# ON SOME INVENTORY MANAGEMENT PROBLEMS IN UNCERTAIN ENVIRONMENTS

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# ON SOME INVENTORY MANAGEMENT PROBLEMS IN UNCERTAIN ENVIRONMENTS

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**VIDYASAGAR UNIVERSITY**

**For the award of degree of**

**DOCTOR OF PHILOSOPHY**
**IN**
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**BY**

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**September, 2016**

**This Thesis is dedicated to**
**my respected Sir**

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**( Former Prof. of Vidyasagar University)**

# CERTIFICATE

This is to certify that the thesis entitled **“ON SOME INVENTORY MANAGEMENT PROBLEMS IN UNCERTAIN ENVIRONMENTS”** being submitted to the **VIDYASAGAR UNIVERSITY** by **Sri Manoranjan De** for the award of degree of **DOCTOR OF PHILOSOPHY in Science** is a record of bona-fide research work carried out by him under our guidance and supervision. **Sri De** has worked in the **Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University** as per the regulations of this University.

In our opinion, this thesis is of the standard required for the award of the degree of **DOCTOR OF PHILOSOPHY IN SCIENCE**.

The results, embodied in this thesis, have not been submitted to any University or Institution for the award of any degree or diploma.

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# DECLARATION

I, Manoranjan De, do hereby declare that, I have not submitted the results embodied in my thesis – "**ON SOME INVENTORY MANAGEMENT PROBLEMS IN UNCERTAIN ENVIRONMENTS**"or a part of it for any degree/ diploma or any other academic award anywhere before.

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# *List of Publications*

## A. List of Published/Accepted Papers

1. An economic production lot size model for randomly imperfect production system with stock-dependent demand and rework, ***International Journal Operational Research (Accepted, 2015)***, Inderscience Enterprises Ltd.

2. An EPL model with reliability-dependent randomly imperfect production system over different uncertain finite time horizons, ***Journal of Intelligent and Fuzzy Systems (Accepted, 2016)***, IOS Press (SCIE).

3. EPL models for complementary and substitute items under imperfect production process with promotional cost and selling price dependent demands, ***OPSEARCH (Published, 2015)***, SPRINGER.

4. Quality and pricing decisions for substitutable items under imperfect production process over a random planning horizon, ***Hacettepe Journal of Mathematics and Statistics (Accepted, 2016)***, Hacettepe University (SCIE).

## B. List of Communicated Papers

1. Optimum ordering for two substitute items in a news-vendor management with promotional effort on demand using Rough Age based Genetic Algorithm
   Communicated to ***International Journal of Industrial Engineering-Theory, Application and Practice***, Univ. Cincinnati Industrial Engineering.

2. Green logistics under imperfect production system: A Rough age based Multi-Objective Genetic Algorithm approach
   Communicated to ***Computers & Industrial Engineering***, ELSEVIER.

3. EPL models with fuzzy imperfect production system including carbon emission : A fuzzy differential equation approach
   Communicated to ***Journal of Intelligent Manufacturing***, SPRINGER.

4. A learning effected imperfect production inventory model for several markets with fuzzy trade credit period and inflation.
   Communicated to ***International journal of Uncertainty, Fuzziness and Knowledge based system***, World Scientific.

5. A fuzzy imperfect EPL model with dynamic demand under bi-level trade credit policy.
   Communicated to ***Applied Mathematical Modelling***, ELSEVIER.

# List of Acronyms

| | |
|---|---|
| **ACEC** | Average Carbon Emission Cost |
| **ATP** | Average Total Profit |
| **CE** | Carbon Emission |
| **CEC** | Carbon Emission Cost |
| **CPI** | Consumer Price Index |
| **DM** | Decision Maker |
| **DV** | Decision Variable |
| **EOQ** | Economic Order Quantity |
| **EPC** | Environment Protection Cost |
| **EPL** | Economic Production Lot size |
| **EPQ** | Economic Production Quantity |
| **FAGA** | Fuzzy Age based Genetic Algorithm |
| **FDE** | Fuzzy Differential Equation |
| **FEMOGA** | Fast and Elitist Multi-Objective Genetic Algorithm |
| **FM** | Fuzzy Model |
| **FNLOP** | Fuzzy Non-Linear Optimization Problem |
| **FRI** | Fuzzy Riemann Integration |
| **FUCS** | Feasible Constrained Solution Space |
| **FUSS** | Feasible Unconstrained Solution Space |
| **GA** | Genetic Algorithm |
| **GMIR** | Graded Mean Integration Representation |
| **GRG** | Generalized Reduced Gradient |
| **H** | High |
| **IFN** | Intuitionistic Fuzzy Number |
| **IFOT** | Intuitionistic Fuzzy Optimization Technique |
| **IFS** | Intuitionistic Fuzzy Set |
| **IODOS** | Inverse Of Degree Of Substitutability |
| **L** | Low |
| **LFN** | Linear Fuzzy Number |
| **M** | Middle |
| **MOGA** | Multi-Objective Genetic Algorithm |

| | |
|---|---|
| **MOLP** | Multi-Objective Linear Programming |
| **MONLP** | Multi-Objective Non-Linear Programming |
| **MOOP** | Multi-Objective Optimization Problem |
| **MOP** | Multi-Objective Problem |
| **MOPP** | Multi-Objective Programming Problem |
| **NLP** | Non-Linear Programming |
| **O** | Old |
| **ODM** | Optimistic Decision Maker |
| **PDM** | Pessimistic Decision Maker |
| **PFFN** | Parabolic Flat Fuzzy Number |
| **PFN** | Parabolic Fuzzy Number |
| **PPI** | Producer Price Index |
| **PSO** | Particle Swarm Optimization |
| **RAGA** | Rough Age based Genetic Algorithm |
| **RD** | Resultant Demand |
| **RMOGA** | Rough Age based Multi-Objective Genetic Algorithm |
| **RV** | Random Variable |
| **SOLOP** | Single-Objective Linear Optimization Problem |
| **SONLOP** | Single-Objective Non-Linear Optimization Problem |
| **SOOP** | Single-Objective Optimization Problem |
| **TC** | Total Cost |
| **TFN** | Triangular Fuzzy Number |
| **TLBO** | Teaching and Learning Based Optimization |
| **TrFN** | Trapezoidal Fuzzy Number |
| **UPC** | Unit Production Cost |
| **VH** | Very High |
| **VL** | Very Low |
| **VMP** | Vector Minimum Problem |
| **VO** | Very Old |
| **VY** | Very Young |
| **Y** | Young |

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# Chapter 1

# Introduction

## 1.1 Introduction of Operations Research

Operations Research or Operational Research in developed countries usage, is a discipline that deals with the application of advanced analytical methods to help for better decisions. It is often considered to be a sub-field of Mathematics. The terms Management Science, Industrial Engineering, Operations Management and Decision Science are sometimes used as synonym of Operations Research (OR).

Utilizing techniques from other mathematical sciences, such as mathematical modelling, statistical analysis and mathematical optimization, operations research arrives at optimal or near-optimal solutions to complex decision-making problems. Because of its emphasis on human-technology interaction and because of its focus on practical applications, operations research has overlap with other disciplines, notably industrial engineering and operations management and draws on psychology and organization science. Operations Research is often concerned with determining the maximum (of profit, performance, or yield) or minimum (of loss, risk, or cost) of some real-world objective. Originating in military efforts before World War II, its techniques have grown to concern problems in a variety of industries or companies. Today's global markets and instant communications mean that customers expect high-quality products and services when they need them, where they need them. Organizations, whether public or private, need to provide these products and services as effectively and efficiently as possible. This requires careful planning and analysis - the hallmarks of good OR. This is usually based on process modelling, analysis of options or business analytic.

### 1.1.1 Origin of Operations Research

The Operations Research (OR) was introduced during World War II, when the British military management (the U.K and the USA) called upon a group of scientists together to apply a scientific approach to the study of military operations to win the battle. The main objective was to formulate specific proposals and plans for aiding the military commanders to arrive at the decisions on optimal utilization of limited military logistical and armament supports and also to implement the decisions effectively. The effectiveness of operations research in military invokes an interest in OR among other government departments, industries, research and development, etc. In India, operations research came into existence with the opening of an OR unit in 1949 at the Regional Research Laboratory at Hyderabad. An OR unit under Professor P. C. Mahalonobis was established in 1953 in the Indian Statistical Institute, Calcutta to apply OR methods in national planning and survey. He made the first important application of OR in India in preparing the draft of the Second Five Year Plan. The draft plan frame is still considered to be the most scientifically formulated plan of massive economic development of India.

### 1.1.2 Examples of OR in action

• **Inventory Control:** Inventory control is the activity concerned with the management of inventory situations. There are two basic functions of inventory control:

1. Maintaining an accounting record to handle the inventory transactions concerning each inventory item.

2. Deciding inventory replenishment decisions. There are two basic replenishment decisions.

    (a) When is it necessary to place an order (or produce) to replenish inventory?

    (b) How much is to be ordered (or produced) in each replenishment?

• **Supply Chain Management:** Supply chain management is a set of approaches utilized to efficiently coordinate and integrate suppliers, manufacturers, warehouses and stores, so that merchandise is produced and distributed at the right quantities, to the right locations and at the right time, in order to minimize system-wide costs while satisfying service level requirements.
• **Scheduling:** of aircrews and the fleet for airlines, of vehicles in supply chains, of orders in a factory and of operating theaters in a hospital.
• **Facility planning:** computer simulations of airports for the rapid and safe processing of travellers, improving appointment systems for medical practitioners.
• Planning and forecasting: identifying possible future developments in telecommunications, deciding how much capacity is needed in a holiday business.
• **Yield management:** setting the prices of airline seats and hotel rooms to reflect changing demand and the risk of no customer.
• **Credit scoring:** deciding which customers offer the best prospects for credit companies.
• **Marketing:** evaluating the value of sale promotions, developing customer profiles and computing the life-time value of a customer.
• **Defence and peace keeping:** finding ways to deploy troops rapidly. OR techniques are

deployed in defence operations ( viz. administration, intelligence, training etc.) of the air force, army and navy in order to arrive at an optimum strategy to achieve consistent goals.

# 1.2 Basic Concepts and Terminologies

The inventory systems depend on several parameters- such as demand, replenishment, resources, lead time and various types of costs, constraints, etc. Detailed descriptions on these parameters are available in the literature on Inventory Control problems(cf., Hadley and Whitin [96], Naddor [184], etc.).

## 1.2.1 Definitions and Terminologies

**Demand:** Demand refers to the quantity of a commodity required at a given time. It usually depends upon the decisions of people outside the organization which has the inventory problem. The size, rate and pattern can classify the demand into following categories.

*Deterministic demand*

- fixed or constant
- dependent on stock
- dependent on price
- dependent on quality
- dependent on trade credit
- etc.

*Stochastic demand*

- with known distribution
- with unknown distribution
- etc.

*Imprecise demand*

- Fuzzy demand
- Rough demand
- Fuzzy-random demand
- Fuzzy-rough demand
- etc.

In some cases, demand may be represented by vague, imprecise and uncertain data. This type of demand is termed as fuzzy demand. Demand also can be treated as fuzzy-stochastic in nature.

**Complementary Product:** A product that is typically used in conjunction with another product, such that a change in the demand for one product results in a change in the demand for the other. Two goods (A and B) are complementary if using more of good A requires the use of more of good B. For example, the demand for one good (printers) generates demand for the other (ink cartridges). If the price of one good falls and people buy more of it, they will usually buy more of the complementary good also, whether or not its price falls. Similarly, if the price of one good rises and reduces its demand, it may reduce the demand for the paired or complementary good as well. Others examples of complementary products are:

- DVD player and DVD
- Flash-light and battery
- Boot and lace
- Tea and sugar
- Mobile Phone and Sim Card
- Car and fuel

**Substitute Product:** One product is called a substitute for another only if it can be used in exactly the same way and serves the same need. For substitutable items, an increase in the price of one of the products will increase the demand for the substitute item. There are several situations when substitution between products is allowed. Fair price shops, ration shops, supermarkets, etc. are the examples where substitutions between the products are very common. Substitution frequently occurs between different brands of rice, wheat, soda, shampoo, toothpaste, hair oil, etc. in a stationary shop. Some examples of substitute products are:

- Petroleum and natural gas
- Tea and coffee
- Margarine and butter
- Sprite and 7-UP

**Replenishment/Supply:** Replenishment can be categorized according to size, pattern and lead time. Replenishment size refers to the quantity or size of the order to be received into inventory. The size may be constant or variable, depending on the type of inventory system. Replenishment patterns refer to how much amount of inventory is added to the inventory stock. The replenishment patterns are usually instantaneous or uniform. Normally, replenishment are made either in once or batch-wise.

**Time/Planning Horizon:** The time period over which the inventory level will be controlled is called the time horizon. It may be finite or infinite depending upon the nature of the inventory system of the commodity.

**Constraints:** Constraints in inventory system deal with various properties that some way place limitations imposed on the inventory system. Constraints may be imposed on the amount of investment, available space, resources and finance, the amount of inventory held, average inventory expenditure, number of orders, etc. These constraints can also be fuzzy, random and fuzzy-random in nature.

**Fully Back-logged/Partially Back-logged Shortages:** During stock-out period, the sales and/or goodwill may be lost either by a delay or complete refusal in meeting the demand of the customers. In the case where the unfulfilled demand for the goods can be satisfied completely at a later date, then it is a case of fully back-logged shortages, i.e., it is assumed that no customer walk away during this period and the demand of all these waiting customers is met at the beginning of the next period. Again, it is normally observed that during the stock-out period, some of the customers wait for the product and others walk away. Such a phenomenon is called partially backlogged shortages.

**Salvage:** During storage, some items get partially spoiled or damaged, i.e., some items loose their utility. But in a developing country, it is normally observed that some of these are sold at a lower price (less than the purchasing price) to a section of customers and this gives some revenue to the management. This revenue is called salvage value.

**Inventory Cost:** The cost relevant to inventory decision making, namely

- Ordering or Setup Cost
- Purchase Cost
- Shortage or Penalty Cost
- Holding or Carrying Cost
- Unit Production Cost
- Unit Transportation Cost

**Ordering or Set-up Cost:** It is the cost associated with the expense of issuing a purchase order to an out-side supplier or setting up machines before internal production. These costs also include clerical and administrative costs, telephone charges, telegram, transportation costs, loading and unloading costs, watch and ward costs, etc. Generally, this cost is assumed to be independent of the quantity ordered for or produced. In the costs like transportation cost, etc., some part of it may be quantity dependent.

**Holding or Carrying Cost:** It is the cost associated with the storage of the inventory until its use or sale. It is directly proportional to the quantity in inventory and the time for which the stocks are held. This cost generally includes the costs such as rent for storage space, interest on the money locked-up, insurance, taxes, handling, etc.

**Purchase or Unit Cost of an Item:** It is the unit purchase price to obtain the item from an external source or the unit production cost for internal production. It may also depend upon the demand when production is done in large quantities as it results in reduction of production cost per unit. Also, when quantity discounts are allowed for bulk orders, unit price is reduced and depends on the quantity purchased or ordered. Unit production cost is also production dependent. For example, if one worker is needed to tend the machine, then as more units are produced per unit time, the wages of the worker spread over more units.

**Shortage or Stock-out Cost or Penalty Cost:** It is the penalty incurred when the stock proves inadequate to meet the demand of the customers. This cost parameter does not depend upon the source of replenishment of stock but upon the amount of inventory not supplied to the customer.

**Unit Transportation Cost:** The cost by which one unit product is transported from source or availability of the product to destination or the retailer.

**Wholesaler :** Wholesaler is a person or firm that buys large quantity of goods from manufacturers/ producer/ supplier. Thus a wholesaler takes the inventory in bulk and delivers a bundle of related product to retailers. Wholesaler is also known as distributor. A distributor is typically an organization that takes ownership of significant inventories of products. A wholesaler is a middleman between a manufacturer and retailers of the product. The wholesaler makes money by buying the product(s) from the manufacturer at a lower price- usually through discounts based on volume buying.

**Manufacturer :** One who makes products through a process involving raw materials, components, or assemblies, usually on a large scale with different operations divided among different workers. Manufacturer is also known as producer.

**Retailer :** One who sells goods or commodities directly to consumers is known as retailer. A retailer purchased items from the manufacturer or wholesaler and sold to the end user at a marked up price. Retailers stock inventory and sell in smaller quantities to the general public.

**Supply chain :** A supply chain is a system of organizations, people, activities, information, and resources involved in moving a product or service from supplier to customer. Supply chain activities transform natural resources, raw materials, and components into a finished product that is delivered to the end customer. In sophisticated supply chain systems, used products may re-enter the supply chain at any point where residual value is recyclable.

**Promotional Effort :** Generally, promotion communicates with the public in an attempt to influence them toward buying products and/or services. Promotion includes all the ways available to make a product and/or service known to and purchased by customers and clients. The particular activity to promote the business, product or service is called promotion. A store might advertise that it's having a big promotion on certain items. For instance, a business person may refer to an ad. as a promotion. Price discount, trade credit, free gifts etc. are well established tools of promotional effort.

**Advertisement :**The non-personal communication of information which is usually paid for and usually persuasive in nature about products, services or ideas by identified sponsors through the various media, is regarded as Advertising.

**Defective Product :** A product is in a defective condition, unreasonably dangerous to the user, when it has a propensity or tendency for causing physical harm beyond that which would be contemplated by the ordinary user, having ordinary knowledge of the product's characteristics commonly known to the foreseeable class of persons who would normally use the product.

**Recycling :** Recycling is a process by which materials (waste) are change into new products to prevent waste of potentially useful materials, reduce the consumption of fresh raw materials, reduce energy usage, reduce air pollution (from incineration) and water pollution (from land filling) by reducing the need for "conventional" waste disposal, and lower greenhouse gas emissions. Recycling is a key component of modern waste reduction. Recyclable materials include many kinds of glass, paper, metal, plastic, textiles, and electronics. Materials to be recycled are either brought to a collection center or picked up from the curb side, then sorted, cleaned, and reprocessed into new materials.

**Trade Credit :** In recent competitive market, manufacturer /wholesaler /retailer frequently offers delay period for settling the account on purchasing amount of units (greater than or equal to a certain amount fixed by the wholesaler/manufacturer). This is termed as trade credit period. Depending upon the credit period, demand of an item increases or decreases. If credit period is offered to the retailers only by the supplier, it is called one level trade credit. On the other hand if both the supplier and the retailer offer credit period to his/her retailer and customers respectively, it is called two level trade credit. Again, if credit period is offered depending upon some conditions (like amount of purchase should exceed some label, frequency of order etc.), it is called conditionally delay in payment or conditional credit period.

**Inflation and time value of money:** Inflation is a present increase in the level of consumer price or a persistent decline in the purchasing power of money, caused by an increase in available currency and credit beyond the proportion of available goods and service. It is the rate at which the general prices for goods and services are rising and subsequently, purchasing power is falling. With the increase of inflation rate, more amount of money is to be paid for the same quantity of commodity. As for example, if the inflation rate is $1\%$, a \$ 5 of pen will cost \$ 5.05 in a year. Mathematically, Buzacott [30] assumed that cost at time $t$, $\phi(t)$, becomes $\phi(t+\delta t) = \phi(t) + i\,\phi(t)\delta t$ at time $(t+\delta t)$ (where $\delta t$ is sufficiently small) when a constant inflation rate $i$ (\$/unit) exists in the market, i.e.,

$$\begin{aligned} & \phi(t+\delta t) = \phi(t) + i\,\phi(t)\,\delta t \text{ as } \delta t \to 0 \\ \Rightarrow \quad & \frac{\phi(t+\delta t) - \phi(t)}{\delta t} \to i\,\phi(t) \text{ as } \delta t \to 0 \\ \Rightarrow \quad & \frac{d\phi(t)}{dt} = i\,\phi(t) \qquad \Rightarrow \qquad \frac{d\phi(t)}{\phi(t)} = i\,dt \end{aligned}$$

which yields a solution as $\phi(t) = \phi(0)e^{i\,t}$, where $\phi(0)$ is the cost at time $t = 0$.

On the other hand, time value of money is one of the basic concept of finance. We know that if we deposit money in a bank we will receive interest. For example, \$ 1 today invested for one year at 7% return would be worth \$1.07 in a year. Because of this, we prefer to receive money today rather than the same amount in the future. Money we receive today is more valuable to us than money received in the future by the amount of interest we can earn with the money. It is the changes in purchasing power of money over time.

So, if $i\%$ and $r\%$ are the annual inflation and interest rate respectively, resultant effect of inflation and time value of money (i.e., increased rate of cost) on purchasing a unit of item in future is $(i - r)\%$. So if $\phi(0)$ is the cost of an item at time $t = 0$, its cost at time $t$, $\phi(t) = \phi(0)e^{-Rt}$, where $R = (r - i)$ is called discount rate of cash flow.

**Selling Price :** Selling price is the price at which something is offered for sale. Generally, it is not fixed for long time for a particular item.

**Learning Effect :** In many realistic situations, because the firms and employees perform the same task repeatedly, they learn how to perform more efficiently. Therefore, the actual production and/or set-up cost of a job is less rather than earlier in the production process. This phenomenon is known as the 'learning effect' in the literature. Different types of learning effects have been demonstrated and extensively studied in a number of areas. One of them is processing/set-up time. Production process of some jobs to be faster than those of others, i.e., the learning is some time job-dependent.

**Carbon Emission :** Carbon emission mainly the green house gas emissions is the release of carbon into the atmosphere. It mainly contributes to climate change. Since greenhouse gas emissions are often calculated as carbon dioxide equivalents, they are often referred to as "carbon emissions" when discussing global warming or the greenhouse effect. Since the industrial revolution the burning of fossil fuels has increased, which directly correlates to the increase of carbon dioxide levels in our atmosphere and thus the rapid increase of global warming.

**Cap and Trade :** A cap and trade system, also known as emission trading, consists of governmental policies and economic tools that try to control pollution. They set limits on the quantity of greenhouse gas emitted and provide credits as incentives in order to achieve set targets.
*Cap :* Large-scale emitters of greenhouse gases such as corporations are given limits in the form of emission permits for how much they can pump into the atmosphere. The total number of permits issued throughout any given region cannot exceed the overall cap for that system. The permits are usually issued in quantities equivalent to tons of carbon dioxide, which are slowly declining as limits become stricter over time.
*Trade :* Some entities will have an easier time staying within the limits of their emission permits. Efficient companies that produce fewer greenhouse gas emissions than they are allotted can sell their excess permits to other companies that need them. The cost for purchasing such permits is determined by the market.

**Carbon Tax :** A carbon tax is usually defined as a tax based on greenhouse gas emissions (GHG) generated from burning fuels, coal, and gas, aimed at reducing the production of greenhouse gases. It puts a price on each tonne of GHG emitted, sending a price signal that will, over time, elicit a powerful market response across the entire economy, resulting in reduced emissions. It has the advantage of providing an incentive without favouring any one way of reducing emissions over another. By reducing fuel consumption, increasing fuel efficiency, using cleaner fuels and adopting new technology, businesses and individuals can reduce the amount they pay in carbon tax, or even offset it altogether.

**Product Quality :** Product quality means to incorporate features that have a capacity to meet consumer needs (wants) and gives customer satisfaction by improving products (goods) and making them free from any deficiencies or defects.

**Reliability :** Reliability is defined as the probability that a device will perform its intended function during a specified period of time under stated conditions. Mathematically, this may be expressed as,

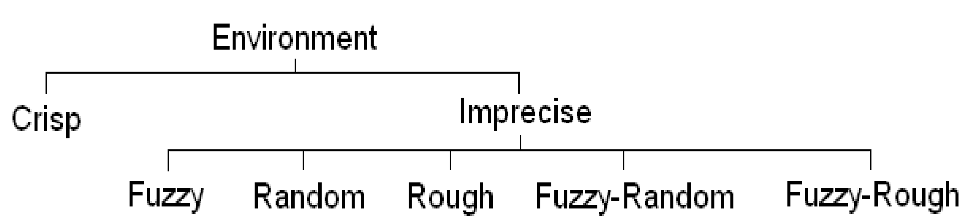
$$R(t) = Pr(T > t) \int_t^\infty f(x)dx$$

where $f(x)$ is the failure probability density function and is the length of the period of time (which is assumed to start from time zero). There are a few key elements of this definition:

- Reliability is predicated on "intended function:" Generally, this is taken to mean operation without failure. However, even if no individual part of the system fails, but the system as a whole does not do what was intended, then it is still charged against the system reliability. The system requirements specification is the criterion against which reliability is measured.
- Reliability applies to a specified period of time. In practical terms, this means that a system has a specified chance that it will operate without failure before time . Reliability engineering ensures that components and materials will meet the requirements during the specified time. Units other than time may sometimes be used.
- Reliability is restricted to operation under stated (or explicitly defined) conditions. This constraint is necessary because it is impossible to design a system for unlimited conditions. A Mars Rover will have different specified conditions than a family car. The operating environment must be addressed during design and testing. That same rover may be required to operate in varying conditions requiring additional scrutiny

### 1.2.2 Different Environments

The parameters, like demand, inventory cost (viz., unit production/purchasing cost, set-up cost, holding cost, shortage cost, transportation cost, advertisement cost, etc.), lead time, quantity, available resources, goals, etc., involved in the inventory system may be deterministic (crisp/precise) or some of them may be non-deterministic (i.e., imprecise like fuzzy, rough, random, fuzzy-rough, fuzzy-random etc.). Thus the environments in which inventory models are developed can be classified as follows:



**Crisp Environment :** The deterministic and precisely defined system parameters and the resources form the environment as a crisp one.

**Fuzzy Environment :** It is an environment in which all or some inventory parameters, resources and/or goal(s) of objective(s), etc., are imprecise and vague (i.e., inexact due to human perception process). It is uncertain in non-stochastic sense and called fuzzy. The fuzzy parameters or quantities are characterized by membership functions.

**Stochastic Environment :** In this environment some of the parameters like, lead time, demand, resources, inventory costs, etc., are random in nature and specified by probability distributions. It may happen that the demand or any factor of a commodity in the society is uncertain, not precisely known, but some past data about it is available. From the available records, the probability distribution of demand or any other factor of the commodity can be determined and with that distribution the problem can be analysed and solved.

**Fuzzy-Stochastic Environment :** It is an combination of both stochastic and fuzzy environments. Here, some parameters, goals, etc., are fuzzy and some others are random. For example, in an inventory control problem, holding cost may be imprecise and demand as random. On the other hand, an inventory parameter/variable may be both fuzzy and random together. The statement - the probability of having large demand of football world cup ticket contains both impreciseness and randomness together. Here large is fuzzy and 'probability' represents randomness.

**Rough Environment :** In this environment some of the parameters like, demand coefficients, inventory costs, planning horizon, etc., are rough in nature and represented by rough sets.

**Fuzzy-Rough Environment :** It is an environment in which some of the parameters are fuzzy-rough in nature. Fuzzy rough variable is a measurable function from a rough space to the set of fuzzy variables. More generally, a fuzzy rough variable is a rough variable taking fuzzy values.

## 1.3 Historical Review on Inventory Models

The control and maintenance of inventory, i.e., over-stocking and under-stocking, is a problem common to all organizations in any sector of the economy. Inventory problems in deterministic environment have been studied by several researchers since early twentieth century. The earliest simple EOQ model has been developed by Ford Harris [99] of Westinghouse Corporation, USA. After few years R.H. Wilson [269] published same type of formula and it has been named as Harris-Wilson formula or Wilson's formula. Since then, lot of research work have been reported by several researchers of inventory control problems in different environments and the process still going on. Full length book in the field of inventory control systems have been presented by several authors [7, 97, 184, 215, 267]. Few of the existing literature in this field are reviewed and presented below.

### 1.3.1 Models with stock dependent demand

The class of inventory models with initial stock or current inventory-level dependent demand rates has recently received considerable attention. In reality, in the context of present competitive market, the inventory/stock is decoratively displayed through electronic media to attract the customers and thus to push the sale. We often see decorative displays of items in shops that are used as "psychic stock" stimulate more sales of some retail items. Baker and Urban [10] and Urban [259] formulated EOQ models with stock dependent demand where the dependency is of simple form. Mandal and Phaujder [168] considered linear form of stock-dependent demand, i.e. D = c + dq, where Mandal and Maiti [169] and Maiti and Maiti [163] and others took the demand as $D = dq^{\beta}$. Recently, Jiangtao *et al.* [117] considered a multi-item inventory model for perishable items where the demand rates of the items are stock-dependent and two level trade credit is allowed with restriction on inventory capacity. Yang *et al.* [277] developed a supply chain policy for a single manufacturer and a single retailer with a single product assuming stock dependent demand for retailer. Tyagi *et al.* [258] and Chakraborty *et al.* [37] investigated models for deteriorating items with stock-dependent demand in crisp and fuzzy environments respectively.

### 1.3.2 Models on imperfect production process

In the literature, few EPL models are available for imperfect units. Rosenblatt and Lee [217] studied the effects of an imperfect production process on the optimal production run time by assuming that time to out-of-control state is exponentially distributed. Salameh and Jaber [225] and Lin [145] studied the EOQ/EPQ model for the items with imperfect quality and proposed discount sales for them. Hayek and Salameh [100] derived an optimal operating policy for the finite production model under the effect of reworking of imperfect quality items. They assumed that all defective units are repairable and allowed back-orders. Chiu [55] extended the work of Hayek and Salameh [100] and examined an EPQ model with defective items reworking the repairable units immediately. Sana [227] presented an EPL model with random imperfect production process and defective units were repaired immediately when they were produced. Sarkar *et al.* [230] obtained the optimal reliability for an EPL model connecting process reliability with imperfect production system. Barzoki *et al.* [11] investigated the effects of imperfect production on the works in process inventory and evaluated the optimum lot size for the minimum total cost. Here, some imperfect products were reworked and others were sold at a reduced price. Krishnamoorthi and Panayappan [133] have studied an EPQ model that incorporated imperfect production quality, not screening out proportion of defects and thereby passing them on to customers and resulting in sales returns. Not all of the defective units are repairable, a portion of them are scrap and discarded beforehand. Chen *et al.* [45] developed an alternative optimization solution process to determine the optimal replenishment lot size considering imperfect rework and multiple shipments. Taleizadeh *et al.* [247] presented an EPQ model with rework process for a single stage production with one machine. Recently Chen [49] investigated a problem with production preventive maintenance, inspection and inventory for an imperfect production process. Pal *et al.* [195] formulated an EPQ model with imperfect production process and stochastic demand. Cárdenas-Barrón *et al.* [35] presented an easy method for the results of Chen *et al.* [45] and Chiu *et al.* [54] deriving the optimal number of replenishment and shipments jointly. Krishnamoorthi and Panayappan [134] evaluated optimal lot size minimizing the total cost for an EPQ model without and with shortages allowing imperfect production system and immediate rework of the imperfect units. Rad *et al.* [207] developed a model of an integrated vendor-buyer supply chain with imperfect production and shortages. Sarkar *et al.* [234] revisited the EPQ model with rework process at a single stage manufacturing system with planned back-orders. The production system was assumed to be imperfect having random defective rates. Recently, Taleizadeh and Wee [248] extended a multi-product single machine manufacturing system with manufacturing capacity limitation and immediate reworking of imperfect products allowing partial back-ordering.

