Total Pages—11 **UG/II/MATH/H/V/17 (Old)**

2017

MATHEMATICS

[Honours]

PAPER – V

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[OLD SYLLABUS]

GROUP – A

(*Real Analysis*)

[*Marks : 64*]

(*Turn Over*)

1. Answer any *two* questions : 15 × 2

(*a*) (*i*) Defined Uniform Convergence and pointwise convergence of a sequence of function $\{f_n\}$ on $[a, b]$. Show that the sequence $\{f_n\}$ where $f_n(x) = nxe^{-nx^2}$ is not uniformly convergent on $[0, 1]$. 2 + 4

(*ii*) For every given $x \in \mathbb{R}$ prove that the series

$$\sum_{n=1}^{\alpha} \frac{\sin nx}{n^3}$$

converges absolutely. 4

(*iii*) Show that the sequence $\{f_n(x)\}_n$ where

$$f_n(x) = \begin{cases} n^2x, & 0 \le x \le 1/n \\ -n^2x + 2n, & \frac{1}{n} < x \le 2/n \\ 0, & \frac{2}{n} < x \le 1 \end{cases}$$

is not uniformly convergent on $[0, 1]$. 5

(b) (i) If $f \in \mathbb{R}[a, b$ and f possesses as primitive ϕ on $[a, b]$ then show that

$$\int_a^b f\,dx = \varphi(b) - \phi(a).$$ 5

(ii) Let $a > 0$. Prove that the integral

$$\int_a^{\alpha} \frac{dx}{x^{\mu}}$$

is convergent if $\mu > 1$ and is divergent to α if $\mu \leq 1$. 5

(iii) Show that

$$\frac{\pi^3}{24\sqrt{2}} \langle \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x}\,dx \langle \frac{\pi^3}{24}.$$ 5

(c) (i) For each natural number n, let

$$f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1].$$

Show that the sequence $\{f_n\}$ converges uniformly on [0, 1]. 4

(ii) Let g be continuous on [0, 1] and $f_n(x) = g(x)x^n$, $x \in [0, 1]$. Prove that the sequence $\{f_n\}$ is uniformly convergent on [0, 1] if and only if $g(1) = 0$. 5

(*iii*) Prove that the series

$$\sum \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all real x. 3

(*iv*) Show that the series

$$(1-x) + x(1-x) + x^2(1-x) + \ldots$$

is not uniformly convergent on [0, 1]. 3

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Show that

$$\int_0^{\alpha} x^{n-1} e^{-x} dx$$

is convergent if and only if $n > 0$. 4

(*ii*) A function $f : [0, 1] > [0, 1]$ is defined as follows :

$$f(x) = \begin{cases} \frac{1}{2^n}, & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} (n = 0,1,2,3 \cdots) \\ 0, & x = 0. \end{cases}$$

Show that f is Riemann integrable over $[0, 1]$. 4

(*b*) (*i*) Prove that

$$\int_0^1 x^{m-1}(1-x)^{x-1}dx$$

exists if and only if $m > 0, n > 0$. 4

(*ii*) Prove that

$$\int_a^b (x-a)^m (b-x)^n dx$$

$$= (b-a)^{m+n+1}\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}(m > -1, n > 4).$$ 4

(*c*) (*i*) Assuming the power series expansion for

$$\frac{1}{\sqrt{1-x^2}} \text{ as } \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

obtain the power series expansion for $\sin^{-1}x$. Deduce that

$$1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots = \frac{\pi}{2}.$$ 4

(*ii*) Let $f(x)$ be the sum of the power series

$$\sum_{n=0}^{\infty} a_n x^n \text{ on } (-R, R) \text{ for } R > 0.$$

If $f(x) + f(-x) = 0$ for all $x \in (-R, R)$ prove that $a_n = 0$ for all even n. 4

