UG/I/MATH/H/II/17

2017

MATHEMATICS

[Honours]

PAPER – II

***Full Marks* : 90**

***Time* : 4 hours**

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

GROUP – A

(*Real Analysis*)

[*Marks* : 35]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) The function $f: R \to R$ and $g: R \to R$ are both continuous on R, then prove

(Turn Over)

that the set $S = \{ x \in R \mid f(x) = g(x) \}$ is a closed set in R. Hence deduce that if $f(x) = g(x)$ at all rational points, then $f(x) = g(x)$ $\forall$ $x \in R$. 4 + 2

(*ii*) If $f : [a, b] \to R$ be differentiable at $c \in R$, show that

$$f'(c) = \lim_{n \to \infty} n \left[f\left(c + \frac{1}{n} \right) - f(c) \right].$$

Show by an example that the existence of this limit does not imply the existence of $f'(c)$, where R is the set of all real numbers and $f'(x) = \frac{d}{dx} f(x)$. 3 + 1

(*iii*) State and prove Darboux theorem. 5

(*b*) (*i*) Let a sequence $\{x_n\}$ be defined by

$$x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n}, n \geq 1, x_1 = 1$$

Does this sequence converge ? Justify your answer. 5

(*ii*) Prove that the function f defined by

$$f(x)=\frac{1}{x^2+1}, x \in R$$ is uniformly continuous on R, where R is the set of all real numbers. 4

(*iii*) Prove that a monotone sequence of real numbers having a convergent subsequence with limit l is convergent with limit l. 6

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) Correct or Justify : A monotone sequence never oscillates. 4

(*ii*) Let $\{x_n\}$, $\{y_n\}$ be two convergent sequences with limits x and y respectively, then prove that

$$\lim_{n\to\infty} \frac{x_n y_1 + x_{n-1} y_2 + \ldots\ldots + x_1 y_n}{n} = xy.$$ 4

(*b*) (*i*) Prove that an infinite bounded set of real numbers has greatest and least limit points. 4

(*ii*) Prove that

$$\lim_{x \to 0} \sin \frac{1}{x} \text{ does not exist.}$$ 4

(*c*) (*i*) State and prove Taylor's theorem with Cauchy's form of remainder after *n* terms. 6

(*ii*) Correct or Justify : The difference of two monotonically increasing function is not necessarily monotonically increasing. 2

3. Answer any *one* question : 4 × 1

(*a*) Let

$$f(x) = x^2 \sin \frac{1}{x} \quad \text{for } x \neq 0$$
$$= 0 \quad \text{for} \quad x = 0$$

and $\phi(x) = x$, prove that $\lim_{x \to 0} \frac{f(x)}{\phi(x)}$ exist but $\lim_{x \to 0} \frac{f'(x)}{\phi'(x)}$ does not exist. 4

(*b*) If

$$\lim_{x \to 0} \frac{ae^{x} + bc^{-x} + 2\sin x}{\sin x + \cos x} = 2$$

find the values of *a* and *b*. 4

GROUP – B

(*Several Variables and Applications*)

[*Marks* : 20]

4. Answer any *two* questions : 8 × 2

(*a*) (*i*) Show that for the function $f(x, y) = |x| + |y|$, partial derivatives f_x and f_y do not exist at (0, 0) but $f(x, y)$ is continuous at (0, 0). 4

(*ii*) State and prove Youngs theorem for the equality of f_{xy} and f_{yx} at some point (a, b) of the domain of definition of $f(x, y)$. 4

(*b*) (*i*) Find the envelope of the lines whose equations are

$$x \sec^2\theta + y \operatorname{cosec}^2\theta = c, \theta$$

being parameter and c is a constant. 4

(*ii*) If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, prove that

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}.$$ 4

(*c*) (*i*) State Euler's theorem for a homogenous function of 2 variables. 2

(*ii*) Show that the function z defined by the equation $F\left(\frac{x}{z}, \frac{y}{z}\right) = 0$

yields $x\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = z$. 6

5. Answer any *one* question : 4 × 1

(*a*) Show that the normal at any point of the curve

$$x = a\cos\theta + a\theta\sin\theta$$
$$y = a\sin\theta - a\theta\cos\theta$$

is at *a* constant distance from the origin. 4

(*b*) For the equiangular spiral $r = ae^{\theta \cot \alpha}$, prove that the radius of curvature subtends a right angle at the pole. 4

GROUP – C

(*Analytical Geometry for two Dimensions*)

[*Marks* : 20]

6. Answer any *two* questions : 8 × 2

(*a*) If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two parallel straight lines, show that the distance between them

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}}.$$ 8

(*b*) Tangents are drawn to the parabola $y^2 = 4ax$ at the points whose abscissae are in the ratio $p : 1$. Show that the locus of the point of intersection is a parabola. 8

(*c*) Show that the condition that the straight line

$\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic

$\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ is

$A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 - l = 0.$ 8

7. Answer any *one* question : 4 × 1

(*a*) A point moves such that the distance between the feet of perpendiculars from it on the lines $ax^2 + 2hxy + by^2 = 0$ is a constant distance $2d$. Show that its locus is

$$(x^2 + y^2)(h^2 - ab) = d^2([a - b]^2 + 4h^2).$$ 4

(*b*) If the polar of a point with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that the locus of the point is the hyperbola itself. 4

GROUP –D

(*Differential Equations*-I)

[*Marks* : 15]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Obtain the differential equation of all circles through the intersection of the circle $x^2 + y^2 = 1$ and the line $x - y = 0$. 4

(*ii*) Solve :

$$(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x. \qquad 4$$

(*iii*) Transforming the differential equation

$$(px - y)(x - py) = 2p$$

to Clairaut's form by substitutions $x^2 = X$ and $y^2 = Y$, solve it and find its singular solution, if any. 7

(*b*) (*i*) Show that the given equation

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x^2)^2}, \;\; 0 < x < 1$$

is exact and hence solve it. 5

(*ii*) Find the solutions of the following

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$$

with $y(0) = 1$ and $y'(0) = 0$. 5

(*iii*) Knowing that $y = x$ is a solution of the differential equation.

$$x^2 \frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0,$$

solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = x^3.$$ 5