### 1.3.3 Models with complementary and substitute products

Now-a-days, due to strong competitive market, retailers prefer the business / production of several items with the hope that due to dull market, if one item does not fetch profit, the other one will save the situation. Among these multi-items, there may be some substitute and / or complementary items. There are several investigations for substitutable items in the newsboy setting. Das and Maiti [60] studied a single period newsboy type inventory problem for two substitutable deteriorating items with resource constraint involving a wholesaler and several retailers. Stavrulaki [244] modelled the joint effect of demand (stock-dependent) stimulation and product substitution on inventory decisions by considering a single period and stochastic demand. Gurler and Yilmaz [93] assumed substitution of a product when the other one is out of stock and presented a two level supply chain newsboy problem with two substitutable products. Kim and Bell [130] investigated the impact of the symmetrical and asymmetrical demand substitution on optimal prices, production levels and revenue and the impact of changes in the production cost on the optimal solutions. Recently Zhao *et al.* [291] developed a two-stage supply chain where two different manufacturers compete to sell substitutable products through a common retailer and analysed the problem using game theory. Here the consumer demand function is defined as a linear form of the two products' retail prices-downward slopping in its own price and increasing with respect to the competitor's price. In marketing substitutable items, the demand of an item is sometimes affected by the other, depending upon the other item's inventory level (Maity & Maiti, [167]). Ahiska and Kurtul [3] presented a one-way product substitution strategy for a stochastic manufacturing/ re-manufacturing system and illustrated using real life data.

### 1.3.4 Models with carbon emission

Recently, in the literature, several inventory models have been analysed under distinct carbon emission policies. Hua *et al.* [107] examined both analytically and numerically the impact of carbon trade, carbon price and carbon cap on order decisions and total cost of an environmental inventory model based on the classical EOQ model. Bouchery *et al.* [25] studied an EOQ model in multi-objective decision making problems with sustainability criteria. Song and Leng [243] studied a news-vendor setting considering a mandatory CE capacity, Carbon Emission (CE) tax and cap and trade systems under stochastic environment. A multi-sourcing deterministic lot-sizing model with carbon constraint was investigated by Absi *et al.* [1]. An EOQ model with a constraint on the emission of carbon was considered by Chen *et al.* [50]. They concluded that without increasing cost significantly, CE can be reduced through operational adjustment. Du *et al.* [75] investigated an emission dependent supply chain consisting of one single emission-dependent manufacturer and one single permit supplier. They derived the influence of ”cap and trade” system on the decision making. Jaber *et al.* [112] examined the effect of different legislative systems such as carbon tax, emission penalty and a combination of tax and penalty on a two echelon supply chain model. Jin *et al.* [118] studied the impact of three CE reduction policies including cap and trade and carbon tax regulations on a major retailer determining its supply network design and choice of transportation. Zakeri *et al.* [287] presented an analytical supply chain planning model that can be used to examine the supply chain performance at the tactical/operational planning level under carbon pricing and trading schemes. He *et al.* [101] examined the production lot sizing issues of a firm under cap and trade and carbon tax regulations. Xu *et al.* [276] studied the joint production and pricing decisions for products of a production firm under cap and trade and carbon tax regulations. They compared the effect of two regulations on the total carbon emissions, the firm’s profit and social welfare.

### 1.3.5 Models allowing credit period

In the last two decades, the inventory models with trade credit have been widely studied by several researchers. The concept of trade credit was first introduced by Haley and Higgins [98]. Goyal [90] was the first who established an EOQ model with a constant demand rate under the condition of permissible delay in payments. Later, Aggarwal and Jaggi [2] generalized the EOQ model from non-deteriorating items to deteriorating items. Jamal *et al.* [115] further extended the EOQ model to allow for shortages. Thereafter, Teng [251] amended the model by using selling price to calculate the revenue instead of unit cost, and obtained an easy analytical closed-form solution. Afterwards, Huang [110, 111] proposed two levels of trade credit policy where the supplier would offer the retailer a delay period for payment and the retailer also adopts the trade credit policy to stimulate his/her customer demand. Furthermore, he also assumed that the retailer’s trade credit period offered by supplier, M is not shorter than the customer’s trade credit period offered by the retailer, N (M≥N). Then Liao [141] further generalized Huang's model to an EPQ model for deteriorating items. Subsequently, Teng [252] established optimal ordering policies for a retailer to deal with bad credit customers as well as good credit customers. Lately, Ouyang *et al.* [192] considered two-level trade credit link to order quantity. Seifert *et al.* [236] organized a review of trade credit literature and provided a detailed agenda for future research. Recently, Chen *et al.* [46] discussed the retailer's optimal EOQ/EPQ when the up-stream trade credit is linked to order quantity or when the down-stream trade credit is only a fraction of the purchase amount. Liao *et al.* [143] derived optimal strategy for deteriorating items with capacity constraints under two-level trade credit. Chung *et al.* [58] established a new EPQ inventory model for deteriorating items under two levels of trade credit, in which the supplier offers to the retailer a permissible delay period and simultaneously the retailer in turn provides a maximal trade credit period to its customers in a supply chain system comprised of three stages. In the next year, Chung *et al.* [59] adopted the rigorous methods of mathematical analysis in order to develop the complete solution procedures to locate the optimal solution removing shortcomings in the earlier investigation by Ouyang *et al.* [191]. Recently, Ouyang *et al.* [190] proposed an integrated inventory model with capacity constraint and a permissible delay payment period that is order-size dependent.

### 1.3.6 Models with inflation and time value of money

At present, the effect of inflation can't be ignored as the economy of any country changes rigorously due to high inflation. Considering this effect on inventory costs, first impetus was given by Buzacott [30]. Among others, Beirman and Thomas [19], Datta and Pal [65], Ray and Chaudhuri [214] studied some EOQ models with linear time-varying demand taking inflation and time value of money into account. Moon and Lee [180] presented an EOQ model with inflation and time value of money. Wee and Law [264] addressed an inventory problem with finite replacement rate of deteriorating items incorporating the effect of inflation and time value of money. In the same year, Chang [42] proposed an inventory model for deteriorating items with trade credit under inflation. In recent years, Chang [42], Jaggi *et al.* [113], Maiti [160, 161], Sana [226, 227] and Sarkar *et al.* [229, 232] and others presented inventory models in this direction.

### 1.3.7 Models with Uncertainty (Impreciseness and Randomness)

The first publication accommodating the uncertainty in non-stochastic sense was in 1965 by Prof. Zadeh [284]. After that extensive research works have been done in this area [43, 76, 148, 201]. But applications of fuzzy sets in inventory control problems are around 25-30 years. Among these works one can refer the works of Park [201], Roy and Maiti [218], Mandal and Maiti [170], Alonso-Ayuso *et al.* [5], Wee *et al.* [265], etc. Lee and Yao [140] developed an EPQ model considering fuzzy demand and fuzzy production quantity. After one year, Yao and Lee [279] presented an inventory model - (i) with back order and (ii) without back order in fuzzy situations considering the fuzzy numbers. Katagiri and Ishii [125] developed their inventory model under fuzzy shortage cost. Ouyang and Chang [189] developed an inventory model with fuzzy lost sales. Dey and Maiti [70] presented an EOQ model with fuzzy lead-time under inflation and time-value of money. Till now, Fuzzy Differential Equation (FDE) and fuzzy integration are little used to solve fuzzy inventory models [92], though the topics on fuzzy differential equations have been rapidly growing in the recent years.

The first impetus on solving FDE was made by Kandel and Byatt [121]. After two years, an extended version of their work has been published by them [122]. Some notable papers in this direction are due to Petrovic *et al.* [202], Buckley and Feuring [27], Chalco-Cano and Roman-Flores [39, 40], etc. On the other hand, study on fuzzy integration was initiated by Sugeno [245]. Dubois and Prade [77, 78] presented two most valuable research papers on fuzzy integration. Sims and Wang [242] gave a good review of this subject. After that several researchers investigated different procedures for fuzzy integration. Wu [270] introduced the concept of fuzzy Riemann integral and its numerical integration.

Petrovic *et al.* [202] considered the newsboy problem with fuzzy demand and fuzzy inventory costs. They considered two fuzzy models one with (i) imprecisely described discrete demand and other with (ii) imprecisely estimated unit holding and unit shortage costs. Generally, fuzzy inventory models are developed considering some of the inventory parameters as fuzzy in nature [18, 43, 140, 163, 223, 265]. To reduce the objective function, they defuzzified the fuzzy parameters to a crisp one by either defuzzification methods or following possibility/necessity measure of fuzzy events. Finally they solved the reduced crisp model to determine decision for the Decision Maker (DM). In the existing literature, little attention has been paid on fuzzy demand and fuzzy production rate. Wee *et al.* [265] developed a multi-objective joint replenishment inventory model of deteriorating items, where demand is stock dependent and fuzzy in nature. They solved the corresponding crisp model and fuzzified the total profit and return on inventory investment for optimal decisions. But occurrence of fuzzy demand/ production rate leads to FDE and fuzzy integration for the formulation of the model. It is better to formulate these types of problems using FDE for more accuracy of inventory decisions at the present day competitive market.

After the second world war, more attention was focused on the stochastic nature of inventory problems. To study the earliest publications, one can follow the research papers [24, 97, 203]. Among others, Kalpakam and Sapan [120], Kodama [131], Hill *et al.* [103], etc. have developed their models with probabilistic lead time or probabilistic time scheduling or uncertain quantity receiving or random supplying. Bookbinder and Cakanyildirim [23] developed a continuous review inventory model under random lead-time. Cakanyildirim *et al.* [31] extended the model to a continuous review inventory model under lot-size dependent random lead-time. Das *et al.* [64] considered a stochastic inventory problem with fuzzy storage cost. Lifetime of the products, specially the seasonal products for which the planning horizon fluctuates every year depending upon the environmental effects, is finite and imprecise (fuzzy or random) in nature. Moon and Yun [181] first developed an EOQ model with Random Planning Horizon. After that large number of research papers have been published incorporating this assumption [180, 219, 221]. According to the author's best knowledge, very few research papers have been published in fuzzy-stochastic environment. Though some researchers have given attention in this field, still there are lot of scope for research work to develop/modify and to solve a number of real-life inventory models in fuzzy-stochastic environment [165].

Recently, new type uncertain variables such as – rough, fuzzy-rough, random-rough are much used in the research field of science and technology including inventory control problems [147, 152, 157, 239, 272–274]. Though some researchers are interested during last few years to deal with the above mentioned uncertainties, still there are lot of scope for the inventory practitioners to develop and solve real-life inventory models in rough, fuzzy-rough or random-rough environments.

## 1.4 Motivation and Objective of the Thesis

### 1.4.1 Motivation of the Thesis

The major phases of OR are identification of the problem correctly, including the objectives, alternative courses of actions and constraints of the system. Based on the above consideration, the next step is to construct a model of the problem or the system under study. The models show the relation and inter-relation between an action and the reaction or between cause and effect. The models also enable to forecast the factors which are crucial for the system. Once a model is formulated, it is possible to analyse the problem. For an appropriate constructed model, it is required to collect some information from well kept records, from current tests, from an experiment. Such information are termed as input data. After formulating the model and collecting the input data, the next step is to obtain a solution procedure.

In OR, inventory process is one of the important application areas. In management system, the studies of inventory control problem have tremendous importances. It protects the system from fluctuation of demand, provide better services to the system, keeps smooth flow of raw-materials, aids, reduces the risk of loss, helps to minimize the workload, manpower and labour, facilitates, cost accounting activities, avoids duplication of order, stock and many more. Inventory problems for fixed (deterministic) parameters have been studied ever since early twentieth century. For these purpose, the readers may refer to Harris [99], Hadley and Whitin [96], Naddor [184].

In inventory control system, high level of inventory attracts more visibility and also may imply that the goods are popular and fresh. Thus the inventory problem with stock dependent demand is addressed by several researchers to reflect realistic circumstances. The problem of stock dependent demand has been studied in both empirical and theoretical papers. Empirical evidence of such demand of specific product has been provided in [10, 163, 169, 259]. Moreover, the theoretical models are proposed for developing optimal policies in [37, 117, 168, 258]. Production disruption is a very familiar event in real life production process. This disruption may be defined in the form of defective units of the item. As mentioned in § 1.3.2, most of the production inventory models are investigated with the assumptions of known proportion of defective units and fixed inspection state. Besides this, at present in case of economic condition throughout the world, one can't ignore the "psychic stock" of goods and period of uncertain out-of-control state.

After all these studies, some lacunas exist in the formulation of inventory models with stock dependent demand and imperfect production.

- In every manufacturing process, it is fact that environment is disturbed to some extent and for that, now-a-days, attention is paid not to pollute the environment. Till now, very few have introduced the environment protection cost (EPC) in EPL models, which again varies with the rate of production.

- Few investigators have considered the non-instantaneous out-of-control state (starting point of imperfect production) to be random during the production time, but none has imposed it as a chance constraint for the system.

- Defective production rate normally increases with the time elapsed from the out-of-control state and the production rate. Sana [227] considered this with constant demand and fully reworked defective units. None has investigated this phenomena in conjunction with stock-dependent demand and environmental protection cost.

Considering these facts, the author has formulated an **EPL model (Model-3.1) with stock dependent demand, random out-of-control state (including chance constraint) and EPC**.

In a production process, the defectiveness of the item depends on the reliability of the machine [230, 233]. For a long run process, learning knowledge [20, 21, 82] is an important factor to the DM. Moreover, it is generally observed that specially for electrical or electronic goods, the life time of the item is not infinite. It is finite, but not predictable, rather it fluctuates for different items. This phenomenon has inspired us to formulate **Model-3.2** which is constructed in terms of an **EPL model with reliability dependent random defective units in out-of-control state over different uncertain finite time**

**horizons.**

In reality, for multi item inventory system, considerable savings may be achieved by the co-ordination of replenishment for a group of items. When inventory for a particular item has been exhausted, price of the item become extremely high and / or quality of the item is reduced to very low, then the demand of that item is met by another substitutable item. Consequently the demand of a complementary item exogenously increase the demand of the relative item, such as mobile phone and sim card, tea and sugar, car and fuel etc. Kim and Bell [130], Maity and Maiti [167] and others researchers (cf. § 1.3.3) investigated the impact of the symmetrical and asymmetrical demands for substitutable and complementary items. But none has considered the following phenomena.

- A production-marketing system for complementary and substitute items under randomly imperfect production process and budget limitation.
- An investigation in conjunction with advertisement / promotional cost, selling price dependent demand and EPC.
- Commencement of imperfect units depends inversely on production rate.
- A part of the set up cost of the business system depends on the production rate.

These concepts motivated us to formulate an **EPL model for complementary and substitute items under imperfect production process with promotional cost and selling price dependent demands (Model-4.1).**

Several authors (Moon and Yun [181], Roy *et al.* [221], Guria *et al.* [94], Manna *et al.* [172]) suggested that for any kind of business process, planning horizon is neither infinite nor finite (fixed or deterministic). They established the concept of random planning horizon for single/multi-item EOQ or EPQ model. But none conceived the idea of random planning horizon for substitutable items. Besides that quality is an important criteria for substitutable items. Moreover, following facts are to be considered in an inventory control system.

- Learning effect on the set up cost of the system and maintenance of the machinery system is realistic phenomena.
- Product substitution depends on the joint effect of price and quality or on the basis of either price or quality.
- Quality improvement cost which is a function of quality of an item, is a part of UPC.
- Development of an efficient heuristic algorithm (FAGA) is necessary to solve such a complex inventory problem in uncertain environment.

Keeping all these in mind, we construct the **Model-4.2 (Quality and pricing decisions for substitutable items under imperfect production process over a random planning horizon).**

In certain situations, the inventory process terminates after a small duration of time which implies that the entire inventory on hand has no value after that duration. In order to control this, news-boy problem is very common in our daily life. Here the DM makes the decisions about the level of inventory for a very short period. This is the case, for example with the products such as daily news paper, magazine, X-mass cards, etc. Several extensions of the news-boy model have been reported in the literature [60, 198, 199, 261] in terms of the assumptions- substitutable item, random demand, budget constraint, etc. But, till now none has considered a news-boy problem with

- Promotional effects on the random demands to boost the demand.
- Substitute items- one substitute the other when the latter is out of stock.
- Budget constraints on purchasing or promotional or both of these costs.
- A heuristic optimization method named as "Rough Age based Genetic Algorithm (RAGA)" is introduced for approximate single objective optimal solutions.

Therefore there is a strong motivation to construct the **Model-4.3** entitled **"Optimum ordering for two substitute items in a news-vendor management with promotional effort on demand using Rough Age based Genetic Algorithm."**

Some developed countries (UK, USA, Germany, etc.) have made a climate policy called "carbon-tariffs policy" on the production firms. This policy has a significant impact on the social welfare. For this reason, the production inventory management system is urged to consider the fact of carbon emission during the production of an item.

Uncertainty is the only certain phenomena in the world of uncertainty. Now-a-days, in the competitive market, the DM of the management system can't take the risk of considering fixed information data. Again the impreciseness (fuzziness) of the parameters / variables is defuzzified in different ways. During the last two decades, a lot of literatures are available in this context [27, 40, 76, 148, 201, 218, 270, 284, 288]. All these motivated us to consider the **Model-5.1** and **Model-5.2**. The other reasons for adaptation of **Model-5.1** are

- A production system producing imperfect quantities after the passage of sometime from the occurrence of production and emitting carbon from the beginning is considered and introduced under all the available carbon rules and regulations of the worldwide countries.
- A heuristic optimization method has been introduced– Rough age based MOGA for multi-objective optimizations in fuzzy environment is presented.

Although there are lot of research works in fuzzy environment, but most of them were not formulated following fuzzy differential equation (FDE) approach. Hence the cost function and sales revenue have become expressions as fuzzy integral (FRI). Moreover, the concept of Intuitionistic Fuzzy Set (IFS) can be seen as an alternative approach to define a fuzzy set instead of normal fuzzy set. Therefore, we consider an **"EPL models with fuzzy imperfect production system including carbon emission: A fuzzy differential equation approach" (Model-5.2)** under the strong motivation of the following.

- As the defective production starts after sometime from the commencement of production and is uncertain, this production time is taken as fuzzy and time dependent fuzzy defective production rate is considered.
- An EPL system is characterised by FDE and solved as MOOP using $\alpha$-cut.
- For optimization, a new technique, IFOT is introduced.

In practical world, to make the co-operative relationship more attractive to boost the demand of the items, a wholesaler / retailer offers several concessions to their customers such as credit period, free transportation, etc. Again a manufacturer produces the items under a defective production process and sells items in different markets. Generally, it is observed that each market has different selling seasons. He *et al.* [102] considered an EOQ model with several markets. Das *et al.* [62] applied this conception in a supply chain model. But still, there are some gaps in the literature, like

- An EPL model for several markets along with the availability of trade credit period. This credit period may be uncertain in non-stochastic sense.
- Model with inflation and transportation cost which is reduced from cycle to cycle with the help of learning inspection.

These gaps motivated us to develop the **Model-6.1** named as **"A learning effected imperfect production inventory model for several markets with fuzzy trade credit period and inflation."**

Again in supply chain management system, there is a scope of bi-level trade credit. However, in practical situation as there is high rate of interest after the credit period, it is more profitable if all the payment is made as early as possible. These ideas influenced us to formulate **"A fuzzy imperfect EPL model with dynamic demand under bi-level trade credit policy" (Model-6.2)** with new realistic assumptions on credit period and its payment.

### 1.4.2 Objective of the Thesis

The main objectives of the thesis are

(i) To develop some inventory / production inventory model(s) for single and / or integrated system i.e. supply chain management system in different types of uncertain environments with few innovative and realistic assumptions which are not considered so far.

(ii) To modify some existing probabilistic or fuzzy programming method and to develop solution techniques (Expectation, FDE, Possibility, Necessity, FAGA, RMOGA, TLBO, IFOT, etc.) as per the requirements of the model described in (i). The models are solved by these methods.

(iii) To convert the uncertain models into the corresponding deterministic single or multi-objective problems by using different appropriate techniques.

(iv) To show different effects or relations of the models' parameters and decision variables through some numerical examples and to perform their sensitivity analyses.

## 1.5 Organization of the thesis

In the proposed thesis, some real life uncertain inventory problems are considered and solved. The proposed thesis is divided into following four parts and seven chapters.

### Part-I : Introduction and Solution Methodologies

Chapter-1

**Introduction**

This Chapter contains an introduction giving an overview of the development on inventory control system in crisp, fuzzy, random, rough, fuzzy-random and fuzzy-rough environments.

Chapter-2

**Solution Methodologies**

In this Chapter, preliminary ideas on crisp set, fuzzy set, intuitionistic fuzzy set, rough set etc, are given. The following techniques/methods have been developed /modified and used to solve the proposed inventory models in uncertain environments.

- Generalized Reduced Gradient Technique (GRG)
- Intuitionistic Fuzzy Optimization Technique (IFOT)
- Fuzzy Age-based Genetic Algorithm (FAGA)

- Rough age based Multi-Objective Genetic Algorithm (RMOGA)
- Teaching and Learning Based Optimization (TLBO)
- Buckley-Feuring method (for solution of FDE)
- Hukuhara derivative and Chalco-Cano technique (for solution of FDE)

## Part-II: Inventory Problems in Uncertain Environment

### Chapter-3

#### Inventory Problems with Stock dependent Demand in Random Environment

**Model-3.1: An EPL model for randomly imperfect production system with stock-dependent demand and rework**

In this model, we consider a single item, imperfect EPL model with stock-dependent demand and partial rework. In real life EPL models, defective production commences from the out-of-control state, after the passage of some time from production commencement. Its occurrence is random and imposed here through a chance constraint. The set-up cost is partly production dependent. Unit production cost (UPC) is also production dependent and a part of it is taken as environment protection cost (EPC). Defective rate is also assumed to be random and production dependent. The model is formulated as an average cost minimization problem subject to a chance constraint and solved using a non-linear optimization technique- GRG method through LINGO 11.0 software. Several special cases are derived and more specifically, the present investigation derives the expressions of Sana [227] and Khouja and Mehrez [127]. Numerical experiments are performed to illustrate the general and particular models. Some sensitivity analyses are presented against few model parameters.

**Model-3.2: An EPL model with reliability-dependent randomly imperfect production system over different uncertain finite time horizons**

Imperfect EPL models are considered over different types of uncertain finite time horizons with stock-dependent demand, reliability dependent defective rate and random out-of-control state. Generally, in EPL models, defective production starts after the passage of some time from production commencement. So occurrence of defective production is random and imposed here through a chance constraint. Reliability of a machinery system affects on the defective rate and production cost to produce an item. Here UPC depends on reliability and production rate and part of it is taken against the EPC. Both linear and non-linear production dependent forms of quality are considered. The problems are formulated as total cost minimization problems with crisp, random, fuzzy, fuzzy-random, rough and fuzzy-rough constraints and solved using GRG method through LINGO 11.0 software. Several special cases are derived and numerical experiments are performed to illustrate the general and particular models.

## Chapter-4

### Inventory Problems on Complementary and Substitute Products in Random Environment

**Model-4.1: EPL models for complementary and substitute items under imperfect production process with promotional cost and selling price dependent demands**

This model considers a multi-item, imperfect EPL model with advertisement/promotional cost and selling price dependent demand and partially rework under a budget constraint. The items are either complementary or substitute to each other. In the EPL models, defective production commences from the out-of-control state, after the passage of some time from production commencement. Its occurrence is random after the lapse of certain time and it is imposed here through a chance constraint. The set-up cost is partly production dependent. UPC is also production dependent and a part of it is taken as EPC. Defective rate is also assumed to be random and production dependent. The model is formulated as an average profit maximization problem subject to chance constraints and a budget constraint and solved using the GRG method. Several special cases are derived and numerical experiments are performed to illustrate the general and particular models. Some sensitivity analyses are presented against few model parameters.

**Model-4.2: Quality and pricing decisions for substitutable items under imperfect production process over a random planning horizon**

This investigation determines the optimum qualities and prices of two substitute products for a manufacturer cum retailer in an imperfect production process over a random planning horizon for maximum profit. In this EPL process, items are produced simultaneously, defective production commences during the "out-of-control" state after the passage of some time from the commencement of production and the defective units are partially reworked. The items are substitutable to each other depending on their prices and qualities jointly or separately. UPC depends directly on raw-material, labour and quality improvement costs and inversely to the production rate. A part of it is spent against environment protection. Here learning effect is introduced in the set-up and maintenance costs. For the whole process, the planning horizon is random with normal distribution, which is treated as a chance constraint. A Fuzzy Age based Genetic Algorithm (FAGA) is introduced for the solution of a single objective problem. The above mentioned models are formulated as profit maximization problems subject to a chance constraint and solved using FAGA. The models are demonstrated numerically and the optimum results are presented graphically.

**Model-4.3: Optimum ordering for two substitute items in a news-vendor management with promotional effort on demand using Rough Age based Genetic Algorithm**

This model presents a single period news-vendor management system for two substitute items. Here substitution is made only when one item is exhausted and then the substitute item is sold at a different price. Here, the News-Vendor problem deals with stochastic uniform demand and a promotional effort to boost the random demand. Different scenarios are formulated and combined to find the expected profit of the system. The profit function is optimized with constraints on purchasing and/ or promotional costs. A Rough-Age based GA (RAGA) is developed for single objective optimization and applied to identify the optimal strategies of the proposed model. Finally, real-life experiments are performed to illustrate the models. Some sensitivity analyses are also presented to stabilize the numerical experiments.

## Chapter-5

### Inventory Problems with Carbon Emission in Fuzzy Environment

**Model-5.1: Green logistics under imperfect production system: A Rough age based Multi-Objective Genetic Algorithm approach**

Imperfect EPL models are considered with time dependent defective rate. Here, defective production starts after the passage of some time from the production commencement. Produced defective units are partially reworked and sold as fresh units. Under the environmental regulation, a cost (carbon tax) is charged by the government to mitigate global warming by reducing carbon emission (CE). Management also uses carbon trading when upper limit of carbon emission is given by the government. This cost brings a contradiction to production manager. For more profit, if more production is decided, then CE and tax due to that are more. The models are formulated as profit maximization problems and solved using Rough age based Multi-Objective Genetic Algorithm (RMOGA). Numerical experiments and graphical presentation are performed to illustrate the models. An algorithm with example for the firm management to achieve the maximum profit is also presented.

**Model-5.2: EPL models with fuzzy imperfect production system including carbon emission : A fuzzy differential equation approach**

This model outlines the production policies for maximum profit of a firm producing imperfect economic lot size with time-dependent fuzzy defective rate under the respective country's carbon emission rules. In this investigation, two criteria in production process are considered : (i) Generally in EPL models, defective production starts after the passage of some time from production commencement. So the starting time of producing defective units is normally uncertain and imprecise. Here it is taken as fuzzy. Thus produced defective units are fuzzy, partially reworked instantly and sold as fresh units. As a result, the inventory level at any time becomes fuzzy and the relation between the production, demand and inventory level becomes a Fuzzy Differential Equation (FDE). (ii) Under the environmental regulation, a cost (say carbon tax) is charged by the government to mitigate global warming by reducing CE. Firm management also uses carbon trading when upper limit of carbon emission is fixed by the government. This cost brings a contradiction to production management. For more profit, if more production is decided, then CE and tax due to that increase. To avoid the carbon penalty, total production may be reduced but in that case, profit will be less. Considering the above two real-life criteria, some production policies are outlined. Here models are formulated as profit maximization problems using FDE, the corresponding inventory and environmental costs are calculated using fuzzy Riemann-integration. $\alpha$-cuts of average profits are obtained and the reduced multi-objective crisp problems are solved using Intuitionistic Fuzzy Optimization Technique (IFOT). Numerical experiments and graphical presentation are performed to illustrate the models. Considering different carbon regulations, an algorithm for a firm management in a country is presented to achieve the maximum profit. Real-life production problems for the firms in Annex I and developing countries are solved.

## Chapter-6

### Inventory Problems with Trade Credit Policy in Fuzzy Environment

**Model-6.1: A learning effected imperfect production inventory model for several markets with fuzzy trade credit period and inflation.**

This model consists of joint relationship among supplier, manufacturer-cum-retailer and multiple markets in which manufacturer-cum-retailer gets a facility of fuzzy credit period for purchasing of raw materials from supplier. The manufacturer produces the finished goods along with defective units at a constant rate. Here the finished product is transported to different markets in different seasons, with a transportation cost that depends on the amount of transportation and learning ratio. Also, the demand of the item is different in each market. Further, the optimal operation policy that maximizes total profit of the integrated system is derived under a constant rate of inflation. But due to impreciseness in trade credit period, profit function seems fuzzy in nature, thereby determining the optimal imprecise values of decision variables. Equivalent crisp profit function is obtained by applying fuzzy expectation method. The closed form solutions of the objective and it's concavity properties have been derived to obtain maximum profit. Finally, the models are illustrated with certain numerical data and graphical solutions are provided with sensitivity analysis with respect to model's parameters.

**Model-6.2: A fuzzy imperfect EPL model with dynamic demand under bi-level trade credit policy**

An imperfect EPL model with fuzzy dynamic demand is developed in a fuzzy production process under bi-level trade credit policy. Supplier offers a delay period (M) to the manufacturer-cum-retailer for payment of raw-material cost. Due to this facility, manufacturer-cum-retailer also offers a trade credit period (N) to the customers to boost the demand. During trade credit period of customers, demand of the item increases with time at a decreasing rate. Different parameters of demand are assumed as fuzzy. Depending upon the values of M and N, twelve scenarios are depicted. In each scenario, the model is represented through FDE whose solution is obtained using Chalco-Cano [39] technique. Thus average profit function is imprecise in nature and its $\alpha$-cut values are maximized for making optimal decision of production run time. All scenarios are illustrated with numerical examples. Further more, an alternative approach of payment for the remaining inventory after the credit period M has also been proposed in the present study. In the new approach, retailer clears all dues before the end of business cycle whenever it is feasible. It has been explained with the help of numerical examples and the outcomes are compared against the above traditional approach.

## Part-III : Summary of the Thesis

Chapter-7

**Summary and Future Extension**

## Part-IV: Appendices, Bibliography and Indices

**Appendix A**

**Appendix B**

**Bibliography**

**Indices**

# Chapter 2

# Solution Methodology

## 2.1 Mathematical prerequisites

### 2.1.1 Crisp Set Theory

**Crisp Set:** By crisp one means dichotomous, that is, yes or no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false- and nothing in between. In set theory, an element can either belongs to a set or not; and in optimization, a solution is either feasible or not. A classical set, $X$, is defined by crisp boundaries, i.e., there is no uncertainty in the prescription of the elements of the set. Normally it is defined as a well defined collection of elements or objects, $x \in X$, where $X$ may be countable or uncountable.