3. Answer any *three* questions : 4 × 3

(*a*) Use Abel's test to show that

$$\int_{\pi}^{\infty} e^{-x} \frac{\sin x}{x} dx$$

is convergent. 4

(*b*) Obtain the Fourier series expansion of the function f defined by

$$f(x) = x \sin x, -\pi \le x \le \pi$$

on the interval $[-\pi, \pi]$. 4

(*c*) Show that the sequence of functions $\{f_n\}_n$ defined by

$$f_n(x) = \frac{x^n}{n}, 0 \le x \le 1$$

is uniformly convergent on [0, 1]. 4

(*d*) Define e. Show that $2 < e < 3$. 1 + 3

(*e*) Using differentiation under the integral sign show that

$$\int_0^{\theta} \log(1 + \tan\theta \tan x)\, dx = \theta \log\sec\theta, \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$ 4

4. Answer any *three* questions : 2 × 3

(*a*) Determine the radius of convergence of the power series :

$$1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$ 2

(*b*) Find $\frac{dy}{dx}$ if

$$\int_{\sqrt{x}}^{2y} e^{-xt^2}\, dt = -2.$$ 2

(*c*) Show that the function

$$f(x, y) = (y - x)^3 + (x - 2)^6$$

has neither a maximum nor a minimum at (2, 2). 2

(*d*) Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx$$

is uniformly convergent on $(-\infty, \infty)$. 2

(*e*) State Abel's test and Dirichlet's test for the convergence of the integral

$$\int_a^{\infty} f(x)\, g(x)\, dx.$$ 2

GROUP – B

(*Metric Space*)

[*Marks* : 14]

5. Answer any *one* question : 8 × 1

(*a*) (*i*) The function $d : A \times A \to \mathbb{R}$ is defined by

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \qquad \forall x, y \in A.$$

Then show that (A, d) is a metric space. 4

(*ii*) Define a metric space on a set X. If x, y, z be any three points on a metric space (X, d), show that $d(x, y) \geq | d(x, z) = - d(z, y) |$. 4

(*b*) (*i*) Let (X, d) be a metric space. Then prove that A $(B =)$ $\delta(A) \leq \delta(B)$, $\forall A, B \in X$. Where δ is the diameter of a set. 4

(*ii*) Define an open set and an open sphere in a metric space (X, d). Show that every open sphere in (X, d) is an open set. 4

6. Answer any *two* questions : 3 × 2

(*a*) Prove that if (X, d) is a metric space and $x_1, x_2, \ldots x_n \in X$ then

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_n) + \cdots + d(x_{n-1}, x_n).$$ 3

(*b*) Prove that the limit of a sequence in a metric space if exists, is unique. 3

(*c*) Let (X, d) be a metric space and $A \subset X$. Show that int A is the largest open set contained in A. 3

GROUP – C

(*Complex Analysis*)

[*Marks* : 12]

7. Answer any *one* question : 8 × 1

(*a*) (*i*) Define analytic function $f(z)$ on a domain D of the complex plane.

(*ii*) If the real and imaginary parts u and v of $f(z)$ are both differentiable at (x, y) and satisty the Cauchy-Riemann partial differential equations $u_n = v_y$, $u_y = -v_x$ then prove that $f(z)$ is differential at $z = x + iy$. 2 + 6

(*b*) (*i*) Use Milne-Thomson method to find an analytic function whose real part is given by $u(x, y) = e^x(x\cos y - y\sin y)$.

(*ii*) Let f be analytic on a region G. If $f'(z) = 0$ on G, show that f is constant on G. 8

8. Answer any *one* question : 4 × 1

(*a*) Let

$$f(z) = \begin{cases} \text{Re}(z) & , \text{if } \text{Re}(z) \neq 0 \\ 0 & , \text{if } \text{Re}(z) = 0. \end{cases}$$

show that f is not continuous at $z = 0$. 4

(*b*) If $f(z)$ is an analytical function of z. Prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2|f'(z)|^2.$$ 4