2017

MATHEMATICS

[Honours]

PAPER – I

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

GROUP – A

(*Classical Algebra*)

[*Marks* : 30]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) If *a*, *b*, *c* be three positive real numbers

(Turn Over)

such that the sum of any two is greater than the third, then prove that

$$(a+b+c)^3 \geq 27\,(a+b-c)\,(b+c-a)\,(c+a-b).$$ 5

(*ii*) Solve the equation $x^4 - x^3 + 2x^2 - x + 1 = 0$ which has four distinct roots of equal moduli. 5

(*iii*) If α, β, γ and δ be the roots of the equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, find the value of 5

(*A*) $(\alpha + \beta - \gamma - \delta)\,(\beta + \gamma - \alpha - \beta)\,(\gamma + \alpha - \beta - \delta)$

(*B*) $(3\alpha - \beta - \gamma - \delta)\,(3\beta - \gamma - \alpha - \delta)\,(3\delta - \alpha - \beta - \gamma)$

(*b*) (*i*) Find a substitution of the form $x = my + n$ which will transform the equation

$$x^4 - 7x^3 + 13x^3 - 12x + 6 = 0$$

into a reciprocal equation. Utilize this to solve the equation. 5

(*ii*) If n be a positive integer, then prove that

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}.$$ 5

(*iii*) Find the least value of $x^{-2} + y^{-2} + z^{-2}$, when $x^2 + y^2 + z^2 = 9$. 5

2. Answer any *one* question : 8 × 1

(*a*) (*i*) Prove that when $0 \le \theta < \pi/2$ the principal value of $\tan^{-1}(\cos\theta + i\sin\theta)$ is

$$\frac{\pi}{2} + \frac{i}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

Deduce that when $0 \le \theta < \pi/2$,

$$\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \cdots$$
$$= \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$ 5

(*ii*) If a, b, c be positive and $a + b + c = 1$, then show that

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \ge 8.$$ 3

(*b*) (*i*) If one root of the equation $x^3 + ax + b = 0$ be twice the difference of the other two, then show that one root is $\frac{13b}{3a}$. 4

(*ii*) If a, z_1, z_2 be complex, examine the validity of the relation

$$a^{z_1} \cdot a^{z_2} = a^{z_1+z_2}.$$ 4

3. Answer any *one* question : 4 × 1

(*a*) If the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ be equal to the product of the other two, prove that $r^2 = p^2 s$. If $p \neq 0$ show that the equation can be solved by the substitution

$$x + \frac{r}{px} = t.$$ 4

(*b*) Prove that

$$\cos 4\phi - \cos 4\theta = 8\prod_{r=0}^{3}\left[\cos\phi - \cos\left(\theta + \frac{r\pi}{2}\right)\right].$$ 4

4. Answer any *one* question : 3 × 1

(*a*) If α be a special root of the equation $x^n - 1 = 0$ then prove that $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ is a complete list of distinct roots of the equation $x^n - 1 = 0$. 3

(*b*) State Descartes' rule of signs. If all the roots of the equation $f(x) = 0$ be non-zero real and v, v' are respectively the number of variations of signs in the sequence of coefficients of $f(x)$ and $f(-x)$ then prove that equation $f(x) = 0$ has v positive roots and v' negative roots. 3

GROUP – B

(*Abstract Algebra*)

[*Marks* : 35]

5. Answer any *three* questions : 8 × 3

(*a*) (*i*) Prove that every subgroup of a cyclic group is cyclic. 4

(*ii*) Show that the mapping f is bijection where $f: S \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{1-|x|}, x \in s \text{ and } s = \{x \in \mathbb{R}; -1 < x < 1\}.$$

Also determine f^{-1}. 4

(*b*) (*i*) Let S be a finite set containing two elements. How many different binary relations can be defined on S ? How many of these are reflexive ? Show that the number of different reflexive relations on a set of n elements is $2^{n^2 - n}$. 4

(*ii*) Prove that every permutation can be expressed as a product of transpositions. 4

(*c*) (*i*) Prove that a skew field contains no divisor of zero. 4

(*ii*) Let (G, o) be a finite group containing an even number of elements. Prove that there exists at least one element a, other than the identity e, in G such that $a \circ a = e$ holds. 4

(*d*) Let a be an element of a group $(G, \circ)$. Then prove the following : 2 × 4

(*i*) $o(a) = o(a^{-1})$;

(*ii*) if $o(a) = n$ and $a^m = e$, then n is a divisor of m;

(*iii*) if $o(a) = n$ then $a, a^2, \ldots, a^n (= e)$ are distinct elements of G;

(*iv*) if $o(a) = n$, then $o(ap) = n$ if and only if p is prime to n.