**Convex Set:** A subset $S \subset \Re^n$ is said to be convex, if for any two points $x_1, x_2$ in S, the line segment joining the points $x_1$ and $x_2$ is also contained in $S$. In other words, a subset $S \subset \Re^n$ is convex, if and only if

$$x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S; \quad 0 \leq \lambda \leq 1.$$

**Convex Combination:** Given a set of vectors $\{x_1, x_2, \cdots, x_n\}$, a linear combination $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ is called a convex combination of the given vectors, if $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ and $\sum\limits_{i=1}^{n} \lambda_i = 1$

**Convex function:** The function $f : S \rightarrow \Re$ is said to be convex if for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$, implies that

$$f\{(1-\lambda)x_1 + \lambda x_2\} \leq (1-\lambda)f(x_1) + \lambda f(x_2).$$

The definition of convex functions can be modified for concave functions by replacing $'\leq'$ by $'\geq'$.

### 2.1.2 Fuzzy Set Theory

The concept of fuzzy set was initialized by Zadeh [284] in 1965. Fuzzy set theory has been well developed and applied in a wide variety of real problems including inventory control problems. It was developed to define and solve the complex system with sources of uncertainty or impreciseness which are non-stochastic in nature. The term "*FUZZY*" was proposed by Prof. L, A. Zadeh in 1962 [283]. A short delineation of the fuzzy set theory is given below.

#### 2.1.2.1 Fuzzy Set

Fuzzy sets deal with objects that are 'matter of degree', with all possible grades of truth between yes or no. So a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. Let X be a collection of objects and x be an element of X, then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))/x \in X\}$, where $\mu_{\widetilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X to the membership space M which is considered as the closed interval [0,u], where $0 < u \leq 1$.

**Note:** When M consists of only two points 0 and 1, $\widetilde{A}$ becomes a non-fuzzy set (or Crisp set) and $\mu_{\widetilde{A}}(x)$ reduces to the characteristic function of the non-fuzzy set (or crisp set).

- **Equality:** Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ in $X$ are said to be equal if and only if $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x), \forall x \in X$.

- **Containment:** A fuzzy set $\tilde{A}$ in $X$ is contained in or is a subset of another fuzzy set $\tilde{B}$ in $X$, written as $\tilde{A} \subset \tilde{B}$ if and only if $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x), \forall x \in X$.

- **Support:** The support of a fuzzy set $\widetilde{A}$ is a crisp set, denoted by $S(\widetilde{A})$, and defined as $S(\widetilde{A}) = \{x \,|\, \mu_{\widetilde{A}}(x) > 0\}$.

- **Height:** The height of a fuzzy set $\widetilde{A}$ is the maximum membership grade value of $\widetilde{A}$ and denoted by $h(\widetilde{A}) = \sup_{x \in X} \mu_{\widetilde{A}}(x)$, where $X$ is universal set.

- **Normal fuzzy set:** A fuzzy set $\widetilde{A}$ is called normal if its height is 1, i.e., if $h(\widetilde{A}) = \sup_{x \in X} \mu_{\widetilde{A}}(x) = 1$

- **Core:** The core of a fuzzy set $\widetilde{A}$ is a set of all points with unit membership degree in $\widetilde{A}$ denoted by $Core(\widetilde{A})$, and defined as $Core(\widetilde{A}) = \{x \in X \,|\, \mu_{\widetilde{A}}(x) = 1\}$.

- **Convexity:** A fuzzy set $\widetilde{A}$ in $X$ is said to be convex if and only if for any $x_1, x_2 \in X$, the membership function of $\widetilde{A}$ satisfies the inequality $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$ for $0 \leq \lambda \leq 1$.

#### 2.1.2.2 Fuzzy Number

A fuzzy number is a convex, normal fuzzy set defined on the real line. Here some definitions of fuzzy numbers are presented below.

A general shape of a fuzzy number following the above definition may be shown pictorially as in Fig. 2.1. Here, $a_1$, $a_2$, $a_3$ and $a_4$ are real numbers. A fuzzy number $\widetilde{A}$ in $X$ is said to be discrete or continuous according as its membership function $\mu_{\widetilde{A}}(x)$ is discrete or continuous. Linear Fuzzy Number (LFN), Triangular Fuzzy Number (TFN), Parabolic Fuzzy Number (PFN) and Trapezoidal Fuzzy Number (TrFN), are special classes of continuous fuzzy numbers.

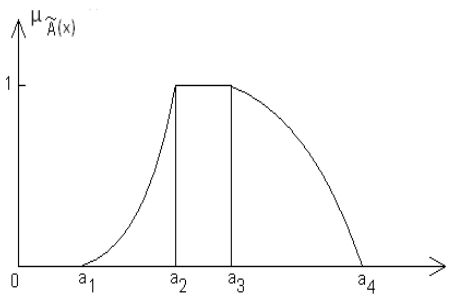



**Figure** 2.1: Membership function of general fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$

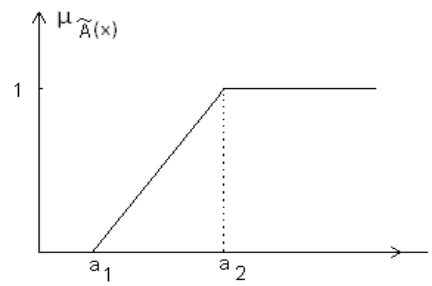



**Figure** 2.2: Membership function of LFN

**Definition 2.1. Linear Fuzzy Number (LFN):** *A LFN $\widetilde{A}$ is specified by two parameters $(a_1, a_2)$ and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : X \to [0, 1]$ as follows (cf. Fig. 2.2):*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if } x \le a_1 \\ \dfrac{a_2 - x}{a_2 - a_1} & \text{if } a_1 \le x \le a_2 \\ 1 & \text{if } x \ge a_2 \end{cases}$$

**Definition 2.2. Triangular Fuzzy Number (TFN):** *A TFN $\widetilde{A}$ is specified by the triplet $(a_1, a_2, a_3)$ and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : X \to [0, 1]$ as follows (cf. Fig. 2.3):*

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases}$$

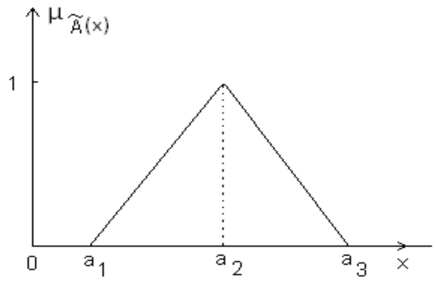



**Figure** 2.3: Membership function of TFN

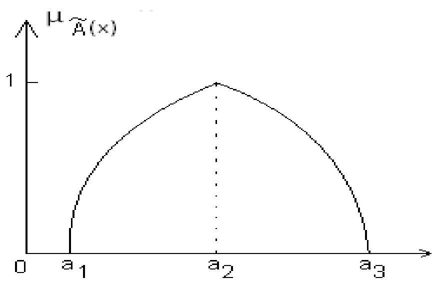



**Figure** 2.4: Membership function of PFN

**Definition 2.3. Parabolic Fuzzy Number (PFN):** *A PFN $\widetilde{A}$ is also specified by the triplet $(a_1, a_2, a_3)$ and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : X \to [0, 1]$ as follows (cf. Fig. 2.4):*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - (\dfrac{a_2 - x}{a_2 - a_1})^2 & \text{for } a_1 \leq x \leq a_2 \\ 1 - (\dfrac{x - a_2}{a_3 - a_2})^2 & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2.4. Trapezoidal Fuzzy Number (TrFN):** *A TrFN $\widetilde{A}$ is specified by four parameters $(a_1, a_2, a_3, a_4)$ and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : X \to [0, 1]$ as follows (cf. Fig. 2.5):*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\ 1 & \text{for } a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{for } a_3 \leq x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$$

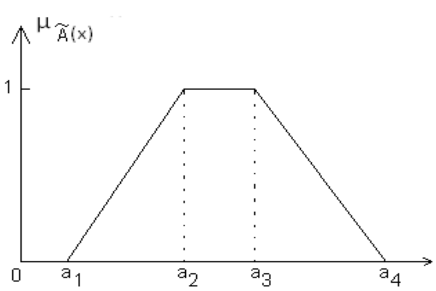



**Figure** 2.5: Membership function of TrFN

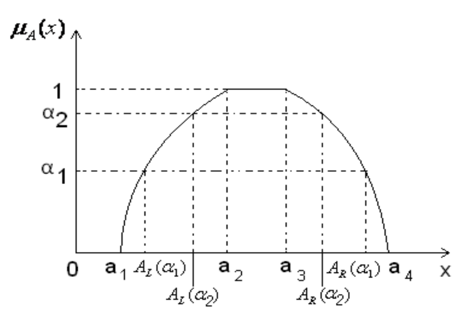



**Figure** 2.6: $\alpha$-cut of general fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$

**Definition 2.5.** $\alpha$ **- Cut of a fuzzy number:** *$\alpha$- cut of a fuzzy number $\tilde{A}$ in $X$ is denoted by $A[\alpha]$ and is defined as the following crisp set (cf. Fig. 2.6):*

$$A[\alpha] = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X\} \text{ where } \alpha \in [0,1]$$

*$A[\alpha]$ is a non-empty bounded closed interval contained in $X$ and it can be denoted by $A[\alpha] = [A_L(\alpha), A_R(\alpha)]$. $A_L(\alpha)$ and $A_R(\alpha)$ are the lower and upper bounds of the closed interval respectively. Fig. 2.6 represents a fuzzy number $\tilde{A}$ with $\alpha$-cuts $A[\alpha_1] = [A_L(\alpha_1), A_R(\alpha_1)], A[\alpha_2] = [A_L(\alpha_2), A_R(\alpha_2)]$. It shows that if $\alpha_2 \geq \alpha_1$ then $A_L(\alpha_2) \geq A_L(\alpha_1)$ and $A_R(\alpha_1) \geq A_R(\alpha_2)$. Here, $A'[\alpha] = \{x \in X | \mu_{\tilde{A}}(x) > \alpha\}$ is called 'strong $\alpha$-level set'*

**Definition 2.6.** $\alpha$**-cut of a function:** *Let $\widetilde{F}(X)$ be the space of all compact and convex fuzzy sets on $X$. If $f : \Re^n \to \Re$ is a continuous function, then $\tilde{f} : \widetilde{F}(\Re^n) \to \widetilde{F}(\Re)$ is well defined function and its $\alpha$-cut $\tilde{f}(u)[\alpha]$ is given by (cf. Roman-Flores et al. [216])*

$$\tilde{f}(u)[\alpha] = f(u[\alpha]), \ \forall \alpha \in [0,1], \forall \tilde{u} \in \widetilde{F}(\Re^n) \tag{2.1}$$

*where $f(A) = \{f(a)/a \in A\}$.*

**Definition 2.7. Fuzzy Extension Principle [285]:** *If $\tilde{a}$ , $\tilde{b} \subseteq \Re$ and $\tilde{c} = f(\tilde{a}, \tilde{b})$, where $f : \Re \times \Re \to \Re$ is a binary operation, membership function $\mu_{\tilde{c}}$ of $\tilde{c}$ is defined as (cf. page 53 of Zimmermann [288], second revised version)*

$$\text{For each } z \in \Re, \ \ \mu_{\tilde{c}}(z) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), \ x, y \in \Re \text{ and } z = f(x,y)\} \tag{2.2}$$

#### 2.1.2.3 Defuzzification Method

In the literature of fuzzy mathematics, several approaches are available to convert a fuzzy number into its equivalent crisp number [47, 48, 91, 280]. Each method has some merits and demerits over the others. In this thesis, used defuzzification methods are discussed below.

**A. Graded Mean Integration Representation (GMIR) of Fuzzy Number:** Chen and Hsieh [47, 48] introduced GMIR method based on the integral value of graded mean $\alpha$-level of a generalized fuzzy number. The graded mean $\alpha$-level value of generalized fuzzy number $\tilde{A} = (A_1, A_2, A_3, A_4)$ is $\alpha[\frac{A_L(\alpha)+A_R(\alpha)}{2}]$, $\alpha \in [0,1]$. Then the GMIR of a general fuzzy number $\tilde{A}$ is

$$P(\tilde{A}) = \int_0^1 \alpha[\frac{A_L(\alpha) + A_R(\alpha)}{2}]d\alpha / \int_0^1 \alpha d\alpha. = \frac{1}{6}[A_1 + 2A_2 + 2A_3 + A_4] \tag{2.3}$$

Here equal weightage has been given to the left and right parts of the membership function. The representation given by (2.3) can be generalized/modified by replacing $\frac{[A_L(\alpha)+A_R(\alpha)]}{2}$,

$\alpha \in [0,1]$ with $[kA_L(\alpha) + (1-k)A_R(\alpha)]$, $\alpha \in [0,1]$, where the value of $k$ depends on the preference of the decision maker. Therefore, the modified form of Eq. (2.3) is

$$\begin{aligned} P_k(\tilde{A}) &= \int_0^1 \alpha[k\,A_L(\alpha) + (1-k)A_R(\alpha)]d\alpha / \int_0^1 \alpha d\alpha \\ &= \frac{1}{3}[k(A_1 + 2A_2) + (1-k)(2A_3 + A_4)]. \end{aligned} \tag{2.4}$$

The method is also known as k-preference integration representation.

**B. Possibility/Necessity Measure of Fuzzy Event**

In order to measure a fuzzy event, Zadeh [285] proposed the concept of possibility measure in the year 1978. Considering the degree of membership $\mu_{\widetilde{F}}(u)$ of an element $\tilde{u}$ in a fuzzy set $\widetilde{F}$, defined on a referential $U$, one can find in the literature, three interpretations of this degree [80].

**Degree of similarity:** According to degree of similarity, $\mu_{\widetilde{F}}(u)$ is the degree of proximity of $\tilde{u}$ to prototype elements of $\widetilde{F}$. Historically, this is the oldest semantics of membership grades since Bellman *et al.* [14].

**Degree of preference:** According to degree of preference, $\widetilde{F}$ represents a set of more or less preferred objects (or values of a decision variable $x$) and $\mu_{\widetilde{F}}(u)$ represents an intensity of preference in favour of object $\tilde{u}$, or the feasibility of selecting $\tilde{u}$ as a value of $x$. Fuzzy sets then represent criteria or flexible constraints. This view is the one later put forward by Bellman and Zadeh [15], it has given birth to an abundant literature on fuzzy optimization, especially fuzzy linear programming and decision analysis.

**Degree of uncertainty:** This interpretation was proposed by Zadeh [285] when he introduced the possibility theory and developed his theory of approximate reasoning [286]. $\mu_{\widetilde{F}}(u)$ is then the degree of possibility that a parameter $x$ has value $\tilde{u}$, given that all that is known about it is that "$x$ is $\widetilde{F}$". Then the values encompasses by the support of the membership functions are mutually exclusive, and the membership degrees rank these values in terms of their respective plausibility. Set functions called possibility and necessity measures can be derived so as to rank-order events in terms of unsurprising-ness and acceptance respectively.

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Dubois and Prade [76], Liu and Iwamura [148], Zadeh [285]

$$pos(\tilde{a} * \tilde{b}) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re, x * y\}, \tag{2.5}$$

where $pos$ represents possibility, $*$ is any one of the relations $>, <, =, \leq, \geq$.

$$nes(\tilde{a} * \tilde{b}) = 1 - pos\overline{(\tilde{a} * \tilde{b})}, \tag{2.6}$$

where $nes$ represents necessity.

Similarly, possibility and necessity measures of $\tilde{a}$ with respect to $\tilde{b}$ are denoted by $\Pi_{\tilde{b}}(\tilde{a})$ and $N_{\tilde{b}}(\tilde{a})$ respectively and are defined as

$$\Pi_{\tilde{b}}(\tilde{a}) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)), x \in \Re\} \tag{2.7}$$
$$N_{\tilde{b}}(\tilde{a}) = min\{sup(\mu_{\tilde{a}}(x), 1 - \mu_{\tilde{b}}(x)), x \in \Re\}. \tag{2.8}$$

According to the definitions of fuzzy numbers, following lemmas can easily be derived.

**Lemma 2.1.** *If $\tilde{a} = (a_1, a_2, a_3)$ be a TFN with $0 < a_1$ and $b$ is a crisp number, $pos(\tilde{a} > b) \geq \alpha$ iff $\dfrac{a_3 - b}{a_3 - a_2} \geq \alpha$.*

**Lemma 2.2.** *If $\tilde{a} = (a_1, a_2, a_3)$ be a TFN with $0 < a_1$ and $b$ is a crisp number, $nes(\tilde{a} > b) \geq \alpha$ iff $\dfrac{b - a_1}{a_2 - a_1} \leq 1 - \alpha$.*

**Lemma 2.3.** *If $\tilde{a} = (a_1, a_2, a_3)$ be a TFN with $0 < a_1$ and $b$ is a crisp number, $pos(\tilde{a} \leq b) \geq \alpha$ iff $\dfrac{b - a_1}{a_2 - a_1} \geq \alpha$.*

**Lemma 2.4.** *If $\tilde{b} = (b_1, b_2, b_3)$ and $\tilde{a} = (a_1, a_2, a_4)$ be TFNs with $0 < a_1$ $0 < b_1$, $pos(\tilde{b} \geq \tilde{a}) \geq \alpha$ iff $\frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} \geq \alpha$.*

**Lemma 2.5.** *If $\tilde{a} = (a_1, a_2, a_3)$ be a TFN and $b$ be a crisp number with $0 < a_1$ and $0 < b$,*

$$\Pi_{\tilde{a}}(b) = N_{\tilde{a}}(b) = \begin{cases} \frac{b-a_1}{a_2-a_1} & for \;\; a_2 \geq b \geq a_1 \\ \frac{a_3-b}{a_3-a_2} & for \;\; a_3 \geq b \geq a_2 \\ 0 & otherwise. \end{cases}$$

**Lemma 2.6.** *[158]: If $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TrFN and $b$ be a crisp number then*

$$pos(\tilde{a} \geq b) = \begin{cases} 1 & if \;\; a_3 \geq b \\ \frac{a_4-b}{a_4-a_3} & if \;\; a_3 \leq b \leq a_4 \\ 0 & otherwise \end{cases}$$

**Lemma 2.7.** *[158]: If $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TrFN and $b$ be a crisp number then*

$$nes(\tilde{a} \geq b) = \begin{cases} 1 & if \;\; a_1 \geq b \\ \frac{a_2-b}{a_2-a_1} & if \;\; a_1 \leq b \leq a_2 \\ 0 & otherwise \end{cases}$$

**C. Credibility Measure [152]:** Let $\tilde{A}$ be any fuzzy number. Then credibility measure of $\tilde{A}$ is denoted by $cr(\tilde{A})$ and defined as

$$cr(\tilde{A}) = \frac{1}{2}[pos(\tilde{A}) + nes(\tilde{A})] \tag{2.9}$$

More generally, according to Maity [158] the above form can be considered as

$$cr(\tilde{A}) = [\rho pos(\tilde{A}) + (1-\rho)nes(\tilde{A})] \;\; where \;\; 0 \leq \rho \leq 1.$$

**D. Fuzzy Expectation [149]:** Let $\tilde{X}$ be any normalized fuzzy variable. The expected value of the fuzzy variable $\tilde{X}$ is denoted by $E[\tilde{X}]$ and defined by

$$E[\tilde{X}] = \int_0^{\infty} cr(\tilde{X} \geq r)\, dr - \int_{-\infty}^{0} cr(\tilde{X} \leq r)\, dr \tag{2.10}$$

provided that at least one of the two integral is finite.

**Lemma 2.8.** *[158]: If $\tilde{A} = (a_1, a_2, a_3)$ be a TFN and $r$ be a crisp number, expected value of $\tilde{A}$, $E(\tilde{A})$ is given by*

$$E[\tilde{A}] = \frac{1}{2}[(1-\rho)a_1 + a_2 + \rho\, a_3]$$

*where $\rho$ ($0 \leq \rho \leq 1$) is the degree of optimism/pessimism for DM.*

**Lemma 2.9.** *[158]: If $\tilde{A} = (a_1, a_2, a_3, a_4)$ is a TrFN and $r$ is a crisp number, then expected value of $\tilde{A}$, $E[\tilde{A}]$, is given by*

$$E[\tilde{A}] = \frac{1}{2}[(1-\rho)(a_1 + a_2) + \rho(a_3 + a_4)]$$

*where $\rho$ ($0 \leq \rho \leq 1$) is the degree of optimism/pessimism for DM.*

#### 2.1.2.4 Fuzzy Differential Equation (FDE)

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [205] and it is based in the H-difference of sets, as follows.

**Definition 2.8.** *Let be u, v$\in \mathscr{F}^n$. If there exists w$\in \mathscr{F}^n$ such that u = v + w, then w is called the H-difference of u and v and it is denoted by u – v.*

**Definition 2.9.** *[205]. Let be T = (a,b) and consider a fuzzy mapping F : (a,b)$\rightarrow \mathscr{F}^n$. We say that F is differentiable at $t_0 \in T$ if there exists an element $F'(t_0) \in \mathscr{F}^n$ such that the limits*

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \;\; and \;\; \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

*exist and are equal to $F'(t_0)$. Here the limit is taken in the metric space ($\mathscr{F}^n$,D).*

**Definition 2.10.** *Let be F : (a,b)→ $\mathscr{F}^n$ and $t_0 \in (a,b)$. We say that F is differentiable at $t_0$ if:*

*(I) it exists an element $F'(t_0) \in \mathscr{F}^n$ such that, for all $h > 0$ sufficiently near to 0, there are $F(t_0+h) - F(t_0)$, $F(t_0) - F(t_0 - h)$ and the limits (in the metric D)*

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h} = F'(t_0) \tag{2.11}$$

*or,*
*(II) it exists an element $F'(t_0) \in \mathscr{F}^n$ such that, for all $h < 0$ sufficiently near to 0, there are $F(t_0+h) - F(t_0)$, $F(t_0) - F(t_0 - h)$ and the limits (in the metric D)*

$$\lim_{h\to 0^-} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0) - F(t_0-h)}{h} = F'(t_0) \tag{2.12}$$

**Theorem 2.1.** *[39]: Let $F : T \to \mathscr{F}$ be a function and denote $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for each $\alpha \in [0,1]$. Then*
*(i) If F is differentiable in the first form (I), then $f_\alpha$ and $g_\alpha$ are differentiable functions and*

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]. \tag{2.13}$$

*(ii) If F is differentiable in the second form (II), then $f_\alpha$ and $g_\alpha$ are differentiable functions and*

$$[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]. \tag{2.14}$$

There are several approaches in the literature [27] to define fuzzy derivative. The concept of fuzzy differentiation was introduced by Dubois and Prade [76]. Motivated by their study [76–78], Seikkala [237] defines fuzzy differentiation and fuzzy integration using inclusion property of $\alpha$-cuts of fuzzy numbers.

**Definition 2.11.** *According to Seikkala [237], if $\tilde{Y}(t)$ be a fuzzy number for each $t \in I(\subseteq \Re)$, having $\alpha$-cut, $\tilde{Y}(t)[\alpha] = [Y_L(t,\alpha), Y_R(t,\alpha)]$ then $\frac{d\tilde{Y}(t)}{dt}$ exists and $\frac{d\tilde{Y}(t)}{dt}[\alpha] = [\frac{dY_L(t,\alpha)}{dt}, \frac{dY_R(t,\alpha)}{dt}]$ provided $[\frac{dY_L(t,\alpha)}{dt}, \frac{dY_R(t,\alpha)}{dt}]$ are $\alpha$-cuts of a fuzzy number for each $t \in I$, i.e., if the following conditions hold:*

$$\begin{cases} \dfrac{dY_L(t,\alpha)}{dt} \text{ and } \dfrac{dY_R(t,\alpha)}{dt} \text{ are continuous on } I \times [0,1]. \\ \dfrac{dY_L(t,\alpha)}{dt} \text{ is an increasing function of } \alpha \text{ for each } t \in I. \\ \dfrac{dY_R(t,\alpha)}{dt} \text{ is a decreasing function of } \alpha \text{ for each } t \in I. \\ \dfrac{dY_L(t,1)}{dt} \leq \dfrac{dY_R(t,1)}{dt}, \forall t \in I. \end{cases} \tag{2.15}$$

*Accordingly, they define fuzzy integral* $\int_a^b \widetilde{Y}(t)dt$ *for all* $a, b \in I$ *having* $\alpha$*-cut*

$$\left(\int_a^b \widetilde{Y}(t)dt\right)[\alpha] = \left[\int_a^b Y_L(\alpha,t)dt, \int_a^b Y_R(\alpha,t)dt\right]$$

*provided that the integrals on the right exist. This definition of fuzzy integration agrees with the definition of Dubois and Prade [77] and Wu [270].*

**A. Fuzzy Differential Equation-1 (FDE-1) [27]:** Consider the first order ordinary differential equation

$$\frac{dY}{dt} = f(t, Y, k), \quad Y(0) = C, \tag{2.16}$$

where $k = (k_1, k_2, ...k_n)$ is a vector of constants and $t$ is in some interval $I$ (closed and bounded) which contains zero. Let the Eq. (2.16) has a unique solution

$$Y = g(t, k, C), \ \text{ for } t \in I, \ k \in K \subset \Re^n, \ C \in \Re \tag{2.17}$$

When $\widetilde{k} = (\widetilde{k}_1, \widetilde{k}_2, ...\widetilde{k}_n)$ is a vector of fuzzy numbers and $\widetilde{C}$ be another fuzzy number, then the Eq. (2.16) reduces to the following FDE

$$\frac{d\widetilde{Y}}{dt} = f(t, \widetilde{Y}, \widetilde{k}), \quad \widetilde{Y}(0) = \widetilde{C} \tag{2.18}$$

assuming that derivative [27, 237] of the unknown fuzzy function $\widetilde{Y}(t)$ exists according to the above definition [238].
According to Buckley and Feuring [27],

$$\widetilde{Y}(t) = g(t, \widetilde{k}, \widetilde{C}) \tag{2.19}$$

is solution of (2.18) if its $\alpha$-cut $\widetilde{Y}(t)[\alpha] = [Y_L(t,\alpha), Y_R(t,\alpha)]$ satisfies the following conditions (2.20) along with the conditions given by the Eq. (2.15).

$$\begin{cases} \dfrac{dY_L(t,\alpha)}{dt} = f_L(t,\alpha), \ \dfrac{dY_R(t,\alpha)}{dt} = f_R(t,\alpha), \ \forall \alpha \in [0,1]. \\ \dfrac{dY_L(0,\alpha)}{dt} = C_L(\alpha), \ \dfrac{dY_R(0,\alpha)}{dt} = C_R(\alpha), \ \forall \alpha \in [0,1]. \end{cases} \tag{2.20}$$

where $\widetilde{f}(t)[\alpha] = [f_L(t,\alpha), f_R(t,\alpha)]$, $\widetilde{C}[\alpha] = [C_L(\alpha), C_R(\alpha)]$ and membership function of $\widetilde{Y}(t)$ is obtained using fuzzy extension principle (Eq. (2.2)). To justify the validity of this solution one can see [27].

**B. Fuzzy Differential Equation-2 (FDE-2) [39]:** Let the fuzzy initial value problem is

$$x'(t) = F(t, x(t)), \qquad x(0) = x_0, \tag{2.21}$$

where $F : [0, a] \times \mathscr{F} \to \mathscr{F}$ is a continuous fuzzy mapping and $x_0$ is a fuzzy number. Following Kaleva [119], we observe that the relations (2.13) and (2.14) in Theorem 2.1 give us an useful procedure to solve the fuzzy differential equation (2.21). In fact, denote $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$, $[x_0]^\alpha = [u^0_\alpha, v^0_\alpha]$ and $[F(t, x(t))]^\alpha$ = $[f_\alpha(t, u_\alpha(t), v_\alpha(t)), g_\alpha(t, u_\alpha(t), v_\alpha(t))]$. Then, we have the following alternatives for solving the problem (2.21):

**Case I**. If we consider $x'(t)$ by using the derivative in the first form (I), then from (2.13) we have $[x'(t)]^\alpha = [u'_\alpha(t), v'_\alpha(t)]$. Now, we proceed as follows:
(i) Solve the differential system

$$\begin{cases} u'_\alpha(t) = f_\alpha(t, u_\alpha(t), v_\alpha(t)), & u_\alpha(0) = u^0_\alpha \\ v'_\alpha(t) = g_\alpha(t, u_\alpha(t), v_\alpha(t)), & v_\alpha(0) = v^0_\alpha \end{cases} \tag{2.22}$$

for $u_\alpha$ and $v_\alpha$;
(ii)Ensure that $[u_\alpha(t), v_\alpha(t)]$ and $[u'_\alpha(t), v'_\alpha(t)]$ are valid level sets;
(iii) By using the Representation Theorem (Remark 1c of Chalco-Cano [39]), we construct a fuzzy solution x(t) such that $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$ for all $\alpha \in [0, 1]$.

**Case II**. If we consider $x'(t)$ by using the derivative in the second form (II), then from (2.14) we have $[x'(t)]^\alpha = [v'_\alpha(t), u'_\alpha(t)]$ and, consequently, now we proceed as follows:

(i) Solve the differential system

$$\begin{cases} u'_\alpha(t) = g_\alpha(t, u_\alpha(t), v_\alpha(t)), & u_\alpha(0) = u^0_\alpha \\ v'_\alpha(t) = f_\alpha(t, u_\alpha(t), v_\alpha(t)), & v_\alpha(0) = v^0_\alpha \end{cases} \tag{2.23}$$

for $u_\alpha$ and $v_\alpha$;
(ii)Ensure that $[u_\alpha(t), v_\alpha(t)]$ and $[v'_\alpha(t), u'_\alpha(t)]$ are valid level sets;
(iii) By using the Representation Theorem again, we obtain a another fuzzy solution x(t) such that $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$ for all $\alpha \in [0, 1]$.

#### 2.1.2.5 Fuzzy Riemann Integration (FRI)

**A. Fuzzy Integral:** The study on fuzzy integral was started before three decades. Sims and Wang [242] gave a good review of this subject. Dubois and Prade [77] defined integral of fuzzy mapping $\widetilde{f}(x)$over a crisp interval $I = [a, b]$ and proved that under certain condition $(\int_I \widetilde{f})[\alpha] = \int_I \widetilde{f}[\alpha]$. In a subsequent paper [78] they define integration of a real mapping $f(x)$ between fuzzy bounds $\widetilde{D} = [\widetilde{a}, \widetilde{b}]$. If $I = [I_L, I_R]$, where $I_L$=infimum of the support of $\widetilde{a}$ and $I_R$=Supremum of the support of $\widetilde{b}$, then according to their definition

$$\forall z \in \Re,\ \mu_{\int_{\widetilde{D}}}(z) = \sup_{x,y \in I} min(\mu_{\widetilde{a}}(x), \mu_{\widetilde{b}}(y)), \text{ under the constraint } z = \int_x^y f(t)dt$$

But their definition does not include integration of fuzzy mapping over a fuzzy domain. In a recent paper Wu [270] defined integration of fuzzy mapping over crisp and fuzzy intervals and accordingly two types of FRIs have been defined by him.

