following the same path it is proved that D_{ζ_M} is also an ideal in V. □

Theorem 2.3.10. *Intersection of any two DIF-ideals in V, such that one is contained in another, is also a DIF-ideal in V.*

Proof. Let $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ be two DIF-ideals in V.

Again let, $R = M \cap N = (\alpha_R, \zeta_R)$, Where $\alpha_R = \alpha_M \wedge \alpha_N$ and $\zeta_R = \zeta_M \vee \zeta_N$.
Let $v_1, v_2 \in V$, then, $\alpha_R(0) = \alpha_M(0) \wedge \alpha_N(0) \leq \alpha_M(v_1) \wedge \alpha_N(v_1) = \alpha_R(v_1)$.

And $\zeta_R(0) = \zeta_M(0) \vee \zeta_N(0) \geq \zeta_M(v_1) \vee \zeta_N(v_1) = \zeta_R(v_1)$. Also,

$$\begin{aligned}
\alpha_R(v_1) &= \alpha_M(v_1) \wedge \alpha_N(v_1) \\
&\leq max[\alpha_M(v_1 * v_2), \alpha_M(v_2)] \wedge max[\alpha_N(v_1 * v_2), \alpha_N(v_2)] \\
&= max\{[\alpha_M(v_1 * v_2) \wedge \alpha_N(v_1 * v_2)], [\alpha_M(v_2) \wedge \alpha_N(v_2)]\} \\
&= max[\alpha_R(v_1 * v_2), \alpha_R(v_2)].
\end{aligned}$$

Similarly, it can be verified that $\zeta_R(v_1) \geq min[\zeta_R(v_1 * v_2), \zeta_R(v_2)]$.

Thus it is proved. □

Theorem 2.3.11. *Union of any two DIF-ideals in V, is also a DIF-ideal in V.*

Proof. Assume that $M = (\alpha_M, \zeta_M)$ and $N = (\alpha_N, \zeta_N)$ are two DIF-ideals in V.

Again let, $R = M \cup N = (\alpha_R, \zeta_R)$. Where $\alpha_R = \alpha_M \vee \alpha_N$ and $\zeta_Q = \zeta_M \wedge \zeta_N$. Let $v_1, v_2 \in V$, then, $\alpha_R(0) = \alpha_M(0) \vee \alpha_N(0) \leq \alpha_M(v_1) \vee \alpha_N(v_1) = \alpha_R(v_1)$.

And $\zeta_R(0) = \zeta_M(0) \wedge \zeta_N(0) \geq \zeta_M(v_1) \wedge \zeta_N(v_1) = \zeta_R(v_1)$.

Also,

$$\begin{aligned}
\alpha_R(v_1) &= \alpha_M(v_1) \vee \alpha_N(v_1) \\
&\leq max[\alpha_M(v_1 * v_2), \alpha_M(v_2)] \vee max[\alpha_N(v_1 * v_2), \alpha_N(v_2)] \\
&= max\{[\alpha_M(v_1 * v_2) \vee \alpha_N(v_1 * v_2)], [\alpha_M(v_2) \vee \alpha_N(v_2)]\} \\
&= max[\alpha_R(v_1 * v_2), \alpha_R(v_2)].
\end{aligned}$$

In the similer way, We can verify that, $\zeta_R(v_1) \geq min[\zeta_R(v_1 * v_2), \zeta_R(v_2)]$. Thus it is proved. □

EXAMPLE 7. *Consider a BCK-algebra $V = \{0, k', l', m', n'\}$ as follows:*

$*$	0	k'	l'	m'	n'
0	0	0	0	0	0
k'	k'	0	0	0	0
l'	l'	l'	0	l'	0
m'	m'	m'	m'	0	0
n'	n'	l'	k'	l'	0

Let an IFS $M = (\alpha_M, \zeta_M)$ in V, is given as follows:

V	0	k'	l'	m'	n'
α_M	0	0.62	0.65	0.62	0.65
ζ_M	1	0.34	0.32	0.34	0.32

Then $M = (\alpha_M, \zeta_M)$ is a DIF-ideal in V.

Again let, $N = (\alpha_N, \zeta_N)$ be an IFS in V defined by

V	0	k'	l'	m'	n'
α_N	0.22	0.56	0.58	0.56	0.58
ζ_N	0.72	0.44	0.42	0.44	0.42

Then $N = (\alpha_N, \zeta_N)$ is a DIF-ideal in V.

Now let $P = M \cap N = (\alpha_P, \zeta_P)$, Where $\alpha_P = \alpha_M \wedge \alpha_N$ and $\zeta_P = \zeta_M \vee \zeta_N$. Then P is an IFS which can be defined by:

V	0	k'	l'	m'	n'
α_P	0	0.56	0.58	0.56	0.58
ζ_P	1	0.44	0.42	0.44	0.42

Then $P = (\alpha_P, \zeta_P)$ is a DIF-ideal in V.

We also assume that $Q = M \cup N = (\alpha_Q, \zeta_Q)$. Where $\alpha_Q = \alpha_M \vee \alpha_N$ and $\zeta_R = \zeta_M \wedge \zeta_N$.

Now $Q = (\alpha_Q, \zeta_Q)$ has been defined as:

V	0	k'	l'	m'	n'
α_Q	0.22	0.62	0.65	0.62	0.65
ζ_Q	0.72	0.34	0.32	0.34	0.32

Then $Q = (\alpha_Q, \zeta_Q)$ is a DIF-ideal in V.

