M.Sc. 1st Semester Examination, 2013

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Real Analysis*)

PAPER—MTM-101

Full Marks : 50

Time : 2 hours

Answer **Q. No. 1** and **three** from **Q. No. 2** to **Q. No. 6**

The figures in the right-hand margin indicate marks

1. Answer any *two* questions : 2 × 2

(*a*) If $f(x) = 3x - 2$ and $g(x) = 7$ find the RS-integral

$$\int_{-3}^{5} f(x)\, d\, g(x).$$

(*b*) Show that a Lipschitz function on $[a, b]$ is a function of bounded variation on $[a, b]$.

(*c*) Let $A_1, A_2, \ldots, A_n, \ldots$ be null sets. Show that $\bigcup_{n=1}^{\infty} A_n$ is a null set.

2. (*a*) Let $f : [a, b] \to R$ be a function of bounded variation on $[a, c]$ and $[c, b]$ where $c \in (a, b)$. Then show that

(*i*) f is of bounded variation on $[a, b]$ and

(*ii*) $V_f[a, c] + V_f[c, b] = V_f[a, b]$.

(*b*) Let $f(x) = x^2, x \in [-1, 1]$.
Show that f is a function of bounded variation on $[-1, 1]$. Find the variation function $V(x)$ on $[-1, 1]$. Express f as the difference of two monotone functions on $[-1, 1]$. 6 + 6

3. (*a*) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$, then show that $f \in R(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.$$

(*b*) Hence show that,

$$\int_0^3 x^2 d([x] - x) = 5.$$

7 + 5

4. (*a*) Let the functions $f(x)$ and $g(x)$ be defined on $[-1, 5]$ as follows :

$$f(x) = 0 \quad \text{for } -1 \le x \le 3$$
$$= 2x + 1 \quad \text{for } 3 < x \le 5$$

$$g(x) = 2 - 3x^2 \quad \text{for } -1 \le x < 3$$
$$= 6 \quad \text{for } 3 \le x \le 5.$$

Show that $f(x)$ is not RS-integrable with respect to $g(x)$ on $[-1, 5]$.

(*b*) If $f(x)$ is monotonic increasing and $g(x)$ is continuous on $[a, b]$ then prove that there exists a point $\xi \in [a, b]$, such that

$$\int_a^b f(x)dg(x) = f(a) \int_a^{\xi} dg(x) + f(b) \int_{\xi}^b dg(x).$$

6 + 6

5. (*a*) Prove that a bounded function $f(x)$ on $[a, b]$ is Lebesgue integrable on $[a, b]$ if and only if for every $\in > 0$ there exists a measurable partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \in.$$

(*b*) If $f(x)$ and $g(x)$ are bounded Lebesgue integrable functions on $[a, b]$ and if $f(x) \leq g(x)$ almost everywhere on $[a, b]$ then prove that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$ 6 + 6

6. State and prove any *two* of the following : 6 × 2

(*i*) Fatous Lemma

(*ii*) Lebesgue's Dominated convergence theorem

(*iii*) Monotone convergence theorem.

[*Internal Assessment* : 10 Marks]