**B. Fuzzy Riemann Integral of type-I [270]:** Let $\tilde{f}(x)$ be a closed and bounded fuzzy valued function on $[a, b]$ and $[f_L(\alpha, x), f_R(\alpha, x)]$ be $\alpha$-cut of $\tilde{f}(x)$ $\forall x \in [a, b]$. If $f_L(\alpha, x)$ and $f_R(\alpha, x)$ are Riemann integrable on $[a, b]$, $\forall \alpha$, then the fuzzy Riemann integral $\int_a^b \tilde{f}(x)dx$ is a closed fuzzy number and its $\alpha$-level set is given by

$$\left(\int_a^b \tilde{f}(x)dx\right)[\alpha] = \left[\int_a^b f_L(\alpha, x)dx, \int_a^b f_R(\alpha, x)dx\right]$$

**C. Fuzzy Riemann Integral of type-II [270]:** Let $\tilde{f}(\tilde{x})$ be a bounded and closed fuzzy valued function defined on the closed fuzzy interval $[\tilde{a}, \tilde{b}]$ and $\tilde{f}(x)$ be induced by $\tilde{f}(\tilde{x})$. $[f_L(\alpha, x), f_R(\alpha, x)]$ be $\alpha$-cut of $\tilde{f}(x)$ and $\tilde{f}(x)$ is either non-negative or non-positive.
**Case-1:** If $\tilde{f}(x)$ is non-negative and $f_L(\alpha, x)$ and $f_R(\alpha, x)$ are Riemann integrable on $[a_R(\alpha), b_L(\alpha)]$ and $[a_L(\alpha), b_R(\alpha)]$ respectively $\forall \alpha$, then the fuzzy Riemann integral $\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})d\tilde{x}$ is a closed fuzzy number and its $\alpha$-level set is given by

$$\left(\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})d\tilde{x}\right)[\alpha] = \begin{cases} \left[\int_{a_R(\alpha)}^{b_L(\alpha)} f_L(\alpha, x)dx, \int_{a_L(\alpha)}^{b_R(\alpha)} f_R(\alpha, x)dx\right] & \text{if } b_L(\alpha) > a_R(\alpha) \\ \left[0, \int_{a_L(\alpha)}^{b_R(\alpha)} f_R(\alpha, x)dx\right] & \text{if } b_L(\alpha) \leq a_R(\alpha) \end{cases}$$

**Case-2:** If $\tilde{f}(x)$ is non positive and $f_L(\alpha, x)$ and $f_R(\alpha, x)$ are Riemann integrable on $[a_L(\alpha), b_R(\alpha)]$ and $[a_R(\alpha), b_L(\alpha)]$ respectively $\forall \alpha$, then the fuzzy Riemann integral $\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})d\tilde{x}$ is a closed fuzzy number and its $\alpha$-level set is given by

$$\left(\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})d\tilde{x}\right)[\alpha] = \begin{cases} \left[\int_{a_L(\alpha)}^{b_R(\alpha)} f_L(\alpha, x)dx, \int_{a_R(\alpha)}^{b_L(\alpha)} f_R(\alpha, x)dx\right] & \text{if } b_L(\alpha) > a_R(\alpha) \\ \left[\int_{a_L(\alpha)}^{b_R(\alpha)} f_L(\alpha, x)dx, 0\right] & \text{if } b_L(\alpha) \leq a_R(\alpha) \end{cases}$$

#### 2.1.2.6 Intuitionistic Fuzzy Set(IFS), [8, 9]

Let $X = x_1, x_2, ..., x_n$ be a finite universal set. An Atanassov's IFS A is a set of ordered triples,

$$A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle : x_i X\}$$

where $\mu_A(x_i)$ and $\nu_A(x_i)$ are functions mapping from X into [0, 1]. For each $x_i \in$X, $\mu_A(x_i)$ represents the degree of membership and $\nu_A(x_i)$ represents the degree of non-membership of the element $x_i$ to the subset A of X. For the functions $\mu_A(x_i)$ and $\nu_A(x_i)$ mapping into [0, 1], the condition $0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1$ holds .

### 2.1.3 Interval and some useful properties

An interval $I$ in $\Re$ is a subset of $\Re$ having two bounds $I_L$, $I_R$ and defined as $I = \{x \in \Re | I_L \leq x \leq I_R\}$. $I_L$ and $I_R$ are termed as left and right bounds of $I$ respectively and the interval is represented by $I = [I_L, I_R]$. It's mean and half width are denoted respectively by $m(I)$ and $w(I)$ and are defined as $m(I) = (I_L + I_R)/2$ and $w(I) = (I_R - I_L)/2$. An interval $I$ can also defined by its mean $m(I)$ and width $w(I)$ as $< m(I), w(I) >$, i.e. $I = [I_L, I_R] \equiv < m(I), w(I) >$. Clearly $\alpha$-cut of a fuzzy number with continuous membership function can be treated as an interval.

**Arithmetic of Interval:** Let $* \in \{+, -, ., /\}$ be a binary operation on the set of positive real numbers. If $A$ and $B$ are closed intervals then $A * B = \{x * y : x \in A, y \in B\}$ defines a binary operation on the set of closed intervals [182]. In the case of division, it is assumed that $0 \notin B$. The operations on intervals used here may be explicitly calculated from the above definition as

$$\begin{aligned}
A + B &= [A_L, A_R] + [B_L, B_R] = [A_L + B_L, A_R + B_R] \\
A - B &= [A_L, A_R] - [B_L, B_R] = [A_L - B_R, A_R - B_L] \\
A.B &= [A_L, A_R].[B_L, B_R] \\
&= [min\{A_L B_L, A_L B_R, A_R B_L, A_R B_R\}, max\{A_L B_L, A_L B_R, A_R B_L, A_R B_R\}] \\
\frac{A}{B} &= \frac{[A_L, A_R]}{[B_L, B_R]} = [A_L, A_R].[\frac{1}{B_R}, \frac{1}{A_R}], \text{ where } 0 \notin B \\
kA &= \begin{cases} [kA_L, kA_R], \text{ for } k \geq 0 \\ (kA_R, kA_L), \text{ for } k < 0, \text{ where } k \text{ is a real number.} \end{cases}
\end{aligned}$$

### 2.1.4 Random Set Theory

**Probability Space:** An order tuple $(S, \Omega, P)$ is said to be Probability Space if

(I) S is a non-empty set of outcomes of a random experiment $E$,

(II) $\Omega$ is a set of all events (i.e., subsets of S), which is a $\sigma$-field, i.e., satisfies the following properties: (i) $\emptyset \in \Omega$ and (ii) $A \in \Omega \Rightarrow A^c \in \Omega$, where $A^c$ is the complement of $A$ in $\Omega$,

(III) $A_1, A_2, ... \in \Omega \Rightarrow A = \bigcup_{i=1}^{\infty} A_i \in \Omega.$

(IV) $P$ is a probability function for the events, i.e., $P : \Omega \to [0,1]$ and $P(\{x_i\}) = p_i$, $0 \le p_i \le 1, \forall x_i \in S(i = 1,2,3,...), \sum_{i=1}^{\infty} p_i = 1.$

**Random Variable (RV):** If $(S, \Omega, P)$ is a probability space associated with a random experiment $E$ and $\Re$ be a set of real numbers then the mapping $\widehat{X} : (S, \Omega, P) \to \Re$ is known as random variable or stochastic variable.

The RV is of two types one is continuous RV and another is discrete RV.

**Probability Distribution of Random Variable:** Let $\widehat{X}$ be an RV defined on $(S, \Omega, P)$. A real valued function $F$ defined on $\Re$ that is nondecreasing, right continuous, satisfies

$$F(-\infty) = 0 \text{ and } F(\infty) = 1$$

and is characterize by

$$F(x) = P\{\omega : \hat{X}(\omega) \le x\} \text{ for all } x \in \Re$$

is called Distribution Function (DF) of RV $\widehat{X}$.

**Probability Density function:** If $b > a$,

$$P(a < \hat{X} \le b) = F(b) - F(a) = \int_b^a F'(x)\,dx$$

$$\text{or,}\;\; P(a < \hat{X} \le b) = \int_b^a f(x)\,dx$$

where $f(x) = F'(x)$. The function $f(x)$ is called probability density function of the random variable $x$.

There are several types of probability distributions for describing various types of discrete and continuous random variables. One of them is normal distribution which is used in this thesis.

**Normal Distribution:** The best known and most widely used probability distribution is the Normal distribution. The density function of the normal distribution is a bell-shaped symmetrical curve about mean (cf. Fig. 2.7) and its probability density function with the parameters $m$ (mean) and $\sigma(> 0)$ (standard deviation) is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{-(x-m)^2}{2\sigma^2}\right\} \quad \text{where } -\infty < x < \infty.$$

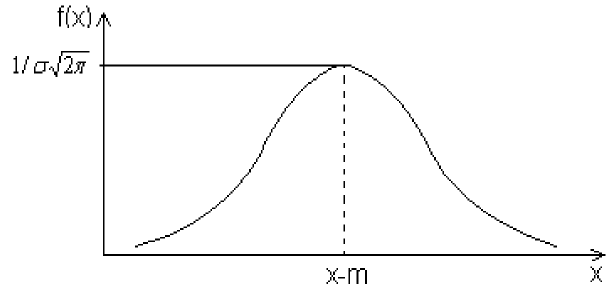



**Figure** 2.7: Density curve of normal distribution

### 2.1.5 Fuzzy-random variable and its properties

**Definition 2.12. Fuzzy-random variable [206]:** *Let R is the set of real numbers, $F_c(R)$ is set of all fuzzy variables and $G_c(R)$ is all of non-empty bounded close interval. In a given probability space $(\Omega, F, P)$, a mapping $\xi : \Omega \to F_c(R)$ is called a fuzzy random variable in $(\Omega, F, P)$, if $\forall\ \alpha \in$(0,1], the set-valued function $\xi_\alpha : \Omega \to G_c(R)$ defined by $\xi_\alpha(\omega) = (\xi(\omega))_\alpha = \{x | x \in R, \mu_{\xi(\omega)}(x) \geq \alpha\}$, $\forall \omega \in \Omega$, is F measurable. Different semantics of fuzzy-random variable are also presented by Xu and Zhou [275].*

**Theorem 2.2.** *Let $\tilde{\bar{\xi}}$ is LR fuzzy random variable, for any $\omega \in \Omega$, the membership function of $\tilde{\bar{\xi}}(\omega)$ is*

$$\mu_{\tilde{\bar{\xi}}(\omega)}(t) = \begin{cases} L\left(\frac{\bar{\xi}(\omega)-t}{\xi_L}\right) & \text{for } t \leq \bar{\xi}(\omega) \\ R\left(\frac{t-\bar{\xi}(\omega)}{\xi_R}\right) & \text{for } t \geq \bar{\xi}(\omega) \end{cases}$$

*where the random variable $\bar{\xi}(\omega)$ is normally distributed with mean $m_\xi$ and standard deviation $\sigma_\xi$ and $\xi_L$, $\xi_R$ are the left and right spreads of $\tilde{\bar{\xi}}(\omega)$. The reference functions L: [0,1]$\to$ [0,1] and R : [0,1]$\to$ [0,1] satisfy that L(1)=R(1)=0,L(0)=R(0)=1, and both are monotone functions. Then*

$$\begin{cases} Pr[Pos\{\tilde{\bar{\xi}}(\omega) \geq t\} \geq \delta] \geq \gamma \\ Pr[Nec\{\tilde{\bar{\xi}}(\omega) \geq t\} \geq \delta] \geq \gamma \end{cases} \quad \textit{are equivalent to}$$

$$t \leq \begin{cases} m_\xi + \sigma_\xi \Phi^{-1}(1-\gamma) + \xi_R R^{-1}(\delta) \\ m_\xi + \sigma_\xi \Phi^{-1}(1-\gamma) - \xi_L L^{-1}(1-\delta) \end{cases}$$

*where $\Phi$ is standard normally distributed, $\delta, \gamma \in$[0,1] are predetermined confidence levels.*

*Proof.* According to definition of possibility we get,

$$Pos[\tilde{\bar{\xi}}(\omega) \geq t] \geq \delta \;\Leftrightarrow\; R\left[\frac{t-\bar{\xi}(\omega)}{\xi_R}\right] \leq \delta \;\Leftrightarrow\; \bar{\xi}(\omega) \geq t - \xi_R R^{-1}(\delta)$$

So for predetermined level $\delta$, $\gamma \in [0, 1]$ we have,

$$Pr[Pos\{\tilde{\bar{\xi}}(\omega) \geq t\} \geq \delta] \geq \gamma$$
$$\Leftrightarrow Pr[\bar{\xi}(\omega) \geq t - \xi_R R^{-1}(\delta)] \geq \gamma$$

$$\Leftrightarrow Pr[\frac{\bar{\xi}(\omega)-m_{\xi}}{\sigma_{\xi}} \geq \frac{t-\xi_R R^{-1}(\delta)-m_{\xi}}{\sigma_{\xi}}] \geq \gamma$$
$$\Leftrightarrow \Phi\left(\frac{t-\xi_R R^{-1}(\delta)-m_{\xi}}{\sigma_{\xi}}\right) \leq 1-\gamma$$
$$\Leftrightarrow t \leq m_{\xi} + \sigma_{\xi}\Phi^{-1}(1-\gamma) + \xi_R R^{-1}(\delta)$$

Similarly from the measure of necessity we have,

$$Nes[\tilde{\bar{\xi}}(\omega) \geq t] \geq \delta \;\Leftrightarrow\; L\left[\frac{\bar{\xi}(\omega)-t}{\xi_L}\right] \geq 1-\delta \;\Leftrightarrow\; \bar{\xi}(\omega) \geq t + \xi_L L^{-1}(1-\delta)$$

So for predetermined level $\delta$, $\gamma \in [0,1]$ we have,

$$Pr[Nes\{\tilde{\bar{\xi}}(\omega) \geq t\} \geq \delta] \geq \gamma$$
$$\Leftrightarrow t \leq m_{\xi} + \sigma_{\xi}\Phi^{-1}(1-\gamma) - \xi_L L^{-1}(1-\delta)$$

The proof is complete.

□

### 2.1.6 Rough Set Theory

Rough set theory, initialized by Pawlak [200], has been proved to be an excellent mathematical tool dealing with vague description of objects. A fundamental assumption in rough set theory is that any object from a universe is perceived through available information, and such information may not be sufficient to characterize the object exactly. One way is the approximation of a set by other sets. Thus a rough set may be defined by a pair of crisp sets, called the lower and the upper approximations, that are originally produced by an equivalence relation (reflexive, symmetric, and transitive). A relation $\simeq$ defined on U is called reflexive if each object is similar to itself, i.e., $x \simeq x$; symmetric if $x \simeq y \Rightarrow y \simeq x$; transitive if $x \simeq y, y \simeq z \Rightarrow x \simeq z$ for any $x, y, z \in U$.

Let $U$ be a universe. Slowinski and Vanderpooten [241] extended the equivalence relation to more general case and proposed a binary similarity relation that has not symmetry and transitivity but reflexivity. Different from the equivalence relation, the similarity relation does not generate partitions on U, for example, the similarity relation defined on $\Re$ as "x is similar to y if and only if $|x-y| \leq 1$".

The similarity class of x, denoted by $R(x)$, is the set of objects which are similar to $x$,

$$R(x) = \{y \in U \,|\, y \simeq x\} \tag{2.24}$$

Let $R^{-1}(x)$ be the class of objects to which $x$ is similar

$$R^{-1}(x) = \{y \in U \,|\, x \simeq y\} \tag{2.25}$$

**Lower and Upper Approximation [241]:** Let $U$ be a universe, and $X$ be a set representing a concept. Then its lower approximation is defined by

$$\underline{X} = \{x \in U \,|\, R^{-1}(x) \subset X\}; \tag{2.26}$$

while the upper approximation is defined by

$$\overline{X} = \bigcup_{x \in X} R(x) \tag{2.27}$$

**Example 2.1.** *Let $\Re$ be a universe. We define a similarity relation $\simeq$ such that $y \simeq x$ if and only if $[y] = [x]$, where $[x]$ represents the largest integer less than or equal to $x$. For the set $[0, 1]$, we have $\underline{[0, 1]} = [0, 1)$, and $\overline{[0, 1]} = [0, 2)$. All sets $[0, r)$ with $0 \leq r \leq 1$ have the same upper approximation* $[0, 1)$.

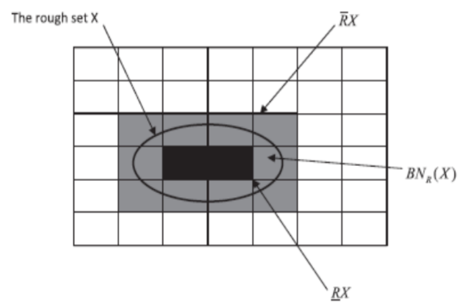



**Figure** 2.8: A graphical representation of a rough set

**Rough Set [200]:** Let $U$ be the universe and and let $R$ be an equivalence relation on $U$. For any subset $X \in U$, the pair $T = (U, R)$ is called an approximation space.
The two subsets:

$$\underline{R}X = \{x \in U \mid [x]_R \subset X\} \tag{2.28}$$

$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \phi\} \tag{2.29}$$

are called the R-lower (2.28) and R-upper (2.29) approximation of $X$, respectively. The $R$-boundary region of $X$ is denoted by $BN_R(X)$ and defined as

$$BN_R(X) = \overline{R}X - \underline{R}X \tag{2.30}$$

$POS_R(X) = \underline{R}X$ is used to denote the $R-$positive region of $X$ (represented by the blackened cell in Fig. 2.8). $NEG_R(X) = U - \overline{R}X$ is used to denote the $R-$negative region of $X$ (represented by the white cell in Fig. 2.8). The cells of Fig. 2.8 represent objects to be evaluated, white cells are considered to be out side the rough set, black cells are definitely within the rough set. Gray cells in Fig. 2.8 may or may not fit within our set. Therefore it is obvious that if $BN_R(X) = 0$, then we have a crisp set, $BN_R(X) > 0$ provides us with a rough set for our evaluation.

**Rough space [146]:** Let $\Lambda$ be a nonempty set, $\kappa$ a $\sigma$-algebra of subset of $\Lambda$ and $\Delta$ be an element in $\kappa$ and $\pi$ nonnegative, real-valued, additive set function. Then $(\Lambda, \Delta, \kappa, \pi)$ is called a rough space.

When we do not have information enough to determine the measure $\pi$ for a real-life problem, we use Laplace criterion which assumes that all elements in $\Lambda$ are equally likely to occur. For this case, the measure $\pi$ may be taken as the cardinality of the set $\Lambda$. This criterion will be used in all examples in this theses for simplicity.

**Rough variable [146]:** A rough variable $\check{\xi}$ is a measurable function from the rough space $(\Lambda, \Delta, \kappa, \pi)$ to the set of real numbers. That is for every Borel set $B$ of $\Re$, we have

$$\{\lambda \in \Lambda \mid \check{\xi}(\lambda) \in B\} \in \kappa$$

The lower and upper approximations of the rough variable are defined as follows,
$\underline{\check{\xi}} = \{\check{\xi}(\lambda) \mid \lambda \in \Delta\}$ and $\overline{\check{\xi}} = \{\check{\xi}(\lambda) \mid \lambda \in \Lambda\}$ respectively.

**Example 2.2.** *Let* $\Lambda = \{x| \, 0 \le x \le 10\}$ *and* $\Delta = \{x| \, 2 \le x \le 6\}$. *Then the function* $\check{\xi}(x) = x^2$ *defined on* $(\Lambda, \Delta, \kappa, \pi)$ *is a rough variable.*

**Example 2.3.** *A rough variable* $([a,b][c,d])$ *with* $c \le a \le b \le d$ *is a measurable function from a rough space* $(\Lambda, \Delta, \kappa, \pi)$ *to the real line, where* $\Lambda = \{x| \, c \le x \le d\}$, $\Delta = \{x| \, a \le x \le b\}$ *and* $\check{\xi}(x) = x$ *for all* $x \in \Lambda$.

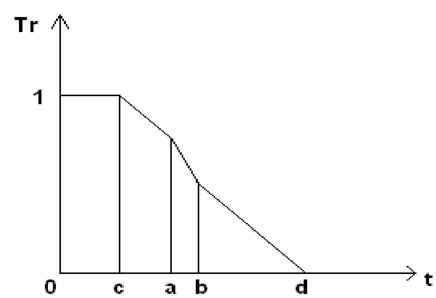



Figure 2.9: Tr $\{\check{\xi} \ge t\}$ function curve

**Trust measure [146]:** Let $(\Lambda, \Delta, \kappa, \pi)$ be a rough space. The trust measure of event $A$ is denoted by $Tr\{A\}$ and defined by $Tr\{A\} = \frac{1}{2}(\underline{Tr}\{A\} + \overline{Tr}\{A\})$, where $\underline{Tr}\{A\}$ and $\overline{Tr}\{A\}$ denote the lower and upper trust measures of event $A$, defined by $\underline{Tr}\{A\} = \frac{\pi\{A \bigcap \Delta\}}{\pi\{\Delta\}}$ and $\overline{Tr}\{A\} = \frac{\pi\{A\}}{\pi\{\Lambda\}}$ respectively. When the sufficient amount of information is given about the measurement of $\pi$ for a real life problem, it may be viewed as the Lebesgue measure. More generally, the above form can be considered as $Tr\{A\} = (1-\eta)\underline{Tr}\{A\} + \eta\overline{Tr}\{A\}$, where $0 < \eta < 1$

Let $\hat{\xi} = ([a,b][c,d])$, $c \le a \le b \le d$ be a rough variable and lebesgue measure is used for trust measure of an rough event associated with $\hat{\xi} \ge t$. Then the trust measure of the rough event $\hat{\xi} \ge t$ is denoted by $Tr\{\hat{\xi} \ge t\}$ and its function curve (cf. Fig. 2.9) is presented below

$$Tr\{\hat{\xi} \ge t\} = \begin{cases} 0 & \text{for } d \le t \\ \frac{\eta(d-t)}{(d-c)} & \text{for } b \le t \le d, \\ \frac{\eta(d-t)}{d-c} + \frac{(1-\eta)(b-t)}{b-a} & \text{for } a \le t \le b, \\ \frac{\eta(d-t)}{d-c} + (1-\eta) & \text{for } c \le t \le a, \\ 1 & \text{for } t \le c \end{cases} \tag{2.31}$$

**Rough Expectation [146]:** let $\check{X}$ be a rough variable. The expected value of the rough variable $\check{X}$ is denoted by $E[\check{X}]$ and defined by

$$E[\check{X}] = \int_0^\infty Tr(\check{X} \geq r)\, dr - \int_{-\infty}^0 Tr(\check{X} \leq r)\, dr \tag{2.32}$$

provided that at least one of the two integrals is finite.

**Lemma 2.10.** *[146]: Let $\check{\xi} = ([a,b][c,d])$ is a rough variable where $c > 0$. Then expected value of $\check{\xi}$ is*

$$E[\check{\xi}] = \frac{1}{4}(a+b+c+d) \tag{2.33}$$

*Proof.* Since $\check{\xi} = ([a,b][c,d])$ is a rough variable and $r$ is a crisp number, then from definition of trust measure (taking $\eta = 0.50$) we have,

$$Tr\{\check{\xi} \geq r\} = \begin{cases} 0 & \text{for } d \leq r \\ \frac{d-r}{2(d-c)} & \text{for } b \leq r \leq d, \\ \frac{1}{2}\left(\frac{d-r}{d-c} + \frac{b-r}{b-a}\right) & \text{for } a \leq r \leq b, \\ \frac{1}{2}\left(\frac{d-r}{d-c} + 1\right) & \text{for } c \leq r \leq a, \\ 1 & \text{for } r \leq c \end{cases} \qquad Tr\{\check{\xi} \leq r\} = \begin{cases} 0 & \text{for } r \leq c \\ \frac{r-c}{2(d-c)} & \text{for } c \leq r \leq a, \\ \frac{1}{2}\left(\frac{r-c}{d-c} + \frac{r-a}{b-a}\right) & \text{for } a \leq r \leq b, \\ \frac{1}{2}\left(\frac{r-c}{d-c} + 1\right) & \text{for } b \leq r \leq d, \\ 1 & \text{for } d \leq r \end{cases}$$

So the expected value of $\check{\xi}$ is calculated using (2.32) as follows:

$$\begin{aligned} E[\check{\xi}] &= \int_0^\infty Tr(\check{\xi} \geq r)\, dr - \int_{-\infty}^0 Tr(\check{\xi} \leq r)\, dr \\ &= \int_0^c 1\, dr + \int_c^a \frac{1}{2}\left(\frac{d-r}{d-c} + 1\right) dr + \int_a^b \frac{1}{2}\left(\frac{d-r}{d-c} + \frac{b-r}{b-a}\right) dr + \int_b^d \frac{d-r}{2(d-c)}\, dr \\ &= \frac{1}{4}(a+b+c+d) \end{aligned} \tag{2.34}$$

□

**Theorem 2.3.** *Let $\hat{\xi} = ([a,b][c,d])$, $c \leq a \leq b \leq d$ be a rough variable and a rough event is $\hat{\xi} \geq t$. Then $Tr\{\hat{\xi} \geq t\} \geq \alpha$ iff*

$$t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta}, & b \leq t \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)}, & a \leq t \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta}, & c \leq t \leq a \\ c & \end{cases}$$

*Proof.* For predetermined level $\alpha \in [0,1]$, we have,
$Tr[\hat{\xi} \geq t] \geq \alpha$
$\Leftrightarrow \alpha \leq Tr[\hat{\xi} \geq t]$,

$$\Leftrightarrow \alpha \leq \begin{cases} \frac{\eta(d-t)}{(d-c)} & \text{for } b \leq t \leq d, \\ \frac{\eta(d-t)}{d-c} + \frac{(1-\eta)(b-t)}{b-a} & \text{for } a \leq t \leq b, \\ \frac{\eta(d-t)}{d-c} + (1-\eta) & \text{for } c \leq t \leq a, \\ 1 & \text{for } t \leq c \end{cases}$$

[using the definition of trust measure of an event]

$$\Leftrightarrow t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta}, & b \leq t \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)}, & a \leq t \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta}, & c \leq t \leq a \\ c \end{cases}$$

The proof is complete. □

**Modified Trust measure :** We consider a modification/ refinement of the rough intervals. Let $\hat{\xi}$ is rough variable given by $\hat{\xi}$=([a,b],[c,d]), $0 \leq c \leq c_1 \leq a \leq b \leq c_2 \leq d$. Here, the trust measures (cf. Eqs. (2.35) and (2.36)) of rough events $\hat{\xi} \geq r$ and $\hat{\xi} \leq r$ are considered on five sub-intervals (extension of three sub-intervals trust measure) and corresponding function curves are depicted in Figs. 2.10 and 2.11.

$$Tr\{\hat{\xi} \geq r\} = \begin{cases} 0, & d \leq r \\ \frac{(d-r)}{3(d-c)}, & c_2 \leq r \leq d \\ \frac{(d-r)}{3(d-c)} + \frac{(c_2-r)}{3(c_2-c_1)}, & b \leq r \leq c_2 \\ \frac{1}{3}[\frac{(d-r)}{(d-c)} + \frac{(c_2-r)}{(c_2-c_1)} + \frac{(b-r)}{(b-a)}], & a \leq r \leq b \\ \frac{1}{3}[\frac{(d-r)}{(d-c)} + \frac{(c_2-r)}{(c_2-c_1)} + 1], & c_1 \leq r \leq a \\ \frac{1}{3}[\frac{(d-r)}{(d-c)} + 2], & c \leq r \leq c_1 \\ 1, & r \leq c. \end{cases} \tag{2.35}$$

$$Tr\{\hat{\xi} \leq r\} = \begin{cases} 0, & r \leq c \\ \frac{(r-c)}{3(d-c)}, & c \leq r \leq c_1 \\ \frac{1}{3}[\frac{(r-c)}{(d-c)} + \frac{(r-c_1)}{(c_2-c_1)}], & c_1 \leq r \leq a \\ \frac{1}{3}[\frac{(r-c)}{(d-c)} + \frac{(r-c_1)}{(c_2-c_1)} + \frac{(r-a)}{(b-a)}], & a \leq r \leq b \\ \frac{1}{3}[\frac{(r-c)}{(d-c)} + \frac{(r-c_1)}{(c_2-c_1)} + 1], & b \leq r \leq c_2 \\ \frac{1}{3}[\frac{(r-c)}{(d-c)} + 2], & c_2 \leq r \leq d \\ 1, & d \leq r. \end{cases} \tag{2.36}$$

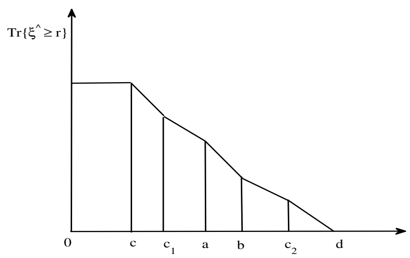



**Figure** 2.10: $Tr\{\hat{\xi} \geq r\}$ function curve.

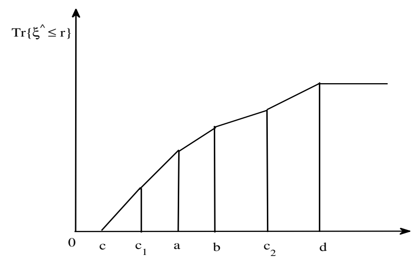



**Figure** 2.11: $Tr\{\hat{\xi} \leq r\}$ function curve.

### 2.1.7 Fuzzy-Rough variable

**Definition 2.13. Fuzzy-Rough variable [146]:** *Fuzzy rough variable is a measurable function from a rough space* $(\Lambda, \Delta, \kappa, \pi)$ *to the set of fuzzy variables. More generally, a fuzzy rough variable is a rough variable taking fuzzy values.*

**A. Fuzzy-Rough Expectation [146]:** Suppose $\check{\check{X}}$ be a fuzzy rough variable. The expected value of the fuzzy-rough variable $\check{\check{X}}$ is denoted by $E[\check{\check{X}}]$ and is defined by

$$E[\check{\check{X}}] = \int_0^\infty Tr(\lambda \in \Lambda \,|\, E[\check{\check{X}}(\lambda)] \geq r)\, dr - \int_{-\infty}^0 Tr(\lambda \in \Lambda \,|\, E[\check{\check{X}}(\lambda)] \leq r)\, dr \quad (2.37)$$

provided that at least one of the two integrals is finite.