(*e*) (*i*) Let H be a subgroup of a group G. Prove that the relation ρ defined on G by "$a \rho b$ if and only if $ba^{-1} \in H$" for $a, b \in G$ is an equivalence relation. 3

(*ii*) Define normal subgroup of a group. Prove that intersection of two normal subgroups is again a normal subgroup of a group. 5

6. Answer any *two* questions : 4 × 2

(*a*) State Lagrange's theorem. Is the converse true ? Justify by supporting an example. Prove that $a^{o(G)} = e$, where G be a finite group and $a \in G$. 1 + 1 + 2

(*b*) Show that the ring of matrices

$$S = \left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$

has divisor of zero and does not contain unity. 4

(*c*) Define integral domain. Prove that if integral domain D is of finite characteristic, then its characteristic must be a prime number. 4

7. Answer any *one* question : 3 × 1

(*a*) Define homomorphism of a group. Let $G = (\mathbb{Z}, +)$ and a mapping $\phi : G \to G$ be defined by $\phi(x) = x + 1, x \in G$. Examine if ϕ is a homomorphism. 1 + 2 (*b*) Prove that if p be a prime and a be a positive integer such that p is not a divisor of a then $a^{p-1} \equiv 1 \pmod{p}$. 3

GROUP – C

(*Linear Algebra*)

[*Marks* : 25]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Prove that

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

and hence deduce that

$$\begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha\beta+\beta\gamma+\gamma\alpha).$$ 5

(*ii*) Prove that the homogeneous system $AX = 0$ containing n equations in n unknowns has a non-zero solution if and only if rank $A < n$. 5

(*iii*) Find the eigen values and the corresponding eigen vectors of the following matrix : 5

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

(*b*) (*i*) Reduce the following quadratic form to its normal form. Find the rank and signature of it.

$$2x^2 + 3y^2 + 4z^2 - 4xy + 4yz.$$ 5

(*ii*) Let U and W be two subspace of a finite dimensional vector space V over a field F. Then prove that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$ 5

(*iii*) Investigate for what values of λ and μ the following equations :

$$x + y + z = 6,$$
$$x + 2y + 3z = 10,$$
$$x + 2y + \lambda z = \mu$$

have (*i*) no solution, (*ii*) a unique solution and (*iii*) an infinite number of solutions. 5

9. Answer any *one* question : 8 × 1

(*a*) (*i*) Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of the vector space V over a field F and a non-zero vector β of V can be expressed as

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n,\ a_i \in F.$$

If $a_i \neq 0$, then prove that $\{\alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_{i+1}, \ldots\ldots, \alpha_n\}$ forms also a basis of V. 4

(*ii*) Let A be a square matrix such that $I + A$ is non-singular. Let $\overline{A} = (I + A)^{-1}(I - A)$. Prove that

(*A*) $(I + \overline{A})$ is non-singular

(*B*) $\overline{\overline{A}} = A$. 4

(*b*) (*i*) Prove that the intersection of two subspaces of a vector space V over a field F is a subspace of V. 4

(*ii*) Examine the linear dependence of the set of vectors $\alpha = (1, 2, -3)$, $\beta = (2, -3, 1)$, $\gamma = (-3, 1, 1)$. Hence find the rank of

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 1 & 1 \end{pmatrix}.$$ 4

10. Answer any *one* question : 2 × 1

(*a*) State Cayley-Hamilton theorem. Using this theorem, how can you find out A^{-1} for some square matrix A. 2

(*b*) Show that $S = \{(x, y, z) \in R^3 : 3x - 4y + z = 0\}$ is a sub-space in R^3. 2