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V, then UC of level c and LC of level d of the IFS $M = (\alpha_M, \zeta_M)$ in V,is as followes:

$$\alpha^{\leq}_{M,c} = \{v_1 \in V/\alpha_M(v_1) \leq c\},$$
$$\zeta^{\geq}_{M,d} = \{v_1 \in V/\zeta_M(v_1) \geq d\}, \text{where } c, d \in [0, 1].$$

Theorem 2.3.12. *If $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V, then $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are ideals in V for $c, d \in [0, 1]$*

Proof. Let $M = (\alpha_M, \zeta_M)$ be a DIF-ideal in V, and let $c \in [0,1]$ with $\alpha_M(0) \leq c$.

Then we have, $\alpha_M(0) \leq \alpha_M(v_1), \forall v_1 \in V$, but $\alpha_M(v_1) \leq c, \forall v_1 \in \alpha^{\leq}_{M,c}$. So $0 \in \alpha^{\leq}_{M,c}$.

Let $v_1, v_2 \in V$ be such that $v_1 * v_2 \in \alpha^{\leq}_{M,c}$ and $v_2 \in \alpha^{\leq}_{M,c}$, then, $\alpha_M(v_1 * v_2) \in \alpha^{\leq}_{M,c}$ and $\alpha_M(v_2) \in \alpha^{\leq}_{M,c}$. So, $\alpha_M(v_1 * v_2) \leq c$ and $\alpha_M(v_2) \leq c$.

Since α_M is a DF-ideal in V, it follow that, $\alpha_M(v_1) \leq \alpha_M(v_1 * v_2) \bigvee \alpha_M(v_2) \leq c$ and hence $v_1 \in \alpha^{\leq}_{M,c}$. Therefore $\alpha^{\leq}_{M,c}$ is ideal in V for $c \in [0,1]$.

In similar way, it is proved that $\zeta^{\geq}_{M,d}$ is ideal in V for $c, d \in [0,1]$. □

Theorem 2.3.13. *If $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are either contained no elements or formed ideals in V for $c, d \in [0,1]$, then $M = [\alpha_M, \zeta_M]$ is a DIF-ideal in V.*

Proof. Let $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ be either contained no elements or formed ideals in V for $c, d \in [0,1]$. For any $v_1 \in V$, let $\alpha_M(v_1) = c$ and $\zeta_M(v_1) = d$. Then $v_1 \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$, so $\alpha^{\leq}_{M,c} \neq \phi \neq \zeta^{\geq}_{M,d}$. Since $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are ideals in V, therefore $0 \in \alpha^{\leq}_{M,c} \bigwedge \zeta^{\geq}_{M,d}$.

Hence, $\alpha_M(0) \leq c = \alpha_M(v_1)$ and $\zeta_M(0) \geq d = \zeta_M(v_1)$, where $v_1 \in V$.

If there exist $v_1', v_2' \in V$ such that $\alpha_M(v_1') > max\{\alpha_M(v_1' * v_2'), \alpha_M(v_2')\}$, then by taking, $c_0 = \frac{1}{2}(\alpha_M(v_1') + max\{\alpha_M(v_1' * v_2'), \alpha_M(v_2')\}$.

We have, $\alpha_M(v_1') > c_0 > max\{\alpha_M(v_1' * v_2'), \alpha_M(v_2')\}$. So, $v_1' \notin \alpha^{\leq}_{M,c_0}$, $(v_1' * v_2') \in \alpha^{\leq}_{M,c_0}$ and $v_2' \in \alpha^{\leq}_{M,c_0}$, that is $\alpha^{\leq}_{M,c_0}$ is not an ideal in V, Hence a contradiction arises.

Eventually, let there exist $p, q \in V$ such that $\zeta_M(p) < min\{\zeta_M(p * q), \zeta_M(q)\}$. Taking $d_0 = \frac{1}{2}(\zeta_M(p') + min\{\zeta_M(p' * q'), \zeta_M(q')\}$, then

$min\{\zeta_M(p' * q'), \zeta_M(q')\} > d_0 > \zeta_M(p')$.

Therefore, $p * q \in \zeta^{\geq}_{M,d}$ and $q \in \zeta^{\geq}_{M,d}$ but $p \notin \zeta^{\geq}_{M,d}$. Again a contradiction. This completes the proof. □

But if an IFS M is not a DIF-ideal in V, then $\alpha^{\leq}_{M,c}$ and $\zeta^{\geq}_{M,d}$ are not ideals in V for $c, d \in [0,1]$, which is illustrated by the example below.

EXAMPLE 8. *Let $V = \{0, d_1, e_1, f_1\}$ be a BCK-algebra [using Example 6] with the following cayley table:*

$*$	0	d_1	e_1	f_1
0	0	0	0	0
d_1	d_1	0	d_1	d_1
e_1	e_1	d_1	0	0
f_1	f_1	d_1	f_1	0

Let $M = (\alpha_M, \zeta_M)$ be an IFS in V defined by

V	0	d_1	e_1	f_1
α_M	0.1	0.5	0.7	0.6
ζ_M	0.8	0.4	0.2	0.4

Which is not a DIF-ideal in V.

For $c = 0.67$ and $d = 0.25$, we get $\alpha^{\leq}_{M,c} = \zeta^{\geq}_{M,d} = \{0, d_1, f_1\}$, which are not ideals in V, as $e_1 * d_1 = d_1 \in \{0, d_1, f_1\}$, and $d_1 \in \{0, d_1, f_1\}$, but $e_1 \notin \{0, d_1, f_1\}$.

2.4 Summary

The purpose of this chapter is to define the notion of DIFSAs, DIF-ideals as an extention of DFSAs and DF-ideals in BCK/BCI-algebras. It is shown that the union of two DIF-ideals in BCK/BCI-algebra is also a DIF-ideal. At the same time we also investigated that every DIF-ideal must be a DIFSA but in general it may not be true in reverse direction, that is illustrated by an example. Further a condition is presented to exibit that a DIFSA becomes a DIF-ideal in BCK/BCI-algebra.