**Lemma 2.11.** *Let* $\check{\check{\xi}} = (\check{\xi} - L, \check{\xi}, \check{\xi} + R)$ *be a Triangular fuzzy rough variable, where* $\check{\xi} = ([a, b][c, d])$ *is a rough variable, then expected value of* $\check{\check{\xi}}$ *is*

$$E[\check{\check{\xi}}] = \frac{1}{4}(a + b + c + d) + \frac{\rho R - (1 - \rho)L}{2} \quad (2.38)$$

*Proof.* Since $\check{\check{\xi}} = (\check{\xi} - L, \check{\xi}, \check{\xi} + R)$ where $\check{\xi} = ([a, b][c, d])$ is a triangular fuzzy rough variable, then using the expectation of fuzzy variable (cf. Lemma 2.8) we get,

$$E[\check{\check{\xi}}] = E[\frac{1}{2}\{(1 - \rho)(\check{\xi} - L) + \check{\xi} + \rho(\check{\xi} + R)\}] = E[\check{\xi} + \theta]$$

where $\theta = \frac{\rho R - (1 - \rho)L}{2}$ and $0 \leq \rho \leq 1$
Again, using Lemma 2.10, we get,

$$E[\check{\xi} + \theta] = \frac{1}{4}(a + b + c + d) + \theta = \frac{1}{4}(a + b + c + d) + \frac{\rho R - (1 - \rho)L}{2}.$$

Therefore,

$$E[\check{\check{\xi}}] = \frac{1}{4}(a + b + c + d) + \frac{\rho R - (1 - \rho)L}{2} \qquad \text{where } 0 \leq \rho \leq 1$$

□

**Lemma 2.12.** *Let $\check{\tilde{\xi}} = (\check{\xi} - L_1, \check{\xi} - L_2, \check{\xi} + R_1, \check{\xi} + R_2)$ be a trapezoidal fuzzy rough variable, where $\check{\xi} = ([a,b][c,d])$ is a rough variable. Then expected value of $\check{\tilde{\xi}}$ is*

$$E[\check{\tilde{\xi}}] = \frac{1}{4}[a+b+c+d] + \frac{\rho(R_1+R_2) - (1-\rho)(L_1+L_2)}{2} \quad where \quad 0 \le \rho \le 1.$$

*Proof.* Since $\check{\tilde{\xi}} = (\check{\xi} - L_1, \check{\xi} - L_2, \check{\xi} + R_1, \check{\xi} + R_2)$ where $\check{\xi} = ([a,b][c,d])$ is a trapezoidal fuzzy rough variable,
then using Lemma 2.9 we get,

$$E[\check{\tilde{\xi}}] = E[\frac{1}{2}[(1-\rho)(2\check{\xi} - L_1 - L_2) + \rho(2\check{\xi} + R_1 + R_2)]] \quad where \quad 0 \le \rho \le 1$$

$$= E[\check{\xi} + \theta] \quad where \quad \theta = \frac{\rho(R_1+R_2) - (1-\rho)(L_1+L_2)}{2}$$

Again using Lemma 2.10 we get,

$$E[\check{\xi} + \theta] = \frac{1}{4}[a+b+c+d] + \theta$$

$$= \frac{1}{4}[a+b+c+d] + \frac{\rho(R_1+R_2) - (1-\rho)(L_1+L_2)}{2}.$$

Therefore,

$$E[\check{\tilde{\xi}}] = \frac{1}{4}[a+b+c+d] + \frac{\rho(R_1+R_2) - (1-\rho)(L_1+L_2)}{2} \quad where \quad 0 \le \rho \le 1.$$

□

**Lemma 2.13.** *[146]: Let $\check{\tilde{\xi}}$ is a fuzzy rough variable and $g_k(\check{\tilde{\xi}})$ be the continuous functions for $k = 1, 2, ..., p$. Then the possibility $pos\{g_k(\check{\tilde{\xi}}) \le 0, k = 1, 2, ..., p\}$ is a rough variable.*

**Lemma 2.14.** *[146]: Let $\check{\tilde{\xi}}$ is a fuzzy rough variable and $g_k(\check{\tilde{\xi}})$ be the continuous functions for $k = 1, 2, ..., p$. Then the necessity $nes\{g_k(\hat{\tilde{\xi}}) < 0, k = 1, 2, ..., p\}$ is a rough variable.*

**B. Chance of Fuzzy Rough event:** A fuzzy rough constraint represents a fuzzy event which frequently occurs in constraint inventory control problem in fuzzy rough environment. There are several approaches to find a feasible region of search space of such problems. Chance measure of such constraints due to solution $X$ can be taken as feasibility measure of the solution. Here two approaches of chance measure is presented which are followed to solve some inventory control models of this thesis.

**Approach-1 [146]:** Let $\check{\tilde{\xi}}$ be a fuzzy rough vector on the rough space $(\Lambda, \Delta, \kappa, \pi)$ and $g_r(\check{\tilde{\xi}})$ be the continuous functions from $\Re^n \longrightarrow \Re$, $r = 1, 2, ..., p$. Then the chance of the fuzzy rough event $g_r(\check{\tilde{\xi}}) \le 0 \;\; for \;\; r = 1, 2, ..., p$. be a function from [0,1] to [0,1], which is defined as

$$Ch\{g_r(\check{\tilde{\xi}}) \le 0, r = 1, 2, ..., p\}(\alpha) = sup\{\beta \mid Tr\{pos\{g_r(\check{\tilde{\xi}}) \le 0, r = 1, 2, ..., p\} \ge \beta\} \ge \alpha\}$$

where $\alpha, \beta \in [0, 1]$.

**Approach-2 [146]:** Let $\check{\tilde{\xi}}$ be a fuzzy rough vector on the rough space $(\Lambda, \Delta, \kappa, \pi)$ and $g_r(\check{\tilde{\xi}})$ be the continuous functions from $\Re^n \longrightarrow \Re$, $r = 1, 2, ..., p$. Then the chance of the fuzzy rough event $g_r(\check{\tilde{\xi}}) \leq 0 \ for \ r = 1, 2, ..., p$. be a function from [0,1] to [0,1], which is defined as

$$Ch\{g_r(\check{\tilde{\xi}}) \leq 0, r = 1, 2, ..., p\}(\alpha) = sup\{\beta \mid Tr\{nes\{g_r(\check{\tilde{\xi}}) < 0, r = 1, 2, ..., p\} \geq \beta\} \geq \alpha\}$$

where $\alpha, \beta \in [0, 1]$.

**Theorem 2.4.** *Let $\tilde{\hat{\xi}} = (\hat{\xi} - L, \hat{\xi}, \hat{\xi} + R)$ is a fuzzy rough variable characterized the following membership function,*

$$\mu_{\tilde{\hat{\xi}}}(t) = \begin{cases} \frac{t-\hat{\xi}+\xi_L}{\xi_L} & \text{for } \hat{\xi} - \xi_L \leq t \leq \hat{\xi} \\ \frac{\hat{\xi}+\xi_R-t}{\xi_R} & \text{for } \hat{\xi} \leq t \leq \hat{\xi} + \xi_R \\ 0 & \text{otherwise.} \end{cases}$$

*where $\xi_L$ and $\xi_R$ are left and right spreads of $\tilde{\hat{\xi}}$, and $\hat{\xi} = ([a, b][c, d])$ rough variable, characterized by the above mentioned trust measure function then for an event $\tilde{\hat{\xi}} \geq t$,*

$$\begin{cases} Tr[Pos(\tilde{\hat{\xi}} \geq t) \geq \beta] \geq \alpha \\ Tr[Nec(\tilde{\hat{\xi}} \geq t) \geq \beta] \geq \alpha \end{cases} \quad \textit{are equivalent to}$$

$$\begin{cases} t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta} + (1-\beta)\xi_R, & \text{for} \quad b \leq t - (1-\beta)\xi_R \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)} + (1-\beta)\xi_R, & \text{for} \quad a \leq t - (1-\beta)\xi_R \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta} + (1-\beta)\xi_R, & \text{for} \quad c \leq t - (1-\beta)\xi_R \leq a \\ c + (1-\beta)\xi_R \end{cases} \\ \\ t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta} - \beta\xi_L, & \text{for} \quad b \leq t + \beta\xi_L \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)} - \beta\xi_L, & \text{for} \quad a \leq t + \beta\xi_L \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta} - \beta\xi_L, & \text{for} \quad c \leq t + \beta\xi_L \leq a \\ c - \beta\xi_L \end{cases} \end{cases}$$

*Proof.* For the given confidence level $\alpha$ and $\beta \ in[0, 1]$ we have,

$$Tr[Pos(\tilde{\hat{\xi}} \geq t) \geq \beta] \geq \alpha$$

$$\Leftrightarrow Tr[\tfrac{\hat{\xi}+\xi_R-t}{\xi_R} \geq \beta] \geq \alpha \qquad \text{(using Lemma 2.1)}$$

$$\Leftrightarrow Tr[\hat{\xi} \geq t - (1-\beta)\xi_R] \geq \alpha$$

$$\Leftrightarrow \alpha \leq \begin{cases} \frac{\eta[d-t+(1-\beta)\xi_R]}{(d-c)} & \text{for } b \leq t - (1-\beta)\xi_R \leq d, \\ \frac{\eta[d-t+(1-\beta)\xi_R]}{d-c} + \frac{(1-\eta)[b-t+(1-\beta)\xi_R]}{b-a} & \text{for } a \leq t - (1-\beta)\xi_R \leq b, \\ \frac{\eta[d-t+(1-\beta)\xi_R]}{d-c} + (1-\eta) & \text{for } c \leq t - (1-\beta)\xi_R \leq a, \\ 1 & \text{for } t - (1-\beta)\xi_R \leq c. \text{ (using trust measure)} \end{cases}$$

$$\Leftrightarrow t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta} + (1-\beta)\xi_R, & \text{for} \quad b \leq t - (1-\beta)\xi_R \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)} + (1-\beta)\xi_R, & \text{for } a \leq t - (1-\beta)\xi_R \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta} + (1-\beta)\xi_R, & \text{for} \quad c \leq t - (1-\beta)\xi_R \leq a \\ c + (1-\beta)\xi_R \end{cases}$$

Similarly for necessity measure with the same confidence level

$$Tr[Nes(\tilde{\hat{\xi}} \geq t) \geq \beta] \geq \alpha$$

$$\Leftrightarrow Tr[\hat{\xi} \geq t + \beta\xi_L] \geq \alpha \qquad \text{(using Lemma 2.2)}$$

$$\Leftrightarrow \alpha \leq \begin{cases} \frac{\eta[d-t-\beta\xi_L]}{(d-c)} & \text{for } b \leq t+\beta\xi_L \leq d, \\ \frac{\eta[d-t-\beta\xi_L]}{d-c} + \frac{(1-\eta)[b-t-\beta\xi_L]}{b-a} & \text{for } a \leq t+\beta\xi_L \leq b, \\ \frac{\eta[d-t-\beta\xi_L]}{d-c} + (1-\eta) & \text{for } c \leq t+\beta\xi_L \leq a, \\ 1 & \text{for } t+\beta\xi_L \leq c. \text{ (using trust measure)} \end{cases}$$

$$\Leftrightarrow t \leq \begin{cases} d - \frac{\alpha(d-c)}{\eta} - \beta\xi_L, & \text{for} \quad b \leq t+\beta\xi_L \leq d \\ \frac{\eta(b-a)+(1-\eta)b(d-c)-\alpha(d-c)(b-a)}{\eta(b-a)+(1-\eta)(d-c)} - \beta\xi_L, & \text{for} \quad a \leq t+\beta\xi_L \leq b \\ d + \frac{(1-\eta-\alpha)(d-c)}{\eta} - \beta\xi_L, & \text{for} \quad c \leq t+\beta\xi_L \leq a \\ c - \beta\xi_L \end{cases}$$

The proof is complete. □

## 2.2 Single-Objective Optimization in Crisp Environment and Solution Techniques

### 2.2.1 Single-Objective Optimization Problem

The problem of optimization concerns with the maximization/minimization of an algebraic or a transcendental equation of one or more variables, known as objective function under some available resources which are represented as constraints. Such type of problem is known as Single-Objective Optimization Problem (SOOP). This can be formulated as:

$$\left.\begin{array}{l} \text{Find} \qquad x = (x_1, x_2, ..., x_n)^T \\ \text{which maximizes/minimizes } f(x) \\ \text{subject to} \qquad x \in X \\ \text{where } X = \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j = 1,2,...,l \\ h_k(x) = 0, & k = 1,2,...,m \\ x_i \geq 0, & i = 1,2,...,n \end{array} \right\} \end{array}\right\} \quad (2.39)$$

where, $f(x)$, $g_j(x)$, $j = 1, 2, ..., l$ and defined on n-dimensional set.

It is noted that, when both the objective function and $h_k(x)$, $k = 1, 2, ..., m$ are functions constraints are linear, the above SOOP becomes a SOLOP. Otherwise, it is a SONLOP.

A decision variable vector $x$ satisfying all the constraints is called a feasible solution to the problem. The collection of all such solutions forms a feasible region. The SOOP (2.39) is to find a feasible solution $x^*$ such that for each feasible point $x, f(x) \leq f(x^*)$ for maximization problem and $f(x) \geq f(x^*)$ for minimization problem. Here, $x^*$ is called an optimal solution or solution to the problem.

**Local Minimum:** $x^* \in X$ is said to be a local minima of (2.39) if there exists an $\epsilon > 0$ such that $f(x) \geq f(x^*), \ \forall x \in X \ : \ |x - x^*| < \epsilon,$.
**Convex Function:** A function $f(x_1, x_2, ..., x_n)$ is convex if the Hessian Matrix, given by $H(x_1, x_2, ..., x_n) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{n\times n}$, is positive semi-definite/positive definite.
**Global Minimum:** $x^* \in X$ is said to be a global minima of (2.39) if $f(x) \geq f(x^*), \ \forall x \in X$. Otherwise, if the function $f(x)$ is convex then the local minimum solution $x \in X$ is global minimum.
**Convex Programming Problem:** The problem defined in (2.39) is to be called convex programming problem if the objective function $f(x_1, x_2, ..., x_n)$ and the constraint functions $g_j(x_1, x_2, ..., x_n), j = 1, 2, ..., m$ are convex.

For solution of SONLOP by any available NLP method, local optimal solutions are guaranteed. Also, it is known that, a local minimum/maximum solution is a global minimum/maximum for a convex/concave optimization (i.e., a NLP problem to minimize a convex function or to maximize a concave function) problem.

Lot of mathematical techniques based on linearization, gradient based techniques, evolutionary algorithms, stochastic search algorithms, etc., are available in the literature to solve such type of SONLOP. Here, few methods are illustrated, which have been used in this thesis to solve the inventory problems, non-linear in nature.

### 2.2.2 Gradient Based Solution Techniques for Single-Objective Optimization

**Necessary Condition for Optimality:** If a function $f(x)$ is defined for all $x \in X$ and has a relative minimum at $x = x^*$, where $x^* \in X$ and all the partial derivatives $\frac{\partial f(x)}{\partial x_r}$ for $r = 1, 2, ..., n$ are exists at $x = x^*$, then $\frac{\partial f(x^*)}{\partial x_r} = 0$.
**Sufficient Condition for Optimality:** The sufficient condition for a stationary point $x^*$ to be an extreme point is that the matrix of second partial derivatives (Hessian Matrix) of $f(x)$ evaluated at $x = x^*$ is (i) positive definite when $x^*$ is a relative minimum point, and (ii) negative definite when $x^*$ is a relatively maximum point.

#### 2.2.2.1 Generalized Reduced Gradient (GRG) Technique

The GRG technique is a method for solving NLP problems for handling equality as well as inequality constraints. Consider the NLP problem:

$$\left.\begin{array}{ll} \text{Find} & x = (x_1, x_2, ...x_n)^T \\ \text{which maximizes} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j = 1, 2, ..., l \\ h_k(x) = 0, & k = 1, 2, ..., m \\ x_i \geq 0, & i = 1, 2, ..., n \end{array} \right\} \end{array}\right\} \tag{2.40}$$

By adding a non-negative slack variable $s_j \, (\geq 0), \;\; j = 1, 2, ..., l$ to each of the above inequality constraints, the problem (2.40) can be stated as,

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_n)^T & \\ g_j(x) + s_j = 0, & j = 1, 2, ..., l \\ h_k(x) = 0, & k = 1, 2, ..., m \\ x_i \geq 0 & i = 1, 2, ...n \\ s_j \geq 0, & j = 1, 2, ...l \end{array} \right\} \end{array}\right\} \tag{2.41}$$

where the lower and upper bounds on the slack variables, $s_j, \;\; j = 1, 2, ..., l$ are taken as a zero and a large number (infinity) respectively.

Denoting $s_j$ by $x_{j+n}$, $g_j(x) + s_j$ by $\xi_j$, $h_k(x)$ by $\xi_{l+k}$, the above problem can be rewritten as,

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_{n+l})^T & \\ \xi_j(x) = 0, & j = 1, 2, ...l + m \\ x_i \geq 0 & i = 1, 2, ...n + l \end{array} \right\} \end{array}\right\} \tag{2.42}$$

This GRG technique is based on the idea of elimination of variables using the equality constraints. Theoretically, $(l + m)$ variables (dependent variables) can be expressed in terms of remaining $(n - m)$ variables (independent variables). Thus one can divide the $(n + l)$ decision variables arbitrarily into two sets as

$$x = (y, z)^T$$

where, $y$ is $(n - m)$ design or independent variables and $z$ is $(l + m)$ state or dependent variables and

$$\begin{array}{rcl} y & = & (y_1, y_2, ..., y_{n-m})^T \\ z & = & (z_1, z_2, ..., z_{l+m})^T \end{array}$$

Here, the design variables are completely independent and the state variables are dependent on the design variables used to satisfy the constraints

$$\xi_j(x) = 0, \;\; (j = 1, 2, ..., l + m).$$

Consider the first variations of the objective and constraint functions:

$$df(x) = \sum_{i=1}^{n-m} \frac{\partial f}{\partial y_i} dy_i + \sum_{i=1}^{l+m} \frac{\partial f}{\partial z_i} dz_i = \nabla_y^T f \, dy + \nabla_z^T f \, dz \tag{2.43}$$

$$d\xi_j(x) = \sum_{i=1}^{n-m} \frac{\partial \xi_j}{\partial y_i} dy_i + \sum_{i=1}^{l+m} \frac{\partial \xi_j}{\partial z_i} dz_i$$

$$\text{or} \quad d\xi = C\,dy + D\,dz \tag{2.44}$$

$$\text{where} \quad \nabla_y^T f = \left( \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, ..., \frac{\partial f}{\partial y_{n-m}} \right)$$

$$\text{and} \quad \nabla_z^T f = \left( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, ..., \frac{\partial f}{\partial z_{l+m}} \right)$$

$$C = \begin{bmatrix} \frac{\partial \xi_1}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_1}{\partial y_{n-m}} \\ \frac{\partial \xi_2}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_2}{\partial y_{n-m}} \\ \cdots & \cdots\cdots\cdots\cdots\cdots & \cdots \\ \cdots & \cdots\cdots\cdots\cdots\cdots & \cdots \\ \frac{\partial \xi_{l+m}}{\partial y_1} & \cdots\cdots\cdots & \frac{\partial \xi_{l+m}}{\partial y_{n-m}} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{\partial \xi_1}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_1}{\partial z_{l+m}} \\ \frac{\partial \xi_2}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_2}{\partial z_{l+m}} \\ \cdots & \cdots\cdots\cdots\cdots\cdots & \cdots \\ \cdots & \cdots\cdots\cdots\cdots\cdots & \cdots \\ \frac{\partial \xi_{l+m}}{\partial z_1} & \cdots\cdots\cdots & \frac{\partial \xi_{l+m}}{\partial z_{l+m}} \end{bmatrix},$$

$$dy = (dy_1,\, dy_2, ..., dy_{n-m})^T$$

$$\text{and} \quad dz = (dz_1, dz_2, ..., dz_{l+m})^T$$

Assuming that the constraints are originally satisfied at the vector $x$ $(\xi(x) = 0)$, any change in the vector $dx$ must correspond to $d\xi = 0$ to maintain feasibility at $x + dx$. Thus, Eq. (2.44) can be solved as

$$Cdy + Ddz = 0$$

$$\text{or} \quad dz = -D^{-1}Cdy \tag{2.45}$$

The change in the objective function due to the change in $x$ is given by the Eq. (2.43), which can be expressed, using Eq. (2.45) as

$$df(x) = (\nabla_y^T f - \nabla_z^T f D^{-1} C) dy$$

$$\text{or} \quad \frac{df(x)}{dy} = G_R$$

$$\text{where} \quad G_R = \nabla_y^T f - \nabla_z^T f D^{-1} C$$

is called the generalized reduced gradient. Geometrically, the reduced gradient can be described as a projection of the original $n-$dimensional gradient into the $(n - l)$

dimensional feasible region described by the design variables.

A necessary condition for the existence of minimum of an unconstrained function is that the components of the gradient vanish. Similarly, a constrained function assumes its minimum value when the appropriate components of the reduced gradient are zero. In fact, the reduced gradient $G_R$ can be used to generate a search direction $S$ to reduce the value of the constrained objective function. Similarly, to the gradient $\nabla f$ that can be used to generate a search direction $S$ for an unconstrained function. A suitable step length $\lambda$ is to be chosen to minimize the value of $f(x)$ along the search direction. For any specific value of $\lambda$, the dependent variable vector $z$ is updated using Eq. (2.45). Noting that Eq. (2.44) is based on using a linear approximation to the original non-linear problem, so the constraints may not be exactly equal to zero at $\lambda$, i.e., $d\xi \neq 0$. Hence, when $y$ is held fixed, in order to have

$$\xi_j(x) + d\xi_j(x) = 0, \quad j = 1, 2, ..., l + m \tag{2.46}$$

following must be satisfied.

$$\xi(x) + d\xi(x) = 0 \tag{2.47}$$

Using Eq. (2.44) for $d\xi$ in Eq. (2.47), following is obtained

$$dz = D^{-1}(-\xi(x) - Cdy) \tag{2.48}$$

The value $dz$ given by Eq. (2.48) is used to update the value of $z$ as

$$z_{update} = z_{current} + dz \tag{2.49}$$

The constraints evaluated at the updated vector $x$ and the procedure of finding $dz$ using Eq. (2.49) is repeated until $dz$ is sufficiently small.

### 2.2.3 Soft Computing Techniques for Optimization

Heuristic optimization provides a robust and efficient approach for solving complex real world problems. Recently, complicated inventory control problems are also solved using heuristic approaches by several researchers [150, 162]. Among basic heuristic algorithms GA and PSO are much used in different areas of science and technology [18, 137]. In this thesis, some soft computing techniques are developed/modified to solve different inventory control problems and are presented below.

#### 2.2.3.1 Genetic Algorithm (GA)

Now-a-days Genetic Algorithm (GA) (Michalewicz [174]; Mondal and Maiti, [179]) is extensively used to solve complex decision making problems in different fields of science and technology. GA is an exclusive search algorithm based on the mechanics of natural selection and genesis which initially was developed by Holland [105], then Goldberg [87].

General structure of GA is presented below:

## GA procedures

**Representation:** A n-dimensional real vector', $X_i = (x_{i1}, x_{i2}, ..., x_{in})$, is used to represent the $i^{th}$ solution, where $x_{i1}, x_{i2}, ..., x_{in}$ represent n decision variables of the decision making problem under consideration. $X_i$ is called $i^{th}$ chromosome and $x_{ij}$ is called $j^{th}$ gene of $i^{th}$ chromosome.

**Initialization:** N such solutions $X_i = (x_{i1}, x_{i2}, ..., x_{in})$, i = 1, 2, ...,N are randomly generated by random number generator within the boundaries of each variable $[B_{jl}, B_{jr}]$, j= 1, 2, ..., n. These bounds are calculated from the nature of the problem and previous experience. Initialized (P(1)) sub-function is used for this purpose.

**Constraint Checking:** For constrained optimization problems, at the time of generation of each individuals $X_i$ of P(1), constraints are checked using a separate sub-function "check constraint($X_i$)", which returns 1 if $X_i$ satisfies the constraints otherwise returns 0. If check constraint ($X_i$) =1, then $X_i$ is included in P(1) otherwise $X_i$ is again generated and it continues until constraints are satisfied.

**Diversity Preservation:** At the time of generation of P(1), diversity is maintained using entropy originating from information theory. Following steps are used for this purpose.

(i) Probability, $pr_{jk}$, that the value of the $i^{th}$ gene (variable) of the $j^{th}$ chromosome which is different from the $i^{th}$ gene of the $k^{th}$ chromosome, is calculated using the formula $pr_{jk} = 1 - \frac{x_{ji} - x_{ki}}{B_{jr} - B_{jl}}$ where $[B_{jl}, B_{jr}]$ is the variation domain of the $i^{th}$ gene.

(ii) Entropy of the $i^{th}$ gene, $E_i(M)$, i=1, 2, ..., n is calculated using the formula: $E_i(M) = \sum_{j=1}^{M-1} \sum_{k=j+1}^{M} -pr_{jk} log(pr_{jk})$, where M is the size of the current population.

(iii) Average entropy of the current population is calculated by the formula: $E(M) = \frac{1}{n} \sum_{i=1}^{n} E_i(M)$

(iv) Incorporating the above three steps, a separate sub-function "check diversity($X_i$)" is developed. Every time a new chromosome $X_i$ is generated, the entropy between this one and previously generated individuals is calculated. If this information quantity is higher than a threshold, $E_T$, fixed at the beginning, $X_i$ is included in the population otherwise $X_i$ is again generated until diversity exceeds the threshold, $E_T$ . This method induces a good distribution of initial population.

**Fitness Value:** This fitness value is measured to check whether the initialised or generated chromosomes are suited for the consideration. Chromosome with higher fitness value receives larger probability of inheritance in subsequent generation, whereas chromosome with low fitness will more likely to be eliminated. In this thesis, the value of the objective function is taken as the fitness of the chromosome.

**Algorithm 1:** GA PSEUDOCODE

1. **Start**
2. Set iteration counter t=0, Maxsize=200, $\epsilon = 0.0001$ and $p_m(0)$ = 0.9.
3. Randomly generate **Initial** population P(t), where diversity in the population is maintained using entropy originating from information theory.
4. **Evaluate** initial population P(t).
5. Set Maxfit= Maximum fitness in P(t) and Avgfit=Average fitness of P(t).
6. **While** $(Maxfit - Avgfit \leq \epsilon)$ **do**
7.     t=t + 1.
8.     Increase age of each chromosome.
9.     **For** each pair of parents **do**
10         Determine probability of crossover $\tilde{p_c}$ for the selected pair of parents
11.         Perform crossover with probability $\tilde{p_c}$.
12.     **End** for
13.     **For** each offspring perform mutation with probability $p_m$ **do**
14.         Store offsprings into offspring set.
15.     **End** for
16.     Evaluate P(t).
17.     Remove from P(t) all individuals with age greater than their lifetime.
18.     Select a percent of better offsprings from the offspring set and insert into P(t), such that maximum size of the population is less than Maxsize.
19.     Remove all offsprings from the offspring set.
20.     Reduce the value of the probability of mutation $p_m$.
21. **End While**
22. Output: Best chromosome of P(t).
23. **End** algorithm.

**Crossover:** For each pair of parent solutions $X_i, X_j$, a random number c is generated from the range [0, 1] and if c ¡ $p_c$, crossover operation is made on $X_i, X_j$. To make crossover operation on each pair of coupled solutions $X_i, X_j$ a random number $c_1$ is generated from the range [0,1] and their offsprings $Y_1$ and $Y_2$ are determined by the formula: $Y_1 = c_1X_i + (1 - c_1)X_j, Y_2 = c_1X_j + (1 - c_1)X_i$.

For constrained optimization problems, if a child solution satisfies the constraints of the problem, then it is included in the offspring set otherwise it is not included in the offspring set.

**Mutation:**

(i) *Selection for mutation:* For each offspring generate a random number r from the range [0, 1]. If $r < p_m$ then the solution is taken for mutation, where $p_m$ is the probability of mutation.

(ii) *Mutation process:* To mutate a solution $X = (x_1, x_2, ..., x_n)$, a random integer I in the range [1,n] has to be selected. Then replace $x_i$ by randomly generated value within the boundary $[B_{il}, B_{ir}]$ of $i^{th}$ component of X. New solution (if satisfies constraints of the problem) replaces the parent solution. If child solution does not satisfy the constraint, then parent solution will not be replaced by child solution. Constraint checking of a child solution $C_i$ is made using "check constraint $(C_i)$" function.

**Reduction process of $p_m$:** According to real world demand as generation increases, $p_m$ will decrease smoothly since the search space was more wide initially and after some iterations, it should move towards the convergence. This concept lead us to reduce the value of $p_m$ in each generation. Let $p_m(0)$ is the initial value of $p_m$. Then probability of mutation in T-th generation $p_m(T)$ is calculated by the formula $p_m(T) = p_m(0)\ exp(-T/\alpha_1)$, where $\alpha_1$ is calculated so that the final value of $p_m$ is small enough ($10^{-2}$ in our case). So $\alpha_1 = Maxgen/log[\frac{p_m(0)}{10^{-2}}]$, where Maxgen is the expected number of generations that the GA can run for convergence.

**Selection of offsprings:** Maximum population growth in a generation is assumed as forty percent. So not all offsprings are taken into the parent set for next generation. At first offspring set is arranged in descending order in fitness. Then better solutions are selected and entered into parent set such that population size does not exceeds Maxsize.

**Termination Condition:** Algorithm terminates when difference between maximum fitness (Maxfit) of chromosome, i.e., fitness of the best solution of the population and average fitness (Avgfit) of the population becomes negligible.

**Implementation:** With the above functions and values, the algorithm is implemented using C-programming language

#### 2.2.3.2 Fuzzy Age based Genetic Algorithm (FAGA)

Following Last and Eyal [137], here, a GA (Roy *et al.* [222]) with varying population size is used where diversity of the chromosomes in the initial population is maintained using entropy originating from information theory and chromosomes are classified into young, middle age and old (in fuzzy sense) according to their age and lifetime. Following comparison of fuzzy numbers using possibility theory (Liu and Iwamura [148]), here crossover probability is measured as a function of parent's age interval (a fuzzy rule base on parents age limit is also used for this purpose).

**Determination of fitness and lifetime:** Value of the objective function due to the solution $X_i$, is taken as fitness of $X_i$. Let it be Z($X_i$). At the time of initialization, age of each solution is set to zero. Following Michalewicz [174] at the time of birth, life-time of $X_i$ is computed using the following formula:

If Avgfit$\geq Z(X_i)$, lifetime($X_i$)=Minlt+$\frac{K(Z(X_i)-Minfit)}{Avgfit-Minfit}$,

If Avgfit$< Z(X_i)$, lifetime($X_i$)=$\frac{Minlt+Maxlt}{2}$+$\frac{K(Z(X_i)-Avgfit)}{Maxfit-Avgfit}$.

where Maxlt and Minlt are maximum and minimum allowed lifetime of a chromosome, K=(Maxlt -Minlt)/2. Maxfit, Avgfit and Minfit represent respectively the best, average and worst fitness of the current population. To optimize objective function, it is assumed that Maxlt=7 and Minlt=1, N=10. According to the age, a chromosome can belongs to any one of age intervals-young, middle-age or old, whose membership functions are presented in Fig. 2.12. For a small positive number $\delta$ given by the user, the common fuzzy age (a,b,c) is described by Eq. (2.50).

$$Age = \begin{cases} Young, & \text{for } a \leq age < b - \delta \\ Middle, & \text{for } b - \delta \leq age \leq b + \delta \\ Old, & \text{for } b + \delta < age \leq c \end{cases} \tag{2.50}$$

**Crossover Process for FAGA:**

*Determination of probability of crossover*( $\tilde{p}_c$ ): Probability of crossover $\tilde{p}_c$ , for a pair of parents $(X_i, X_j)$ is determined as:

(i) Following Maiti [161], at first age intervals (young, middle-age, old) of $X_i$ and $X_j$ are determined by making possibility measure of fuzzy numbers young, middle-age, old with respect to their ages.

(i) After determination of age intervals of the parents, their crossover probability ( $\tilde{p}_c$ ) is determined as a linguistic variable (low, medium or high) using a fuzzy rule base as presented in Table 2.1. Membership function of these linguistic variables are presented in Fig. 2.13.

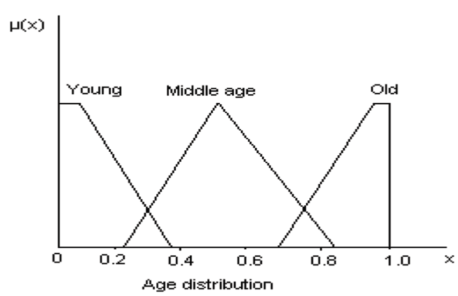



**Figure** 2.12: Membership functions of age intervals.

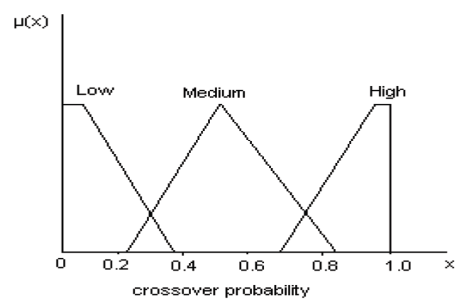



**Figure** 2.13: Membership functions of crossover probabilities.

**Table** 2.1: Fuzzy rule base for crossover probability

| Parent-2 | Parent-1 | | |
|---|---|---|---|
| | Young | Middle-age | Old |
| Young | Low | Medium | Low |
| Middle-age | Medium | High | Medium |
| Old | Low | Medium | Low |

#### 2.2.3.3 Teaching-Learning-Based Optimization (TLBO)

TLBO is a teaching-learning based process developed by Rao *et al.* [211, 212] that intents to study the influence of a teacher on the output of learners in a class. This algorithm is basically helpful in mimicking the teaching-learning ability of teachers and learners in a class room where there are two phases as Teacher phase (highly learned person) and learners phase (learners). The output of the **TLBO** algorithm concerns with results or grades of a learner that depend on the quality of teacher. Meanwhile, the interaction between learners can also be helpful in terms of achieving better outcomes.

Initially, a group of learners is considered as population in this TLBO algorithm. Followed by different subjects offered to the learners using different design variables in such a way that the result will be resemblance to the 'fitness' value of the optimization problem. From the complete set of population, best solutions are treated as teacher. The working principles of learner's phase and teacher's phase are described below:

**Teacher phase:** First step in this phase is learners, learning through a teacher and that teacher will try to increase the mean result of the classroom of value $M_1$ at their potential level (i.e. $T_A$). But in general to get the better value depending on the value of $M_1$, a teacher can move to other mean value $M_2$ depending on their capability. To formulate it, further consider $M_j$ be the mean and $T_i$ be the teacher at any iteration i, whilst $T_i$ will try to enhance existing mean $M_j$ to the new mean as $M_{new}$. Finally the difference between existing mean and new mean is shown as [211, 212]

$$Difference\_Mean_i = r_i(M_{new} - T_F M_j), \tag{2.51}$$

where $T_F$ is the teaching factor which decides the value of mean to be changed and $r_i$ is the random number in the range [0, 1]. Moreover, value of $T_F$ can be either 1 or 2 to decide randomly with equal probability as a heuristic approach

$$T_F = round[1 + rand(0, 1)\{2 - 1\}]. \tag{2.52}$$

Depending on this $Difference\_Mean$, the updated solution can be written as:

$$X_{new,i} = X_{old,i} + Difference\_Mean_i. \tag{2.53}$$

**Learner phase:** Second part of the algorithm where learners gains their knowledge by interacting it in-between. Randomly a learner interacts with other learners for enhancing their knowledge. New things can be learned if the learners gets more knowledge from other learners and mathematically it can be expressed below.
At any iteration i, taking two different learners $X_i$ and $X_j$ where $i \neq j$

$$X_{new,i} = X_{old,i} + r_i(X_i - X_j) \qquad \text{If} \quad f(X_i) < f(X_j), \tag{2.54}$$

$$X_{new,i} = X_{old,i} + r_i(X_j - X_i) \qquad \text{If} \quad f(X_j) < f(X_i) \tag{2.55}$$

Accept $X_{new}$ if it gives better function value. It is necessary to show in the basic TLBO algorithm the updated solution of both the teacher and learner phase. Also, if duplicate solutions are present, they are randomly modified. Hence in the TLBO algorithm, the total number of function evaluations is = $\{$(2 $\times$ population size $\times$ number of generations) + (function evaluations required for duplicate elimination)$\}$. Besides, it may be noted that if duplicate solutions are not removed, then it does not lead to final solution but at times, it may show pre-mature convergence.

**The sketch of TLBO algorithm:** As explained above, the step-wise procedure for implementing TLBO can be noted down as follows.

Step 1 Define the optimization problem and initialize its parameters.

Step 2 Initialize the population.

Step 3 Learners are learning from the teacher in the context to teacher phase

Step 4 With mutual interaction among learners, they increase their knowledge in the context to learner phase.

Step 5 Finally the terminating criterion is to stop if the maximum generation number is achieved; otherwise repeating from Step 3.

The flow chart of the Elitist TLBO algorithm is shown in Fig. 2.14.

**Implementation of TLBO for optimization:** The procedure for implementing TLBO is given below.

**Step 1:** *Define the optimization problem and initialize its parameters*

Initializing the parameters as population size ($P_n$), number of generations ($G_n$), number of design variables ($D_n$), and limits of design variables. ($U_L, L_L$).

Defining the optimization problem as: Minimize f(X). subject to $X_i \in x_i = 1, 2, ..., D_n$ where f(X) is the objective function, X is a vector for design variables such that $L_{L,i} \leq x_{,i} \leq U_{L,i}$.

**Step 2:** *Initialize the population*

According to the population size and number of design variables generate a random population. In TLBO, as population size indicates the number of learners and thus design variables determines the subjects (i.e. courses) offered. Thus, population can be expressed as:

$$Population = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,D} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,D} \\ \vdots & \vdots & \vdots & \vdots \\ x_{P_n,1} & x_{P_n,2} & \cdots & x_{P_n,D} \end{bmatrix}$$

**Step 3:** *Teacher phase*

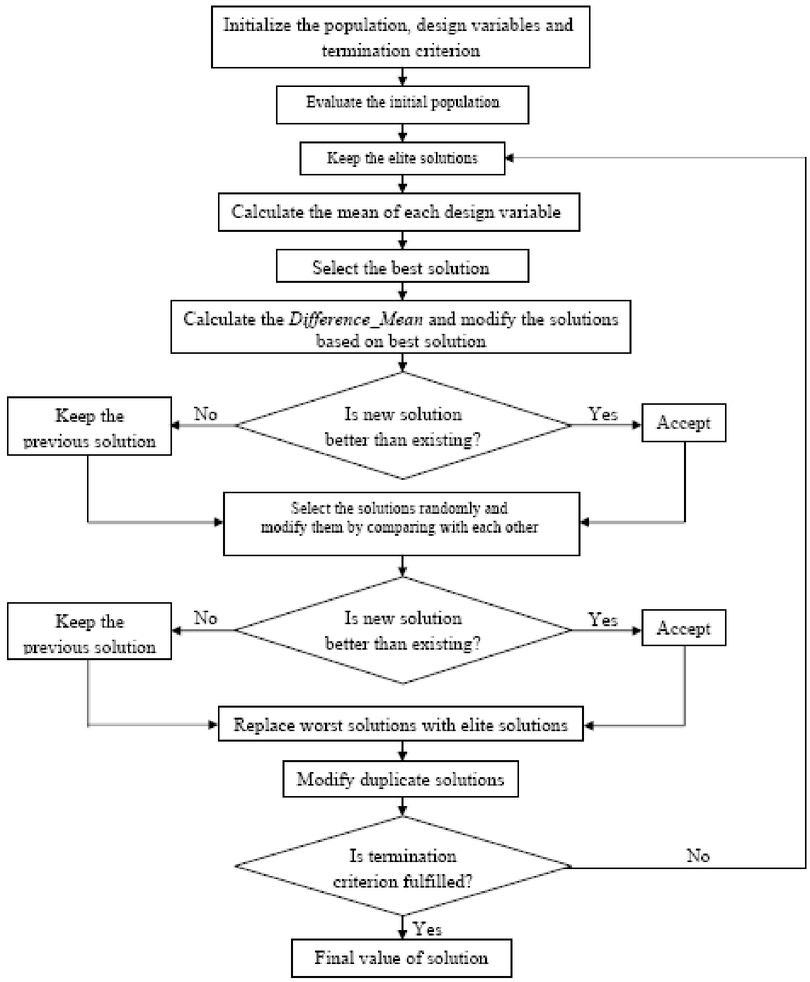



**Figure** 2.14: Flowchart of TLBO algorithm

To calculate the mean of a particular subject, calculate mean of the population column-wise

$$M_{,D} = [m_1, m_2, ..., m_D]. \tag{2.56}$$

The best solution will act as a teacher for that particular iteration

$$X_{teacher} = X_{f(X)=min}. \tag{2.57}$$

Further, the teacher will try to shift the mean from $M_{,D}$ towards $X_{,teacher}$, which act as a new mean for the iteration. Hence,

$$M_{new,D} = X_{teacher,D}. \tag{2.58}$$

The difference between two means is expressed as

$$Difference_{,D} = r(M_{new,D} T_F M_{,D}). \tag{2.59}$$

The value of $T_F$ is selected as 1 or 2. To get the updated value the obtained difference must be added to the current solution using

$$X_{new,D} = X_{old,D} + Difference_{,D} \tag{2.60}$$

Accept $X_{new}$ if it gives better function value.

**Step 4:** *Learner phase*
As explained above, with mutual interaction among learners they increase their knowledge in the context to Learner phase.

**Step 5:** *Termination criterion*
Finally the terminating criterion is to stop if the maximum generation number is achieved; otherwise repeating from Step 3.

It has been observed that there isn't any particular approach to manage the constraints in an optimization problem. Alike other heuristic algorithms (e.g. GA, PSO, ACO, etc.), TLBO algorithm does not have best mechanism to handle the constraints. So it is necessary to inculcate constraint handling technique with TLBO algorithm besides having its own potentials for solving. In this calculation, Deb's heuristic constrained handling method [67] is used to handle the constraints with the TLBO algorithm. Deb's method uses tournament selection operator in which two solutions are considered and compared with each other. The three rules used for selection are

- Out of two solutions if one solution is feasible and the other solution is infeasible, prefer the feasible solution,
- Solution having better objective function must be preferred if both the solutions are feasible.
- Solution having the least constraint violation must be preferred if both the solution provides infeasible solutions.

Above rules are useful at the end of Steps 2 (teacher phase) and 3 (learner phase). Based on three heuristic rules, $X_{new}$ should be selected as per Deb's constraint handling rules [67] as there is need to accept solution of $X_{new}$ if and only if it provides better function value at the end of Steps 2 and 3.

Previously work done on TLBO algorithm by Rao *et al.* [211, 212], Rao and Savsani [210] and Rao and Patel [209], the aspect of 'elitism' was not considered and only two common controlling parameters, i.e. population size and number of generations were used. Also, the detailed investigation of the algorithms performance was not provided with the controlling parameters such as population size and number of generations. Presently, 'elitism' is introduced in the TLBO algorithm to identify effects having its own potentials for solving the algorithm.

To replace the worst solutions during every generation, elite solutions are used as it has been in the limelight of the most of evolutionary and swarm intelligence algorithms. In TLBO algorithm, elite solution can be replaced after the end of learner's phase so as to avoid trapping from local optima whilst having duplicate solutions. Meanwhile, those duplicate solution are modified for better result by mutation on randomly selected dimensions. Apart from it, in the present work, the effect of common controlling parameters of algorithms including elite-size on the performance of algorithm are investigated by taking different population sizes, number of generations and elite sizes.

**Algorithm 2:** TLBO PSEUDOCODE

```
begin
g ⟵ 0;
initialize_population(P, pop_size)
evaluate(P)
repeat
    Elite ⟵ select_best(P, elite)
    for i = 1 ⟶ pop_size do                                  // Teacher Phase
        T_F = round(1 + r)
        X_mean ⟵ calculate_mean_vector(P)
        X_teacher ⟵ best_solution(P)
        X_new = X_i + r.(X_teacher − (TF.X_mean))
        evaluate(X_new)
        if f(X_new) better than f(X_i) then
            X_i ⟵ X_new                                      // End of Teacher Phase
        j⟵ random(pop_size)j ≠ i
        // Learner Phase
        if f(X_i) better than f(X_j) then
            X_new,i = X_old,i + r.(X_i − X_j)
        else
            X_new,i = X_old,i + r.(Xj − X_i)
        evaluate(f(X_new,i))
        if f(X_new,i) better than f(X_i) then
            X_i ⟵ X_new,i                                    // End of Learner Phase
    P ⟵ replace_worst_with_elite(P, Elite)
    P ⟵ remove_duplicate_individuals(P)
    g ⟵ g + 1
until (g≠num_gen)                                            // termination_condition
print_best_result(P)
end
```

## 2.3 Multi-Objective Optimization Problem

### 2.3.1 Multi-Objective Programming Problem

Development of single objective mathematical programming problems and methods for their solutions have been presented in the earlier section. But, the world has become more complex and almost every important real-world problem involves more than one objective. In such cases, decision makers find imperative to evaluate best possible approximate solution alternatives according to multiple criteria.

A general multi-objective programming problem (MOPP) is of the following form:

$$\left.\begin{array}{ll} minimize & f_m(X), \qquad m = 1,2,...,M; \\ subject\ to & g_j(X) \geq 0, \quad j = 1,2,...,J; \\ & x_i^L \leq x_i \leq x_i^U, \quad i = 1,2,...,n; \end{array}\right\} \tag{2.61}$$

where the solution $X$ is a vector of $n$ decision variables (DV). i.e. $X = (x_1, x_2, ..., x_n)^T$. The last set of constraints are called variable bounds, restricting each DV $x_i$ to take a value within a lower $x_i^L$ and an upper $x_i^U$ bound. These bounds constitute the decision space. Here $f_1(x), f_2(x), ..., f_M(x)$ are $M\ (\geq\ 2)$ objectives. It is noted that, if the objectives of the original problem are: minimize $f_i(x)$, for $i = 1, 2, ...., m_0$ and maximize $f_i(x)$ for $i = m_0 + 1, m_0 + 2, ...., M$, then the objective in the mathematical formulation will be

$$\text{Min } F(x) = (f_1(x), f_2(x)...., f_{m_0}(x), -f_{m_0+1}(x), -f_{m_0+2}(x), ...., -f_M(x))^T.$$

subject to the same constraints as in (2.61).

If $f_i(x),\ (i = 1, 2, ...., M)\ and\ g_j(x),\ (j = 1, 2, ...., J)$ are linear, the corresponding problem is called Multi-Objective Linear Programming (MOLP) problem. When all or any one of the above functions is non-linear, it is referred as a Multi-Objective Non-linear Programming (MONLP) problem. Here, the problem is often referred to as a Vector Minimum Problem (VMP).

**Convex and non-convex MOPP:** The multi-objective optimization problem (2.61) is said to be convex if all the objective functions and the feasible region are convex, otherwise it is called non-convex.

**Ideal Objective Vector:** An objective vector minimizing each of the objective functions is called an ideal (or perfect) objective vector.

**Complete optimal solution:** $x^*$ is said to be a complete optimal solution to the MONLP in (2.61) iff there exists $x^* \in X$ such that $f_i(x^*) \leq f_i(x),\ i = 1, 2, ...., k$ for all $x \in X$.

In general, the objective functions of the MONLP conflict with each other, a complete optimal solution does not always exist and so Pareto (or non dominated) optimality concept is introduced.

**Pareto optimal solution:** $x^*$ is said to be a Pareto optimal solution to the MONLP iff there does not exist another $x \in X$ such that $f_i(x) \leq f_i(x^*)$ for all $i,\ i = 1, 2, ...., k$ and $f_j(x) <$

$f_j(x^*)$ for at least one index $j, \ j = 1, 2, ...., k.$

An objective vector $F^*$ is Pareto-optimal if there does not exist another objective vector $F(x)$ such that $f_i \leq f_i^*$ for all $i = 1, 2, \cdots, k$ and $f_j < f_j^*$ for at least one index $j$. Therefore, $F^*$ is Pareto-optimal if the decision vector corresponding to it is Pareto optimal.

Unless an optimization problem is convex, only locally optimal solution is guaranteed using standard mathematical programming techniques. Therefore, the concept of Pareto-optimality needs to be modified to introduce the notion of a locally Pareto-optimal solution for a non-convex problem as defined by Geoffrion [85].

**Locally Pareto optimal solution:** $x^* \in X$ is said to be a locally Pareto optimal solution to the MONLP if and only if there exists an $r > 0$ such that $x^*$ is Pareto optimal in $X \cap N(x^*, r)$, where $N(x^*, r)$ is a r-neighborhood of $x^*$, i.e, there does not exist another $x \in X \cap N(x^*, r)$ such that $f_i(x) \leq f_i(x^*)$.

**Concept of Domination:** Most evolutionary multi-objective optimization algorithms use the concept of domination. In these algorithms, two solutions are compared on the basis of whether one dominates the other solution or not. Let us use the operator $\sqsupseteq$ between two solutions $i$ and $j$ as $i \sqsupseteq j$ denotes that solution $i$ is better than solution $j$ on a particular objective. Similarly $i \sqsubseteq j$ for a particular objective implies that solution $i$ is worse than solution $j$ on this objective. With this assumption a solution x is said to dominate the other solution y, if both the following conditions hold.

- The solution x is not worse than the solution y in all the objectives.
- The solution x is strictly better than the solution y in at least one objective, i.e., $f_j(x) \sqsupseteq f_j(y)$ for at least one $j \in \{1, 2, ..k\}$

Now, let us introduce some non-linear programming techniques which have been used in this thesis to achieve at least local Pareto optimal solutions.

### 2.3.2 Solution Techniques for Multi-Objective Programming Problem in Crisp Environment

#### 2.3.2.1 Multi-Objective Genetic Algorithm (MOGA) :

Genetic algorithm approach was first proposed by Holland [105]. Because of its generality and its several advantages over conventional optimization methods it has been successfully applied to many optimization problems. There are several approaches using genetic algorithms to deal with the multi-objective optimization problems. These algorithms can be classified into two types-(i) Non-Elitist MOGA and (ii) Elitist MOGA. A fast and elitist MOGA was developed following Deb *et al.* [69] and is named as Fast and Elitist Multi-objective Genetic Algorithm (FEMOGA).

#### 2.3.2.2 Fast and Elitist Multi-Objective Genetic Algorithm

This multi-objective genetic algorithm has the following two important components.

**(a) Division of a population of solutions into subsets having non-dominated solutions:** Consider a problem having $M$ objectives and take a population $P$ of feasible solutions of the problem of size $N$. We like to partition $P$ into subsets $F_1, F_2, \cdots, F_k$, such that every subset contains non-dominated solutions, but every solution of $F_i$ is not dominated by any solution of $F_{i+1}$, for $i = 1, 2, ..k-1$. To do this for each solution, $x$, of $P$, calculate the following two entities.

(i) Number of solutions of $P$ which dominate $x$, let it be $n_x$.

(ii) Set of solutions of P that are dominated by $x$. Let it be $S_x$.

The above two steps require $O(MN^2)$ computations. Clearly $F_1$ contains every solution $x$ having $n_x = 0$. Now for each solution $x \in F_1$, visit every member $y$ of $S_x$ and decrease $n_y$ by 1. In doing so if for any member $y$, $n_y = 0$, then $y \in F_2$. In this way $F_2$ is constructed. The above process is continued to every member of $F_2$ and thus $F_3$ is obtained. This process is continued until all subsets are identified. For each solution $x$ in the second or higher level of non-dominated subsets, $n_x$ can be at most $N-1$. So each solution $x$ will be visited at most $N-1$ times before $n_x$ becomes zero. At this point, the solution is assigned a subset and will never be visited again. Since there is at most $N-1$ such solutions, the total complexity is $O(N^2)$. So overall complexity of this component is $O(MN^2)$.

**(b) Determine distance of a solution from other solutions of a subset:** To determine distance of a solution from other solutions of a subset following steps are followed:

(i) First sort the subset according to each objective function values in ascending order of magnitude.

(ii) For each objective function, the boundary solutions are assigned an infinite distance value (a large value).

(iii) All other intermediate solutions are assigned a distance value for the objective, equal to the absolute normalized difference in the objective values of two adjacent solutions.

(iv) This calculation is continued with other objective functions.

(v) The overall distance of a solution from others is calculated as the sum of individual distance values corresponding to each objective. Since $M$ independent sorting of at most $N$ solutions (In case the subset contains all the solutions of the population) are involved, the above algorithm has $O(MNlogN)$ computational complexity.

Using the above two operations proposed multi-objective genetic algorithm takes the following form:

1. Set probability of crossover $p_c$ and probability of mutation $p_m$.
2. Set iteration counter $T = 1$.
3. Generate initial population set of solution $P(T)$ of size $N$.

4. Select solution from $P(T)$ for crossover and mutation.
5. Made crossover and mutation on selected solution and get the child set $C(T)$.
6. Set $P_1 = P(T)UC(T)$ // Here U stands for union operation.
7. Divide $P_1$ into disjoint subsets having non-dominated solutions. Let these sets be $F_1, F_2, \cdots F_k$.
8. Select maximum integer $n$ such that order of $P_2 (= F_1UF_2U \ ... \ UF_n) \leq N$.
9. if $O(P_2) < N$ sort solutions of $F_{n+1}$ in descending order of their distance from other solutions of the subset. Then select first $N - O(P_2)$ solutions from $F_{n+1}$ and add with $P_2$, where $O(P_2)$ represents order of $P_2$.
10. Set $T = T + 1$ and $P(T) = P_2$.
11. If termination condition does not hold go to step-4.
12. Output: P(T)
13. End algorithm.

MOGAs that use non-dominated sorting and sharing are mainly criticized for their

- $O(MN^3)$ computational complexity
- non-elitism approach
- the need for specifying a sharing parameter to maintain diversity of solutions in the population.

In the above algorithm, these drawbacks are overcame. Since in the above algorithm computational complexity of step-7 is $O(MN^2)$, step-9 is $O(MNlogN)$ and other steps are $\leq O(N)$, so overall time complexity of the algorithm is $O(MN^2)$. Here selection of new population after crossover and mutation on old population, is done by creating a mating pool by combining the parent and offspring population and among them, best N solutions are taken as solutions of new population. By this way, elitism is introduced in the algorithm. When some solutions from a non-dominated set $F_j$ (i.e., a subset of $F_j$) are selected for new population, those are accepted whose distance compared to others (which are not selected) are much i.e., isolated solutions are accepted. In this way taking some isolated solutions in the new population, diversity among the solutions is introduced in the algorithm, without using any sharing function. Since computational complexity of this algorithm $< O(MN^3)$ and elitism is introduced, this algorithm is named as FEMOGA.

#### 2.3.2.3 Rough age based Multi-Objective Genetic Algorithm (RMOGA) :

In this thesis RMOGA is used where where diversity of the chromosomes in the initial population is maintained using entropy originating from information theory and chromosomes are classified into young, middle age and old (in fuzzy sense) according to their age and lifetime. Here, imprecise nature of the age lifetime of the chromosomes are considered following rough sense and a five levels rough age based (Very Young (VY), Young (Y), Middle (M), Old (O) and Very Old (VO)) probability of crossover is considered. There are several approaches using genetic algorithms to deal with the multi-objective optimization problems. The better known ones include the plain aggregation approach, the population-based non-pareto approach, the pareto-based approach and Niche induction approach [68, 69]. Proposed multi-objective genetic algorithm has been developed following Deb *et al.* [69] with the help of rough age based probability of crossover.

**Rough Set Based Age Dependent Selection:** For the solution of an optimization problem, in our proposed RMOGA, the age of a chromosome is determined by a new mechanism based on weighted mean of their two objective values i.e. fitness values and then a Rough age based selection is applied. Here the age of each chromosome lie in a region of the common age represented by a rough set using five linguistic expressions. These regions are termed as Very Young (VY), Young (Y), Middle (M), Old (O) and Very Old (VO). Now according to the age distributions of the members (in pair) of the mating pool, similar linguistic variables such as Very Low (VL), Low (L), Medium (M), High (H) and Very High (VH) are generated for the said chromosomes to fix $p_c$s. Using the trust measure of rough set, the probability of crossover, $p_c$ for each chromosome is assigned by the corresponding linguistic variables.

*Determination of age:* The above M such two-objective solutions have fitnesses represented by $Z_1(X_i)$ and $Z_2(X_i)$ of the $i^{th}$ chromosomes. Now $Z(X_i) = \lambda Z_1(X_i)+(1-\lambda)Z_2(X_i), \lambda \in rand[0,1]$. At the time of initialization, each chromosome age is defined as null. Now in every generation, the age is counted using the following mechanism (cf. Michalewicz [174]):

If Avgfit$\geq Z(X_i)$, age($X_i$)=Minage+$\frac{K(Z(X_i)-Minfit)}{Avgfit-Minfit}$,

If Avgfit$< Z(X_i)$, age($X_i$)=$\frac{Minage+Maxage}{2}$+$\frac{K(Z(X_i)-Avgfit)}{Maxfit-Avgfit}$.

where Maxage and Minage are maximum and minimum allowed ages of a chromosome, K=(Maxage - Minage)/2. Maxfit, Avgfit, Minfit represent the best, average and worst fitness of the current population. We calculated average age (Avgage) in each generation. Now since age calculated as crisp values, we construct the common rough values form it as:

Rough Age=([$r_1$*Avgage, $r_2$*Avgage], [$r_3$*Avgage, $r_4$*Avgage]),

where $r_1$=$\frac{Maxage-Avgage}{Avgage}$, $r_2$=$\frac{Maxage+Minage}{2}$, $r_3$=$\frac{Maxage-Minage}{2}$, $r_4$=$\frac{Avgage-Minage}{Avgage}$

According to the age, a chromosome can belongs to any one of age intervals- VY, Y, M, O or VO. The common rough age ([a,b],[c,d]) is extended to $0 \leq c \leq c_1 \leq a \leq b \leq c_2 \leq d$ and is described by Eq. (2.62) and depicted in Fig. 2.15.

$$Age = \begin{cases} Very\,Young, & \text{for } c \leq Very\,Young < c_1 \\ Young, & \text{for } c_1 \leq Young < a \\ Middle, & \text{for } a \leq Middle \leq b \\ Old, & \text{for } b < Old \leq c_2 \\ Very\,Old, & \text{for } c_2 < Very\,Old \leq d \end{cases} \tag{2.62}$$

*Determination of probability of crossover*( $p_c$ ): Probability of crossover $p_c$, for a pair of parents $(X_i, X_j)$ is determined as:

Now we consider the age in a different linguistic code Very Low (VL), Low (L),

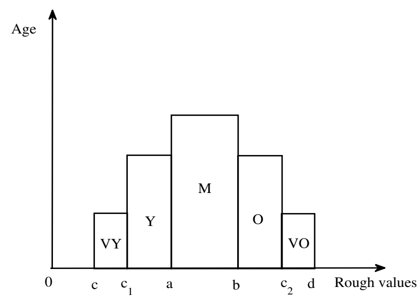



**Figure** 2.15: Rough extended age distribution of interval.

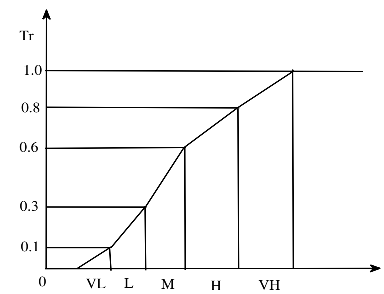



**Figure** 2.16: Rough extended age distribution of $p_c$

Medium (M), High (H) and Very High (VH) scale so that it is more realistic in the sense of classification and acceptable to design for the real world problems. The parent's crossover probability ($p_c$) is determined as a linguistic variables as presented in Table 2.2. Determined $p_c$ values of the extended linguistics are also given in the Fig. 2.16.

**Table** 2.2: Rough extended trust based linguistic

| Chromosomes | Very Young | Young | Middle | Old | Very Old |
|---|---|---|---|---|---|
| Very Young | Very Low | Low | Medium | Low | Very Low |
| Young | Low | Low | High | Low | Very Low |
| Middle | Medium | High | Very High | High | Medium |
| Old | Low | Low | High | Low | Very Low |
| Very Old | Very Low | Very Low | Medium | Very Low | Very Low |

**Division of $P(T)$ into disjoint subsets having non-dominated solutions:** According to Deb *et al.* [69], the following procedure is adapted.

For every $x \in P(T)$ do
    Set $S_x = \Phi$ ,where $\Phi$ represents null set
    $n_x = 0$
    For every $y \in P(T)$ do
        If x dominates y then
            $S_x = S_x U\{y\}$
        Else if y dominates x then
            $n_x = n_x + 1$
        End if
    End For
    If $n_x = 0$ then

$F_1 = F_1 U\{x\}$
End If
End For
Set i=1
While $F_i \neq \Phi$ do
$F_{i+1} = \Phi$
For every $x \in F_i$ do
For every $y \in S_x$ do
$n_y = n_y - 1$
If $n_y = 0$ then
$F_{i+1} = F_{i+1} U\{y\}$
End If
End For
End For
i=i+1
End While
Output:$F_1, F_2, ...F_{i-1}$.

**Determine distance of a solution of subset F from other solutions:** According to Deb *et al.* [69], some modifications are made to evaluate the distance of Pareto solutions which are given as
Set n=number of solutions in F
For every $x \in F$ do
$x_{distance} = 0$
End For
For every objective $m$ do
Sort $F$, in ascending order of magnitude of $m^{th}$ objective.
$F[1] = F[n] = M$, where $M$ is a big quantity.
For i=2 to n-1 do
$F[i]_{distance} = F[i]_{distance} + (F[i+1].objm - F[i-1].objm)/(f_m^{max} - f_m^{min})$
End For
End For
In the algorithm $F[i]$ represents $i^t h$ solution of $F$, $F[i].objm$ represent $m^{th}$ objective value of $F[i]$. $f_m^{max}$ and $f_m^{min}$ represent the maximum and minimum values of $m^{th}$ objective function.

**Complexity analysis:** MOGAs, that use non-dominated sorting and sharing are mainly criticized for their $O(MN^3)$ complexity, but fast and elitist non-dominated sorting algorithm has $O(MN^2)$ computational complexity where N is the popsize and M is the number of objectives. Here also the proposed RMOGA has the same $O(MN^2)$ computational complexity.

## 2.4 Optimization in Fuzzy Environment

### 2.4.1 Single-Objective Optimization in Fuzzy Environment and Solution Techniques

In most of the programming model, the decision maker is not able to define precisely different parameters of the optimization problem under consideration. In these cases, the parameters are defined either as non-stochastic sense, i.e., as fuzzy numbers with feasible membership functions or in stochastic sense, i.e., as random numbers with feasible probability distributions. In case of non-stochastic sense, the problem belongs to the class of SOOP in fuzzy environment and is termed as FNLOP. A crisp non-linear SOOP may be defined as follows:

$$\left.\begin{array}{lll} \text{Max} & f(x,a) & \\ \text{subject to} & g_r(x,a) \le b_r & r=1,2,...,m \\ & x_i \ge 0 & i=1,2,...,n \end{array}\right\} \tag{2.63}$$

where, $x=(x_1,x_2,...,x_n)^T$ is crisp decision vector, $a=(a_1,a_2,...,a_k)^T$ is crisp parameter vector, $b=(b_1,b_2,...,b_m)^T$ is crisp resource vector.
When the vectors $a$, $b$ are fuzzy in nature, i.e., $\tilde{a}$, $\tilde{b}$ , the above problem (2.63) reduces to a FNLOP as

$$\left.\begin{array}{lll} \text{Max} & \tilde{f}(x,\tilde{a}) & \\ \text{subject to} & \tilde{g}_r(x,\tilde{a}) \le \tilde{b}_r & r=1,2,...,m \\ & x_i \ge 0 & i=1,2,...,n \end{array}\right\} \tag{2.64}$$

where, $x=(x_1,x_2,...,x_n)^T$ is crisp decision vector, $\tilde{a}=(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_k)^T$ is fuzzy parameter vector, $\tilde{b}=(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_m)^T$ is fuzzy resource vector (where the symbol $\sim$ represents fuzziness of the parameters).

#### Solution Techniques

As optimization of fuzzy objective is not well defined, it can be transformed to an equivalent crisp problem using different procedures. In this thesis two approaches are used which are presented below:

**Fuzzy Expected Method:** The above problem (2.64) can be transformed to an equivalent crisp problem using expected value of fuzzy objective (cf. Eq. (2.10)) and possibility/necessity measure of fuzzy events for constraints following Liu and Iwamura [148], Maiti [160] for optimistic and pessimistic DMs respectively as below:

$$\left.\begin{array}{rll} \text{Maximize} & E[\tilde{Z}] & \\ \text{subject to} & pos(\tilde{g}_r(x,\tilde{a}) \le \tilde{b}_r) \ge \beta_r & r=1,2,...,m \end{array}\right\} \tag{2.65}$$

$$\left.\begin{array}{rl} \text{Maximize } E[\tilde{Z}] & \\ \text{subject to } \; nes(\tilde{g}_r(x,\tilde{a}) \leq \tilde{b}_r) \geq \beta_r & \quad r = 1,2,...,m \end{array}\right\} \tag{2.66}$$

where $\beta_r$(r=1,2,...,m) are degree of optimism (pessimism) for ODM (PDM). Constraint of the problems (2.65) and (2.66) can be transformed to a deterministic inequality using definition of possibility/necessity measure of fuzzy events. Then transformed problem can be solved using any classical / heuristic optimization technique.

### 2.4.2 Multi-Objective Optimization in Fuzzy Environment and Solution Techniques

#### 2.4.2.1 Intuitionistic Fuzzy Optimization Technique (IFOT) [38]:

To solve multi-objective maximization problems, we have used the following IFOT. For each of the objective functions $f_L(x)$, $f_C(x)$, $f_R(x)$, we first find the upper bounds $U_L, U_C, U_R$ (best values) and the lower bounds $L_L, L_C, L_R$(worst values), where $U_L, U_C, U_R$ are the aspired level achievements and $L_L, L_C, L_R$ are the lowest acceptable level achievements for the objectives $f_L(x)$, $f_C(x)$, $f_R(x)$ respectively and $d_k = U_k - L_k$ is the degradation allowance, or leeway, for objective $f_k(x)$, k = L, C, R. Once the aspiration levels and degradation allowances for each of the objective function have been specified, we form a fuzzy model and then transformed the fuzzy model into a crisp model. The steps of intuitionistic fuzzy programming technique is given below.

**Step 1:** Solve the multi-objective cost function as a single objective cost function using one objective at a time and ignoring all others.

**Step 2:** From the results of step 1, determine the corresponding values for every objective at each solution derived.

**Step 3:** From step 2, we find for each objective, the best $U_k$ and worst $L_k$ values corresponding to the set of solutions. The initial fuzzy models can then be stated as, in terms of the aspiration levels for each objective, as follows : Find $P$ and $t_1$ satisfying $f_k(x)\tilde{>}U_k$, k = L,C,R, subject to the non negatively conditions and associate constraints.

**Step 4:** Define membership function $(\mu_{f_k}; k = L, C, R)$ and a non membership function $(\nu_{f_k}; k = L, C, R)$ for each objective function.

An exponential membership function is defined as

$$\mu_{f_k} = \begin{cases} 1, & f_k \geq U_k \\ \frac{e^{-w\left(\frac{U_k - f_k}{U_k - L_k}\right)} - e^{-w}}{1 - e^{-w}}, & L_k \leq f_k \leq U_k \\ 0, & f_k \leq L_k \end{cases} \tag{2.67}$$

where $w(> 0)$ is the distribution parameter.
A quadratic non-membership function is defined as

$$\nu_{f_k} = \begin{cases} 1, & f_k \leq L_k \\ \left(\frac{U_k - f_k}{U_k - L_k}\right)^2, & L_k \leq f_k \leq U_k \\ 0, & f_k \geq U_k \end{cases} \tag{2.68}$$

$\mu_{f_k}$ is strictly monotonic increasing function with property $\mu_{f_k}(U_k) = 1$, $\mu_{f_k}(L_k) = 0$, where as $\nu_{f_k}(U_k) = 0$ and $\nu_{f_k}(L_k) = 1$. These two functions are continuous within $[L_k, U_k]$. Here, $\mu_{f_k} + \nu_{f_k} \leq 1$ for $L_k \leq f_k \leq U_k$. Therefore, quite naturally the functions meet at a point somewhere in $[L_k, U_k]$.

**Step 5:** After determining the membership and non-membership function defined in ( 2.67) and (2.68) for each objective function following Angelov [6] the problems can be formulated as an equivalent crisp model as

$$\left.\begin{array}{ll} & \max \mu, \quad \min \nu \\ \text{sub to} & \mu \leq \mu_{f_k};\ \text{for all } k = L, C, R \\ & \nu \geq \nu_{f_k};\ \text{for all } k = L, C, R \\ & \mu \geq \nu; \quad \text{and } \mu + \nu \leq 1; \quad \mu, \nu \geq 0 \end{array}\right\} \tag{2.69}$$

where $\mu$ denotes the minimal acceptable degree of objective(s) and constraints and $\nu$ denotes the maximal degree of rejection of objective(s) and constraints. The problem (2.69) can be rewritten as:

$$\left.\begin{array}{ll} & \max\ (\mu - \nu) \\ \text{sub to} & \mu \leq \mu_{f_k};\ \text{for all } k = L, C, R \\ & \nu \geq \nu_{f_k};\ \text{for all } k = L, C, R \\ & \mu \geq \nu; \quad \text{and } \mu + \nu \leq 1; \quad \mu, \nu \geq 0 \end{array}\right\} \tag{2.70}$$

**Step 6:** Now the above problem can be solved by a non-linear optimization technique and optimal solution of $\mu$, (say $\mu^*$) and $\nu$, (say $\nu^*$.) are obtained.

**Step 7: Pareto-Optimal Solution**
After deriving the optimum decision variables, Pareto-optimality test is performed according

to Sakawa [224]. Let the optimum decision vector $x^*$ and the optimum values $f_k^* = f_k(x^*)$, k=L, C, R, are obtained from (2.70). With these values, the following problem is solved using a non-linear optimization technique

$$\left.\begin{array}{rl} & \min V = \sum_k \epsilon_k, \qquad k = L, C, R; \\ \text{sub to} & f_k + \epsilon_k = f_k^*; \text{ for all } k = L, C, R \\ & \epsilon_k \geq 0; \text{ for all } k = L, C, R. \end{array}\right\} \tag{2.71}$$

where $\epsilon_k$ is the deviation parameter from the optimal values.

The optimal solutions of (2.71) are called strong Pareto optimal solution provided V is very small otherwise it is called weak Pareto solution.

# Part II

# Inventory Problems in Uncertain Environments

# Chapter 3

# Inventory Problems with Stock dependent Demand in Random Environment

## 3.1 Introduction

An interesting phenomenon observed in any supermarket is that display of consumer goods in large quantities attracts more and more customers and generates higher demand. Balakrishnan et al. [28] suggested that "high inventories might stimulate demand for a variety of reasons such as- increasing product visibility, kindling latent demand, signalling a popular product, or providing consumers an assurance of future availability". Many researchers [10, 37, 117, 163, 168, 169, 259] have given considerable attention to the situation where the demand rate depends on the level of on-hand inventory (cf. § 1.3.1).

In EPL models, normally UPC of a manufacturing system is assumed as constant. But, in reality, it depends on the combination of different production factors such as raw materials, technical knowledge, resources, production procedure, wear and tear of machineries, firm size, quality of product, environmental pollution etc. Khouja and Mehrez [127] assumed a UPC involving raw material, labour/energy, and wear and tear costs. After that, several authors [169, 171] have implemented this in their EPL models. Das and Maiti [61] used this type of production cost in terms of volume flexibility. Since no manufacturing company can ensure that every machinery systems will remain in good condition for the whole life-time, the reliability parameter takes an important role on the integrated cost function. For more accuracy, we consider the development cost (a part of unit production cost) as a function of reliability parameter which varies with the levels of technology and resources, etc. Sarkar *et al.* [231] considered the UPC as a function of reliability parameter in imperfect production process with safety stock. In the same year, they [230] investigated a model by introducing optimal reliability, production lot size and safety stock. but, till now, none considered the cost to be incurred by the manufacturer against the measures for environment protection.

In a situation where some or all parameters are described by random variables, the problem is expressed as a constraint called chance constraint, which helps to transform the problem to a crisp one. Charnes and Cooper [44] were first to develop the Chance Constraint technique. The problem with fuzzy parameters was solved by Liu and Iwamura [148] in the form of a chance constraint type. In the recent years, it has been extended for various applications in several directions [197]. Similarly following Katagiri *et al.* [126] and Liu [146], the problem with fuzzy-random, rough and fuzzy-rough parameters can be reduced to crisp ones with the help of constraints like chance Constraints.

Classical inventory models are usually developed over the infinite planning horizon. According to Gurunani [95] and Chung and Kim [56], the assumption of infinite planning horizon is not realistic due to several reasons such as variation of inventory costs, change in product specifications and designs, technological developments, etc. Moreover, for seasonal products like fruits, vegetables, warm garments, etc., business period is not infinite, rather fluctuates with each season. Hence the planning horizon for seasonal products varies over the years depending upon the environmental effects. Therefore, it is better to estimate this type finite time horizon as random or fuzzy or fuzzy-random or rough or fuzzy-rough in nature. Moon and Yun [181] developed an EOQ model in random planning horizon. Maiti and Maiti [163] developed inventory models with stock dependent demand and two storage facilities over a random planning horizon. Some more research works are available in this direction [94, 221].

For a manufacturing system, it is difficult to produce perfect quality product for all times, because any system is not same percent perfect. The effect of the presence of defective units in the lot size and rework of these units are studied by several researchers [217, 225, 227, 234, 247, 248] as mentioned in § 1.3.2. Biskup [20] pointed out that repeated processing of similar tasks improves workers skills, e.g., workers are able to perform setups, deal with machine operations or soft-wares, or handle raw materials and components at a faster pace. Scheduling in this setting is known as scheduling with learning effects. This concept of learning was also introduced by Cheng and Wang [52] into the field of scheduling. Recently, Biskup [21] presented a comprehensive review of research on scheduling with learning effects. Eren [82] proposed a non-linear mathematical programming model for the single-machine scheduling problem with unequal release dates and learning effects. But very few researchers have used the concept of learning in EPL with finite time horizon.

Summarizing the above mentioned literature, the systematic chronological developments in the related areas of the present investigation of Model 3.1 are presented in Table 3.1.

Though several inventory models are available with imperfect production as described in

**Table** 3.1: Literature Review for Model-3.1

| Authors with year | Model type | Demand | Out of control state | Rework | EPC | Production dependent quality |
|---|---|---|---|---|---|---|
| Rosenblatt and Lee [217], 1986 | EPQ | Constant | Random | Yes | No | No |
| Mandal and Phaujder [168], 1989 | EPQ | Stock dependent | No | No | No | No |
| Urban [259], 1992 | EOQ | Stock dependent | No | No | No | No |
| Khouja and Mehrez [127], 1994 | EPL | Constant | Random | Yes | No | Yes |
| Mandal and Maiti [169], 1999 | EPQ | Stock dependent | No | No | No | No |
| Salameh and Jaber [225], 2000 | EPQ | Constant | Random | No | No | No |
| Maiti and Maiti [163], 2006 | EOQ | Stock dependent | No | No | No | No |
| Sana [227], 2010 | EPL | Constant | Random | Yes | No | No |
| Sarkar *et al.* [230], 2010 | EPL | Constant | Random | Yes | No | No |
| Taleizadeh *et al.* [247], 2012 | EPQ | Constant | Crisp | Yes | No | No |
| Pal *et al.* [195], 2013 | EPQ | Random | Random | Yes | No | No |
| Krishnamoorthi and Panayappan [134], 2014 | EPL | Time dependent | Crisp | No | No | No |
| Sarkar *et al.* [234], 2014 | EPQ | Constant | Random | Yes | No | No |
| Present model 3.1 | EPL | Stock dependent | Random | Yes | Yes | Yes |

Table 3.2, in comparison of those, our main considerations are also presented here.

**Table** 3.2: Literature Review for Model-3.2

| Authors with year | Model type | Demand type | Out of control state | UPC | Reliability | Time horizon |
|---|---|---|---|---|---|---|
| Rosenblatt and Lee [217], 1986 | EPQ | Constant | Random | No | No | Infinite |
| Mandal and Phaujder [168], 1989 | EPQ | Stock dependent | No | Constant | No | Infinite |
| Khouja and Mehrez [127], 1994 | EPL | Constant | Random | Production rate dependent | No | Infinite |
| Salameh and Jaber [225], 2000 | EPQ | Constant | Random | Constant | No | Infinite |
| Roy *et al.* [221], 2009 | EOQ | Stock dependent | No | No | No | Random and finite |
| Sana [227], 2010 | EPL | Constant | Random | Production rate dependent | No | Infinite |
| Sarkar *et al.* [230], 2010 | EPL | Constant | Random | Constant | Yes | infinite |
| Sarkar *et al.* [231], 2010 | EPL | Constant | Random | Reliability dependent | Yes | Infinite |
| Sarkar [233], 2012 | EPQ | Price and advertising dependent | Time dependent | Production and reliability dependent | Yes | Infinite |
| Guria *et al.* [94], 2013 | EOQ | Inflation and selling price dependent | No | No | No | Random and finite |
| Present model 3.2 | EPL | Stock dependent | Random | Production and reliability dependent | Yes | Uncertain and finite |

Hence, in this chapter, we developed two models. In the first model, we formulate a randomly imperfect single item production-inventory system with production dependent set-up cost, stock dependent demand, partial rework, disposal of defective units, chance-constraint for commencement of imperfect production and variable production cost including EPC. In real life EPL models, a production system remains in control at the beginning and after some time, it goes to out-of-control state. Thus, the occurrence of production of imperfect units is random and is imposed here through a chance constraint. The set-up cost, UPC and defective rate are production dependent and part of UPC is taken as EPC. The problem is formulated as a cost minimization problem and solved using GRG method using LINGO 11.0. Several special cases are derived and more specifically, the present investigation extends the earlier works in this area. As particular cases, the expression of Sana [227] and Khouja and Mehrez [127] are derived. Numerical experiments are performed to illustrate the general and particular models. Some sensitivity analyses are presented against few model parameters.

In the second model, we consider a randomly imperfect single-item production inventory model over imprecise time horizon with learning effect on set-up cost, stock dependent demand, reliability dependent defective production rate, partially reworked and disposal of defective units, chance-constraint for commencement of imperfect production and variable production cost including environmental protection cost. The problem is formulated as a cost minimization problem with crisp, random, fuzzy, fuzzy-random, rough and fuzzy-rough constraints and solved using GRG method. Several special cases are derived and numerical experiments are performed to illustrate the general and particular models.

## 3.2 Model-3.1 : An EPL model for randomly imperfect production system with stock dependent demand and rework [1]

### 3.2.1 Assumptions and Notations

*The following assumptions are used to develop the proposed models:*

(i) Replenishment rate is finite and taken as a DV.

(ii) Lead time is zero.

(iii) No shortages are allowed.

(iv) The inventory system considers a single item and the demand rate is linearly stock-dependent.

(v) The time horizon is infinite and the production time is taken as a DV.

[1]This model has been accepted for publication in **International Journal Operational Research**, Inderscience Enterprises Ltd., Y. 2015

(vi) The production process shifts from "In-control" state to "Out-of-control" state at a time, which is a random variable. Imperfect units are produced in the "Out-of-control" state.

(vii) Production of defective units commences at a random time after the commencement of production. Defective rate depends on production rate and time duration from the starting of defective units' production.

(viii) A part of the defective units are reworked at a cost immediately when they are produced in "out-of-control" state and the defective units which are not reworked, are disposed off by a cost.

(ix) UPC is production dependent and one part of it is used as EPC.

(x) Set up cost is considered as partly production dependent.

*The following notations are used to develop the proposed models:*

$P$ Production rate (units per time unit) (DV).
$q(t)$ Inventory level at time t (units).
$D[q(t)]$ The demand rate (units per time unit) of perfect products at time t, $D[q(t)] = d_0 + d_1 q(t)$ where $d_0 > 0$, $d_1$ is the stock-dependent consumption rate parameter, $0 \leq d_1 \leq 1$.
$T$ Length of the inventory cycle (time unit).
$t_1$ Production run-time in one period (time unit)(DV).
$C_s$ Set up cost per cycle (unit of money per set up), $C_s = C_{s0} + C_{s1} P^{\rho}$, where $\rho > 0$.
$C_h$ Holding cost (money per unit per time unit).
$C_d$ Cost to dispose an imperfect unit (money per unit).
$C_r$ Cost to rework an imperfect unit (money per unit).
$\tau$ An exponential random variable that depends on P and denotes the time at which the process shifts to the "out-of-control" state. The distribution function of "out-of-control" state is $G(\tau) = 1 - e^{-f(P)\tau}$ such that $\int_0^{\infty} dG(\tau) = f(P) \int_0^{\infty} e^{-f(P)\tau} d\tau = 1$. The exponential distribution has often been used to describe the elapsed time to failure of many components of the machinery system.
$\frac{1}{f(P)}$ The mean and standard deviation of the random variable $\tau$ . Here, f(P) is an increasing function of P and the mean time of failure, $1/f(P)$ is a decreasing function of P.
$\theta$ Percentage of rework of defective units.

$\lambda(t, \tau, P)$ Rate of defective units produced at time t when the machine is in the "out-of-control" state. Here $\lambda(t, \tau, P)$ is defined as $\lambda(t, \tau, P) = \alpha P^{\beta}(t-\tau)^{\gamma}$. where $\beta \geq 0$,$\gamma \geq 0$ and $t \geq \tau$. Generally speaking, the rate of defective units increases with increase of production rate and production-run time. The formulation of the function $\lambda(t, \tau, P)$ shows that it is an increasing function of production rate and production-run time simultaneously.

$C(P)$ UPC (money per unit) which is considered as $C(P) = r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3} \ \ where, \ \delta_1, \delta_2, \delta_3 > 0$ and $r_m$ is the raw material cost per unit item, $g$ is the total labour/energy costs per unit time in a production system which is equally distributed over the unit item. So, $(\frac{g}{P^{\delta_1}})$ decreases with increases of $P$. The third term $\eta_1 P^{\delta_2}$ is the wear and tear cost, proportional to the positive power of production rate $P$. In a thermal electricity plant, the inject of 'ash' in the atmosphere depends upon the rate of production. If the production is more, the amount of required raw material i.e., impure coal is more and hence the amount of 'fly-ash' is more. Now-a-days, some measures are taken to reduce the 'fly-ash' amount. Thus the cost due to this measure varies with the production rate. The fourth term $\eta_2 P^{\delta_3}$ is EPC assuming that the cost due to the measures taken for the environment protection is proportional to a positive power of production rate $P$, where the power term varies with the nature of production firms.

$N$ Defective units in a production cycle (units).

$Q$ Expected production lot size (or inventory) (units) without defective units at the end of production period.

$TC(P, t_1)$ Expected total cost (unit of money).

$ATC(P, t_1)$ Average expected total cost (unit of money).

### 3.2.2 Mathematical Model Development

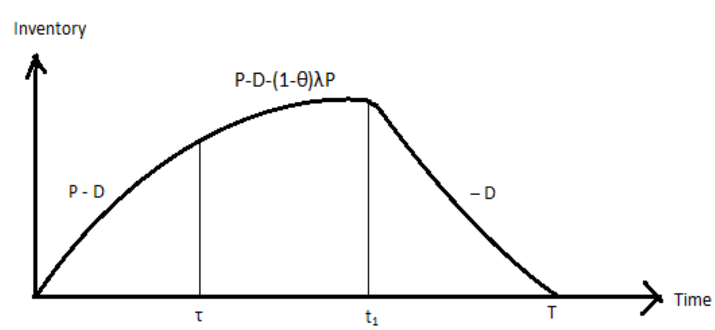



**Figure** 3.1: Inventory versus time

We consider here a production process in which an item is produced at the rate of P per unit time. Initially the production process is "in-control" state upto time $\tau$ and after $t = \tau$, it goes to "out-of-control" state i.e. production of imperfect units is commenced and $\lambda$ percentage of P units are imperfect. Out of these imperfect units, $\theta$ percentage units are reworked and taken as fresh units. The production process is continued up $t = t_1(> \tau)$ and then discontinued. Here, demand is assumed to be stock dependent, $D[q(t)]$ say. In this process, the stock is build up at the rate $P - D(q)$ upto $t = \tau$ and $P - D(q) - (1-\theta)\lambda P$ upto $\tau \le t \le t_1$. After $t = t_1$, the stock is depleted at the rate $D(q)$ and becomes zero at $t = T$ (say). Under these assumptions, the above production process with necessary end conditions can be mathematically expressed by the Eqs. (3.1)- (3.6) (cf. Fig. 3.1). For this process, we consider production, holding, set-up, rework and disposal costs. Hence, our objective is to find the optimum production rate, P and the production period, $t_1$ so that average total cost ATC incurred in the system is minimum.

Hence, we have

$$\frac{dq(t)}{dt} = P - D[q(t)], \quad 0 \le t \le \tau \tag{3.1}$$

$$\frac{dq(t)}{dt} = P - D[q(t)] - (1-\theta)\lambda P, \quad \tau \le t \le t_1 \tag{3.2}$$

$$\frac{dq(t)}{dt} = -D[q(t)], \quad t_1 \le t \le T \tag{3.3}$$

with the boundary conditions

$$q(t) = 0, \qquad at \;\; t = 0 \tag{3.4}$$

$$q(t) = q(\tau), \qquad at \;\; t = \tau \tag{3.5}$$

$$q(t) = 0, \qquad at \;\; t = T \tag{3.6}$$

The solution of the differential equations (3.1), (3.2) and (3.3) is given by,

$$q(t) = \frac{P - d_0}{d_1}(1 - e^{-d_1 t}), \qquad 0 \le t \le \tau \tag{3.7}$$

$$q(t) = \frac{P - d_0}{d_1}(1 - e^{-d_1 t}) - (1-\theta)\alpha P^{\beta+1} e^{-d_1(t-\tau)}\phi(t,\tau,\gamma), \quad \tau \le t \le t_1 \tag{3.8}$$

where,

$$\phi(t,\tau,\gamma) = \frac{(t-\tau)^{\gamma+1}}{(\gamma+1)} + \frac{d_1(t-\tau)^{(\gamma+2)}}{1!(\gamma+2)} + \frac{d_1^2(t-\tau)^{(\gamma+3)}}{2!(\gamma+3)} + \frac{d_1^3(t-\tau)^{(\gamma+4)}}{3!(\gamma+4)} + \cdots$$
$$= \sum_{i=1}^{\infty} \frac{d_1^{i-1}(t-\tau)^{\gamma+i}}{(i-1)!(\gamma+i)}$$

$$q(t) = \frac{d_0}{d_1}\left(e^{d_1(T-t)} - 1\right), \qquad t_1 \le t \le T \tag{3.9}$$

The total defective units during $[\tau, t_1]$ is

$$N = \int_{\tau}^{t_1} \lambda P dt = P \int_{\tau}^{t_1} \alpha P^{\beta}(t-\tau)^{\gamma} dt = \frac{\alpha}{\gamma+1} P^{\beta+1}(t_1-\tau)^{\gamma+1} \tag{3.10}$$

Total expected number of defective units in a production lot size is

$$\begin{aligned} E(N) &= \frac{\alpha}{\gamma+1} P^{\beta+1} \int_0^{t_1} (t_1-\tau)^{\gamma+1} d\Big(G(\tau)\Big) \\ &= \frac{\alpha}{\gamma+1} P^{\beta+1} f(P) e^{-f(P)t_1} \zeta(P, t_1) \end{aligned} \tag{3.11}$$

where $\zeta(P, t_1)$ is given by the equation as

$$\begin{aligned} \zeta(P, t_1) &= \frac{t_1^{\gamma+2}}{(\gamma+2)} + \frac{f(P) t_1^{\gamma+3}}{1!(\gamma+3)} + \frac{f^2(P) t_1^{\gamma+4}}{2!(\gamma+4)} + \frac{f^3(P) t_1^{\gamma+5}}{3!(\gamma+5)} + \cdots \\ &= \sum_{i=1}^{\infty} \frac{f^{i-1}(P) t_1^{\gamma+i+1}}{(i-1)!(\gamma+i+1)} \end{aligned} \tag{3.12}$$

Now at time $t = t_1$ the expected production lot size without defective units is

$$\begin{aligned} &Q = E[q(t_1)] = \textstyle\int_0^{\infty} q(t_1) d\Big(G(\tau)\Big) \\ &= \tfrac{P-d_0}{d_1}(1-e^{-d_1 t_1}) \textstyle\int_0^{\infty} d\Big(1-e^{-f(P)\tau}\Big) \\ &-(1-\theta)\alpha P^{\beta+1} f(P) e^{-f(P)t_1} \textstyle\int_0^{t_1} e^{[f(P)-d_1](t_1-\tau)} \phi(t_1, \tau, \gamma) d\tau \\ &= \tfrac{P-d_0}{d_1}(1-e^{-d_1 t_1}) - (1-\theta)\alpha P^{\beta+1} f(P) e^{-f(P)t_1} \Big[\tfrac{\psi_1}{\gamma+1} + \tfrac{d_1\psi_2}{1!(\gamma+2)} + \tfrac{d_1^2\psi_3}{2!(\gamma+3)} + \cdots\Big] \end{aligned} \tag{3.13}$$

where,

$$\psi_i = \sum_{j=1}^{\infty} \frac{[f(P)-d_1]^{j-1} t_1^{\gamma+j+i}}{(j-1)!(\gamma+j+i)} \text{for all i=1, 2, 3, 4} \cdots \tag{3.14}$$

Again from the Eq. (3.9) we get,

$$\begin{aligned} &Q = E[q(t_1)] = \frac{d_0}{d_1}\Big(e^{d_1(T-t_1)} - 1\Big) \\ &\text{or, } \frac{d_0}{d_1}\Big(e^{d_1(T-t_1)} - 1\Big) = Q = \frac{P-d_0}{d_1}(1-e^{-d_1 t_1}) \\ &\qquad -(1-\theta)\alpha P^{\beta+1} f(P) e^{-f(P)t_1} \Big[\frac{\psi_1}{\gamma+1} + \frac{d_1\psi_2}{1!(\gamma+2)} + \frac{d_1^2\psi_3}{2!(\gamma+3)} + \cdots\Big] \\ &\text{or, } T = t_1 + \frac{1}{d_1}\log\Big[1 + \frac{1}{d_0}\Big\{(P-d_0)(1-e^{-d_1 t_1}) \\ &-(1-\theta)\alpha d_1 P^{\beta+1} f(P) e^{-f(P)t_1} \sum_{i=1}^{\infty} \frac{d_1^{i-1}\psi_i}{(i-1)!(\gamma+i)}\Big\}\Big] \end{aligned} \tag{3.15}$$

Now during the period $(0, t_1)$ the inventory which are to be hold, is

$$\begin{aligned} Q_{h_1} &= \int_0^{t_1} q(t)dt = \int_0^{\tau} q(t)dt + \int_{\tau}^{t_1} q(t)dt \\ &= \frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) \\ &- (1-\theta)\alpha P^{\beta+1}\left[\frac{\xi_1}{\gamma+1} + \frac{d_1\xi_2}{1!(\gamma+2)} + \frac{d_1^2\xi_3}{2!(\gamma+3)} + \frac{d_1^3\xi_4}{3!(\gamma+4)} + \cdots\right] \\ &= \frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - (1-\theta)\alpha P^{\beta+1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\xi_i}{(i-1)!(\gamma+i)} \end{aligned} \tag{3.16}$$

$$\text{where, } \xi_i = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}d_1^{j-1}(t_1-\tau)^{\gamma+j+i}}{(j-1)!(\gamma+j+i)} \text{for all i=1, 2, 3, 4, } \cdots \tag{3.17}$$

Now during the period $(0, t_1)$ the expected inventory which are to be hold, is

$$\begin{aligned} E[Q_{h_1}] &= \int_0^{\infty} Q_{h_1} d\Big(G(\tau)\Big) \\ &= \frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right)\int_0^{\infty} d\left(1-e^{-f(P)\tau}\right) \\ &- (1-\theta)\alpha P^{\beta+1}f(P)e^{-f(P)t_1}\int_0^{t_1} e^{f(P)(t_1-\tau)} \\ &\left(\frac{\xi_1}{\gamma+1} + \frac{d_1\xi_2}{1!(\gamma+2)} + \frac{d_1^2\xi_3}{2!(\gamma+3)} + \frac{d_1^3\xi_4}{3!(\gamma+4)} + \cdots\right)d\tau \end{aligned} \tag{3.18}$$

$$\begin{aligned} &= \frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - (1-\theta)\alpha P^{\beta+1}f(P)e^{-f(P)t_1} \\ &\left[\frac{1}{\gamma+1}\left(\frac{\xi_{11}}{\gamma+2} - \frac{d_1\xi_{12}}{1!(\gamma+3)} + \frac{d_1^2\xi_{13}}{2!(\gamma+4)} - \frac{d_1^3\xi_{14}}{3!(\gamma+5)} + \cdots\right)\right. \\ &+ \frac{d_1}{1!(\gamma+2)}\left(\frac{\xi_{21}}{(\gamma+3)} - \frac{d_1\xi_{22}}{1!(\gamma+4)} + \frac{d_1^2\xi_{23}}{2!(\gamma+5)} - \frac{d_1^3\xi_{24}}{3!(\gamma+6)} + \cdots\right) \\ &+ \frac{d_1^2}{2!(\gamma+3)}\left(\frac{\xi_{31}}{(\gamma+4)} - \frac{d_1\xi_{32}}{1!(\gamma+5)} + \frac{d_1^2\xi_{33}}{2!(\gamma+6)} - \frac{d_1^3\xi_{34}}{3!(\gamma+7)} + \cdots\right) \\ &\left.+ \frac{d_1^3}{3!(\gamma+4)}\left(\frac{\xi_{41}}{(\gamma+5)} - \frac{d_1\xi_{42}}{1!(\gamma+6)} + \frac{d_1^2\xi_{43}}{2!(\gamma+7)} - \frac{d_1^3\xi_{44}}{3!(\gamma+8)} + \cdots\right) + \cdots\cdots\right] \end{aligned} \tag{3.19}$$

where all $\xi_{ij}(P, t_1, \gamma)$[ for all $i, j = 1, 2, 3, 4, \cdots$ ] are given by the equations as

$$\xi_{ij}(P, t_1, \gamma) = \sum_{k=1}^{\infty}\frac{f^{k-1}(P)t_1^{\gamma+k+j+i}}{(k-1)!(\gamma+k+j+i)} \quad \text{for all i, j=1, 2, 3,} \cdots$$

We have the relations among $\xi_{ij}$'s as

$$\xi_{ij} = \xi_{ji} \qquad \text{for all i,j=1, 2, 3, 4,}\cdots$$
$$\text{and} \qquad \xi_{ij} = \xi_{mn} \text{ whenever, i+j=m+n for all i,j,m,n=1, 2, 3, 4, } \cdots$$

Now using the above relations, the Eq. (3.19) is expressed in the form as

$$E[Q_{h_1}] = \frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - (1-\theta)\alpha P^{\beta+1}f(P)e^{-f(P)t_1}$$
$$\left[\frac{\xi_{11}}{(\gamma+1)(\gamma+2)} - \frac{d_1\xi_{12}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{d_1^2\xi_{13}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} - \frac{d_1^3\xi_{14}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \cdots\right] \tag{3.20}$$

Now during the period $[t_1, T]$ the inventory which are to be hold, is

$$Q_{h_2} = \int_{t_1}^{T} q(t)dt = \frac{d_0}{d_1}\left[\frac{1}{d_1}\left(e^{d_1(T-t_1)}-1\right) - (T-t_1)\right] \tag{3.21}$$

Now during the period $(0, T)$ the total expected number of storage units can be obtained as,

$$E[Q_h] = E[Q_{h_1}] + Q_{h_2} \tag{3.22}$$

where $E[Q_{h_1}]\, and\, Q_{h_2}$ are given by the Eqs. (3.20) and (3.21) respectively.

In a cycle (0, T), the expected total cost = Expected holding cost + Rework cost + Disposal cost + Set-up cost + Production cost.

i.e. $TC(P, t_1) = C_hE(Q_h) + \theta C_rE(N) + (1-\theta)C_dE(N) + C_s + C(P)Pt_1$

$$= C_h\left[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - (1-\theta)\alpha P^{\beta+1}f(P)e^{-f(P)t_1}\left(\frac{\xi_{11}}{(\gamma+1)(\gamma+2)} - \frac{d_1\xi_{12}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{d_1^2\xi_{13}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} - \cdots\right) + \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\right] + \left(\theta C_r + (1-\theta)C_d\right)\frac{\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P,t_1)$$
$$+ C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3})Pt_1 \tag{3.23}$$

The expected average total cost is $\qquad ATC(P, t_1) = \dfrac{TC(P, t_1)}{T} \qquad (3.24)$

where T is given by Eq. (3.15)

**Chance constraint**

In this production system, it is expected to have total production time greater than the time of beginning of "out-of-control" state. This requirement acts as a constraint and expressed here as a chance constraint. Hence, the chance constraint is $\quad Prob\Big(t_1 - \tau \geq \epsilon\Big) \geq r$, where $t_1 \geq 0 \; and \; r \in (0,1)$ is a specified permissible probability. Here $m\Big(= \dfrac{1}{f(P)}\Big)$ and $\sigma\Big(= \dfrac{1}{f(P)}\Big)$ are the mean and standard deviation of the exponential random variable $\tau$.

Then the constraint can be written as $Prob(\frac{\tau - m}{\sigma} \leq \frac{t_1 - \epsilon - m}{\sigma}) \geq r$

where $\frac{\tau - m}{\sigma}$ is a random normal variate. Considering $z$, where $\int_0^z \phi(t)dt = r$, $\phi(t)$, being the standard normal density function, we have $\frac{t_1 - \epsilon - m}{\sigma} \geq z$

$$\text{or, } t_1 \geq \frac{1}{f(P)}[1 + z] + \epsilon \tag{3.25}$$

where $z$ is obtained from the normal distribution table for a particular value of $r$.

**Optimization Problem**

Therefore, the production-inventory model is finally reduced to the minimization of expected average total cost given by Eq. (3.24) subject to the chance constraint given by Eq. (3.25). i.e.

$$\text{Min } ATC(P, t_1)\Big(= \frac{TC(P, t_1)}{T}\Big)$$

$$\text{s.t. } t_1 \geq \frac{1}{f(P)}[1 + z] + \epsilon \tag{3.26}$$

### 3.2.3 Particular Cases

**Table** 3.3: Different models deduced from Model-3.1

| Characteristics of model | Models name: Model -3.1A | -3.1B | -3.1C | -3.1D | -3.1E | -3.1F | -3.1G | -3.1H | -3.1I | -3.1J | -3.1K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Stock-dependent demand ($d_1 \neq 0$) | √ | √ | √ | √ | √ | | | | | | |
| Constant demand ($d_1 = 0$) | | | | | | √ | √ | √ | √ | √ | √ |
| Random defective rate ($\beta \neq 0, \gamma \neq 0$) | √ | √ | | | | √ | √ | √ | | | |
| Constant defective rate ($\beta = 0, \gamma = 0$) | | | √ | √ | √ | | | | √ | √ | √ |
| Partial Rework ($0 < \theta < 1$) | | | √ | | | √ | | | √ | | |
| Fully Rework ($\theta = 1$) | √ | | | √ | | | √ | | | √ | |
| No Rework ($\theta = 0$) | | √ | | | √ | | | √ | | | √ |

Putting different values of $\theta$, $\beta$, $\gamma$ and $d_1$, we get the different variations of Model-3.1, which are summarized in Table 3.3 and the fresh units (Q), cycle period (T) and total expected cost (TC) for different models are derived as bellow:

**Model-3.1A(Same as Model-3.1 with full rework)**

Taking $\theta = 1$ in the Model-3.1, we have

$$Q = \frac{d_0}{d_1}\left(e^{d_1(T-t_1)} - 1\right) = \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) \tag{3.27}$$

$$\text{or, } T = t_1 + \frac{1}{d_1}\log\left[1 + \frac{1}{d_0}(P-d_0)(1-e^{-d_1t_1})\right] \tag{3.28}$$

$$\begin{aligned}
&TC(P,t_1) = C_hE(Q_h) + C_rE(N) + C_s + C(P)Pt_1\\
&= C_h\left[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) + \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\right]\\
&+ \frac{C_r\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P,t_1) + C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3})Pt_1
\end{aligned} \tag{3.29}$$

**Model-3.1B (Same as Model-3.1 with No rework)**

Taking $\theta = 0$ in the Model-3.1, the expected production lot size without defective units at the end of production is

$$Q = \frac{d_0}{d_1}\left(e^{d_1(T-t_1)} - 1\right) = \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) - \alpha P^{\beta+1}f(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!(\gamma+i}$$

$$\text{or, } T = t_1 + \frac{1}{d_1}\log\left[1 + \frac{1}{d_0}\left((P-d_0)(1-e^{-d_1t_1}) - \alpha d_1P^{\beta+1}f(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!(\gamma+i}\right)\right] \tag{3.30}$$

where, $\psi_i = \sum_{k=1}^{\infty}\frac{[f(P)-d_1]^{k-1}t_1^{\gamma+k+i}}{(k-1)!(\gamma+k+i)}$; for all $i = 1, 2, 3, ....$

$$\begin{aligned}
&TC(P,t_1) = C_hE(Q_h) + C_dE(N) + C_s + C(P)Pt_1\\
&= C_h\left[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - \alpha P^{\beta+1}f(P)e^{-f(P)t_1}\left(\frac{\xi_{11}}{(\gamma+1)(\gamma+2)}\right.\right.\\
&\left. - \frac{d_1\xi_{12}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{d_1^2\xi_{13}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} - \cdots\right)\\
&\left. + \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\right] + \frac{C_d\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P,t_1)\\
&+ C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3})Pt_1
\end{aligned} \tag{3.31}$$

where, $\xi_{1k} = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{\gamma+i+k+1}}{(i-1)!(\gamma+i+k+1)}$, $\zeta(P,t_1) = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{\gamma+i+1}}{(i-1)!(\gamma+i+1)}$ for all $i = 1, 2, 3, \cdots$

**Model-3.1C (Same as Model-3.1 with Constant defective rate)**

Taking $\beta = 0$ and $\gamma = 0$ in the Model-3.1, expected production lot size without defective units at the end of production is

$$Q = \frac{d_0}{d_1}\left(e^{d_1(T-t_1)} - 1\right) = \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) - (1-\theta)\alpha Pf(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!i}$$

$$\text{or, } T = t_1 + \frac{1}{d_1}\log\left[1+\frac{1}{d_0}\left((P-d_0)(1-e^{-d_1t_1}) - (1-\theta)\alpha d_1 Pf(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!i}\right)\right] \tag{3.32}$$

$$\text{where, } \psi_i = \sum_{k=1}^{\infty}\frac{[f(P)-d_1]^{k-1}t_1^{k+i}}{(k-1)!(k+i)}; \text{ for all } \ i = 1,2,3,\cdots$$

$$\begin{aligned} TC(P,t_1) &= C_hE(Q_h) + \theta C_rE(N) + (1-\theta)C_dE(N) + C_s + C(P)Pt_1 \\ &= C_h\Bigg[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - (1-\theta)\alpha Pf(P)e^{-f(P)t_1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}d_1^{k-1}\xi_{1k}}{(k+1)!} \\ &+ \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\Bigg] + \left(\theta C_r + (1-\theta)C_d\right)\alpha Pf(P)e^{-f(P)t_1}\zeta(P,t_1) \\ &+ C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3})Pt_1 \end{aligned} \tag{3.33}$$

$$where, \xi_{1k} = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{i+k+1}}{(i-1)!(i+k+1)} \ and \ \zeta(P,t_1) = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{i+1}}{(i-1)!(i+1)} \text{ for all } \ i = 1,2,3,\cdots$$

**Model-3.1D (Same as Model-3.1 with Constant defective rate and full rework)**

Taking $\theta = 1$ in the Model-3.1C, the expected production lot size without defective units at the end of production and expected total cost are

$$\begin{aligned} Q &= \frac{d_0}{d_1}\left(e^{d_1(T-t_1)} - 1\right) = \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) \\ \text{or, } T &= t_1 + \frac{1}{d_1}\log\left[1+\frac{1}{d_0}(P-d_0)(1-e^{-d_1t_1})\right] \end{aligned} \tag{3.34}$$

$$\begin{aligned} TC(P,t_1) &= C_hE(Q_h) + C_rE(N) + C_s + C(P)Pt_1 \\ &= C_h\left[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) + \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\right] \\ &+ C_r\alpha Pf(P)e^{-f(P)t_1}\zeta(P,t_1) + C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3})Pt_1 \end{aligned} \tag{3.35}$$

$$\text{where, } \zeta(P,t_1) = \sum_{i=1}^{\infty} \frac{f^{i-1}(P)t_1^{i+1}}{(i-1)!(i+1)} \text{ for all } i = 1,2,3,\cdots$$

**Model-3.1E (Same as Model-3.1 with Constant defective rate and No rework)**

Taking $\theta = 0$ in the Model-3.1C, the expected production lot size without defective units at the end of production is

$$Q = \frac{d_0}{d_1}\left(e^{d_1(T-t_1)} - 1\right) = \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) - \alpha P f(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!i}$$

$$\text{or, } T = t_1 + \frac{1}{d_1}\log\left[1+\frac{1}{d_0}\left((P-d_0)(1-e^{-d_1t_1}) - \alpha d_1 P f(P)e^{-f(P)t_1}\sum_{i=1}^{\infty}\frac{d_1^{i-1}\psi_i}{(i-1)!i}\right)\right] \tag{3.36}$$

$$\text{where, } \psi_i = \sum_{k=1}^{\infty}\frac{[f(P)-d_1]^{k-1}t_1^{k+i}}{(k-1)!(k+i)}; \text{ for all } i = 1,2,3,\cdots$$

$$\begin{aligned}
TC(P,t_1) &= C_hE(Q_h) + C_dE(N) + C_s + C(P)Pt_1 \\
&= C_h\left[\frac{P-d_0}{d_1}\left(t_1 + \frac{1}{d_1}(e^{-d_1t_1}-1)\right) - \alpha P f(P)e^{-f(P)t_1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}d_1^{k-1}\xi_{1k}}{(k+1)!}\right. \\
&\left. + \frac{d_0}{d_1}\left(\frac{e^{d_1(T-t_1)}-1}{d_1} - (T-t_1)\right)\right] + C_d\alpha P f(P)e^{-f(P)t_1}\zeta(P,t_1) \\
&+ C_s + (r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3})Pt_1
\end{aligned} \tag{3.37}$$

$$\text{where, } \xi_{1k} = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{i+k+1}}{(i-1)!(i+k+1)} \; and \; \zeta(P,t_1) = \sum_{i=1}^{\infty}\frac{f^{i-1}(P)t_1^{i+1}}{(i-1)!(i+1)} \text{for all } i = 1,2,3,\cdots$$

**Model-3.1F (Same as Model-3.1 with Constant demand)**

Letting $d_1 \to 0$ in Model-3.1 we have the expected total holding inventory in a cycle (0, T) as

$$\begin{aligned}
E[Q_h] &= \lim_{d_1\to 0}(E[Q_{h_1}] + Q_{h_2}) \\
&= \left[\frac{P-d_0}{2}t_1^2 - \frac{(1-\theta)\alpha}{(\gamma+1)(\gamma+2)}P^{\beta+1}f(P)e^{-f(P)t_1}\xi_{11}(P,t_1)\right] + \frac{d_0}{2}(T-t_1)^2
\end{aligned}$$

$$\text{and } \lim_{d_1\to 0}\left[\frac{d_0}{d_1}\left(e^{d_1(T-t_1)}-1\right)\right] = Q = \lim_{d_1\to 0}\left[\frac{P-d_0}{d_1}(1-e^{-d_1t_1})\right.$$
$$\left. - (1-\theta)\alpha P^{\beta+1}f(P)e^{-f(P)t_1}\left(\frac{\psi_1}{\gamma+1} + \frac{d_1\psi_2}{1!(\gamma+2)} + \frac{d_1^2\psi_3}{2!(\gamma+3)} + \frac{d_1^3\psi_4}{3!(\gamma+4)} + .....\right)\right]$$

or, $d_0(T-t_1) = Q = (P-d_0)t_1 - (1-\theta)E(N)$
where $\psi_1(d_1=0) = \zeta(P,t_1)$ and E(N) is given by the Eq. (3.11)
or, $d_0T = d_0t_1 + (P-d_0)t_1 - (1-\theta)E(N) = Pt_1 - (1-\theta)E(N)$

$$\text{or, } T = \frac{Pt_1}{d_0}\left[1 - \frac{(1-\theta)E(N)}{Pt_1}\right] \tag{3.38}$$

$$\begin{aligned}TC(P,t_1) &= C_hE(Q_h) + \theta C_rE(N) + (1-\theta)C_dE(N) + C_s + C(P)Pt_1\\ &= C_h\left[\frac{P-d_0}{2}t_1^2 - \frac{(1-\theta)\alpha}{(\gamma+1)(\gamma+2)}P^{\beta+1}f(P)e^{-f(P)t_1}\xi_{11}(P,t_1) + \frac{d_0}{2}(T-t_1)^2\right]\\ &+ \left(\theta C_r + (1-\theta)C_d\right)\frac{\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P,t_1)\\ &+ C_s + [r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3}]Pt_1\end{aligned} \tag{3.39}$$

**Model-3.1G (Same as Model-3.1 with Constant demand and full rework)**

Taking $\theta = 1$ in the Model-3.1F, expected production lot size without defective units at the end of production is

$$Q = (P-d_0)t_1 = d_0(T-t_1) \qquad \text{or, } T = \frac{Pt_1}{d_0} \tag{3.40}$$

$$\begin{aligned}&\text{and } TC(P,t_1) = C_h\left[\frac{P-d_0}{2}t_1^2 + \frac{d_0}{2}(T-t_1)^2\right] + C_s\\ &+C_r\frac{\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P,t_1) + [r_m + \frac{g}{P^{\delta_1}} + \eta_1P^{\delta_2} + \eta_2P^{\delta_3}]Pt_1\end{aligned} \tag{3.41}$$

Now letting $Q_1$ is the total production lot size and putting $t_1 = Q_1/P,\ T = Q_1/D,\ \delta_1 = 1,\ \delta_2 = \delta,\ d_0 = D,\ \eta_1 = \eta$ and $\eta_2 = 0$ in the Eqs. (3.40) and (3.41) respectively, we have average expected total cost as

$$\begin{aligned}ATC(P,Q_1/P) &= C_h\left[\frac{P-D}{2}.\frac{Q_1^2}{P^2}.\frac{D}{Q_1} + \frac{D}{2}\left(\frac{Q_1}{D} - \frac{Q_1}{P}\right)^2.\frac{D}{Q_1}\right] + C_rE(N).\frac{D}{Q_1}\\ &+ C_s.\frac{D}{Q_1} + D(r_m + \frac{g}{P} + \eta P^{\delta})\\ &= C_h.\frac{(P-D)Q_1}{2P^2}(D+P-D) + C_rE(N).\frac{D}{Q_1} + C_s.\frac{D}{Q_1} + D(r_m + \frac{g}{P} + \eta P^{\delta})\\ &= \frac{C_h}{2}(1-\frac{D}{P})Q_1 + \frac{C_sD}{Q_1} + C_r\left[\frac{D\alpha}{Q_1(\gamma+1)}\right]P^{\beta+1}f(P)e^{-f(P)Q_1/P}\zeta(P,Q_1/P)\\ &+ D(r_m + \frac{g}{P} + \eta P^{\delta})\end{aligned} \tag{3.42}$$

This is same as the average expected total cost function in the profit function of Sana (2010).

**Model-3.1H (Same as Model-3.1 with Constant demand and No rework)**

Taking $\theta = 0$ in the Model-3.1F, expected production lot size at the end of production is

$$Q = d_0(T - t_1) = (P - d_0)t_1 - E(N) \qquad \text{or, } T = \frac{Pt_1}{d_0}\left[1 - \frac{E(N)}{Pt_1}\right] \tag{3.43}$$

$$TC(P, t_1) = C_h E(Q_h) + C_d E(N) + C_s + C(P)Pt_1$$
$$= C_h\left[\frac{P - d_0}{2}t_1^2 - \frac{\alpha}{(\gamma+1)(\gamma+2)}P^{\beta+1}f(P)e^{-f(P)t_1}\xi_{11}(P, t_1) + \frac{d_0}{2}(T - t_1)^2\right]$$
$$+ \frac{C_d\alpha}{\gamma+1}P^{\beta+1}f(P)e^{-f(P)t_1}\zeta(P, t_1) + C_s + [r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}]Pt_1 \tag{3.44}$$

**Model-3.1I (Same as Model-3.1 with Constant demand and defective rates)**

Taking $\beta = 0$ and$\gamma = 0$ in the Model-3.1F, expected production lot size without defective units at the end of production is

$$Q = (P - d_0)t_1 - (1 - \theta)E(N) = d_0(T - t_1) \qquad \text{or, } T = \frac{Pt_1}{d_0}\left[1 - \frac{(1-\theta)E(N)}{Pt_1}\right] \tag{3.45}$$

$$TC(P, t_1) = C_h E(Q_h) + \theta C_r E(N) + (1-\theta)C_d E(N) + C_s + C(P)Pt_1$$
$$= C_h\left[\left(\frac{P - d_0}{2}t_1^2 - \frac{(1-\theta)\alpha}{2}Pf(P)e^{-f(P)t_1}\xi_{11}\right) + \frac{d_0}{2}(T - t_1)^2\right]$$
$$+\left[\theta C_r + (1-\theta)C_d\right]\alpha Pf(P)e^{-f(P)t_1}\zeta(P, t_1) + C_s$$
$$+[r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}]Pt_1 \tag{3.46}$$

where,$\xi_{11} = \frac{t_1^3}{3} + \frac{f(P)t_1^4}{1!4} + \frac{f^2(P)t_1^5}{2!5} + \cdots$ and $\zeta(P, t_1) = \frac{t_1^2}{2} + \frac{f(P)t_1^3}{1!3} + \frac{f^2(P)t_1^4}{2!4} + \cdots$

**Model-3.1J (Same as Model-3.1 with Constant demand, Constant defective rates and full rework)**

Taking $\theta = 1$ in the Model-3.1I, the expected production lot size without defective units at the end of production is

$$Q = (P - d_0)t_1 = d_0(T - t_1) \qquad \text{or, } T = \frac{Pt_1}{d_0} \tag{3.47}$$

$$TC(P, t_1) = C_h E(Q_h) + C_r E(N) + C_s + C(P)Pt_1$$
$$= C_h\left[\frac{P - d_0}{2}t_1^2 + \frac{d_0}{2}(T - t_1)^2\right] + C_r\alpha Pf(P)e^{-f(P)t_1}\zeta(P, t_1) + C_s$$
$$+[r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}]Pt_1 \tag{3.48}$$

Now letting $Q_1$ is the total production lot size and putting $t_1 = Q_1/P,\ T = Q_1/D,\ \delta_1 = 1,\ r_m = 0$ and $\eta_2 = 0$ in the Eqs.(3.47) and (3.48) approximating up to the 2nd degree term of the expansion of $e^{-f(P)t_1}$, we have average expected total cost as

$$ATC(P, Q_1/P) = C_h\left[\frac{P-D}{2}.\frac{Q_1^2}{P^2}.\frac{D}{Q_1} + \frac{D}{2}\left(\frac{Q_1}{D} - \frac{Q_1}{P}\right)^2.\frac{D}{Q_1}\right] + C_r.\frac{1}{2}\alpha P f(P).\frac{Q_1^2}{2P^2}.\frac{D}{Q_1}$$
$$+ C_s.\frac{d_0}{Q_1} + D(\frac{g}{P} + \eta_1 P^{\delta_2})$$
$$= C_h.\frac{(P-D)Q_1}{2P^2}(D + P - D) + C_r\alpha D f(P)\frac{Q_1}{2P} + C_s.\frac{D}{Q_1} + D(\frac{g}{P} + \eta_1 P^{\delta_2})$$

$$= \frac{C_h}{2}(1 - \frac{D}{P})Q_1 + \frac{C_s D}{Q_1} + C_r\alpha D f(P)\frac{Q_1}{2P} + D(\frac{g}{P} + \eta_1 P^{\delta_2}) \qquad (3.49)$$

This is same as the expected average total cost function of Khouja and Mehrez, (1994).

**Model-3.1K (Same as Model-3.1 with Constant demand, Constant defective rates and No rework)**

Taking $\theta = 0$ in the Model-3.1I, the expected production lot size without defective units at the end of production is

$$Q = (P - d_0)t_1 - E(N) = d_0(T - t_1) \qquad \text{or}, T = \frac{Pt_1}{d_0}\left[1 - \frac{E(N)}{Pt_1}\right] \qquad (3.50)$$
$$TC(P, t_1) = C_h E(Q_h) + C_d E(N) + C_s + C(P)Pt_1$$

$$= C_h\left[\left(\frac{P-d_0}{2}t_1^2 - \frac{\alpha}{2}Pf(P)e^{-f(P)t_1}\xi_{11}\right) + \frac{d_0}{2}(T - t_1)^2\right]$$
$$+ C_d\alpha P f(P)e^{-f(P)t_1}\zeta(P, t_1) + C_s + [r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}]Pt_1 \qquad (3.51)$$

where, $\xi_{11} = \frac{t_1^3}{3} + \frac{f(P)t_1^4}{1!4} + \frac{f^2(P)t_1^5}{2!5} + \cdots$ and $\zeta(P, t_1) = \frac{t_1^2}{2} + \frac{f(P)t_1^3}{1!3} + \frac{f^2(P)t_1^4}{2!4} + \cdots$

Thus, the corresponding cost minimization problem reduces to (3.26) with the appropriate $TC(P, t_1)$ and $T$ for different models.

### 3.2.4 Solution Methodology

The above non-linear optimization problems of Models 3.1 to 3.1K are solved by a gradient based non-linear optimization method- GRG method (cf. Lasdon *et al.* [136] using LINGO Solver 11.0 for particular sets of data.

Table 3.4: Collected data to find mean of $\tau$

| Production rate P | Average Unit production cost | Occurrences of out of control state ($\tau$ hours) at different runs | | | | | | | | | | Average of $\tau(m)$ | f(P)= $\frac{1}{m}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th | | |
| 520 | 261.52 | 0.26 | 0.24 | 0.24 | 0.25 | 0.27 | 0.26 | 0.21 | 0.23 | 0.28 | 0.27 | 0.25 | 4.0 |
| 575 | 261.53 | 0.27 | 0.21 | 0.24 | 0.24 | 0.29 | 0.29 | 0.24 | 0.27 | 0.26 | 0.28 | 0.26 | 3.8 |
| 440 | 261.50 | 0.30 | 0.29 | 0.25 | 0.30 | 0.29 | 0.26 | 0.27 | 0.28 | 0.27 | 0.30 | 0.28 | 3.6 |
| 370 | 261.49 | 0.34 | 0.35 | 0.37 | 0.31 | 0.29 | 0.39 | 0.33 | 0.32 | 0.36 | 0.34 | 0.34 | 2.9 |
| 395 | 261.50 | 0.29 | 0.25 | 0.29 | 0.29 | 0.26 | 0.27 | 0.28 | 0.29 | 0.28 | 0.30 | 0.28 | 3.6 |
| 415 | 261.50 | 0.29 | 0.30 | 0.30 | 0.26 | 0.25 | 0.29 | 0.28 | 0.33 | 0.29 | 0.32 | 0.29 | 3.4 |
| 480 | 261.51 | 0.25 | 0.29 | 0.28 | 0.33 | 0.29 | 0.32 | 0.37 | 0.31 | 0.27 | 0.29 | 0.30 | 3.3 |
| 448 | 261.50 | 0.29 | 0.28 | 0.28 | 0.26 | 0.27 | 0.28 | 0.27 | 0.29 | 0.25 | 0.24 | 0.27 | 3.7 |
| 450 | 261.50 | 0.26 | 0.27 | 0.30 | 0.29 | 0.29 | 0.25 | 0.29 | 0.27 | 0.28 | 0.30 | 0.28 | 3.6 |
| 505 | 261.52 | 0.26 | 0.25 | 0.29 | 0.33 | 0.28 | 0.29 | 0.30 | 0.30 | 0.32 | 0.28 | 0.29 | 3.4 |
| 510 | 261.54 | 0.26 | 0.28 | 0.29 | 0.24 | 0.27 | 0.26 | 0.27 | 0.21 | 0.24 | 0.29 | 0.26 | 3.8 |
| 362 | 261.48 | 0.41 | 0.39 | 0.42 | 0.44 | 0.35 | 0.39 | 0.43 | 0.37 | 0.38 | 0.43 | 0.40 | 2.5 |

### 3.2.5 Numerical Experiments and Results

**Experiment-1:** Linearly production dependent quality [$f(P) = a + bP$] :
**Practical implication:** The model deals with a realistic problem of production. A production house (toy company) produces one type of toys at different rates of production and starting of defective units is observed at different times. The collected data are shown in Table 3.4. Here, mean time of out-of-control state $(m = 1/f(P)$ depends on the production rate. Using the regression analysis f(P) is estimated as $f(P) = 1.25 + 0.005P$. The collected relevant data for the proposed EPL models are given bellow in appropriate units:
$\alpha = 0.05$, $\beta = 0.25$, $\theta = 0.25$, $\gamma = 1.5$, $\rho = 1$, $z = 0.04$, $\epsilon = 0.02$, $d_0 = 350$, $d_1 = 0.01$, $C_h = 2.0$, $C_{s0} = 1000$, $C_{s1} = 0.5$, $C_r = 100$, $C_d = 50$, $f(P) = 1.25 + 0.005P$ and UPC as:

Case 1: $C_1(P) = 250 + \frac{2500}{P} + 0.01P + 0.03P^{1/2}$
Case 2: $C_1(P) = 250 + \frac{2500}{P} + 0.01P$
Case 3: $C_2(P) = 250 + \frac{2500}{P} + 0.000027P^2 + 0.03P^{1/2}$
Case 4: $C_2(P) = 250 + \frac{2500}{P} + 0.000027P^2$

With the above parameters and expressions, the Models -3.1 and 3.1A-3.1K are formulated and optimized using a gradient based non-linear optimization technique- GRG method (LINGO 11.0 software). The corresponding optimum values of production rate($P^*$), production run time($t_1^*$) for minimum cost($ATC^*$) and the related cycle time($T^*$), total expected defective units($E(N^*)$), inventory level of good units($Q^*$), instant of defective production($m^*$), average holding($AHC^*$), set-up($ASC^*$), production($APC^*$), rework($ARC^*$) and disposal($ADC^*$) costs are evaluated for Cases 1-3 and presented in Table 3.5. The minimum average total costs $ATC$ for Model-3.1 due to Case-1 are plotted in Fig. 3.2 against the different values of $t_1$ and P. In this experiment, for Models 3.1 and 3.1A-3.1K due to Case-4, only optimal average costs are presented in Table 3.5 as the behaviour of the other parameters of the models can be inferred from the Case-3 following the behaviour of the models for Cases-1 and 2.

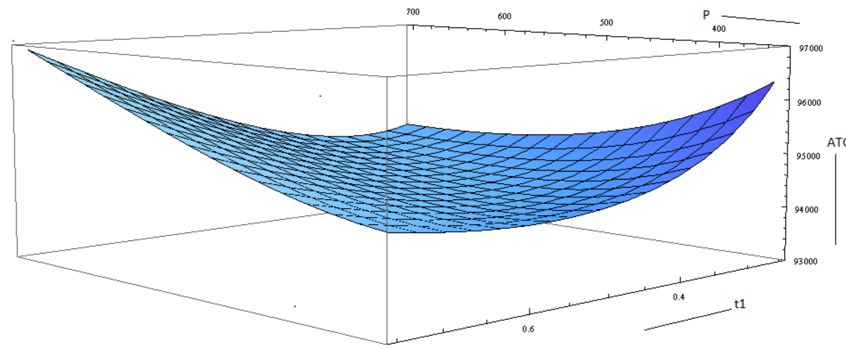



**Figure** 3.2: Expected average total cost w.r.t $t_1$ and $P$

**Experiment-2:** production independent quality $[f(P) = a]$ :

In this experiment, constant f(P)(=1.25) is considered, i.e. quality is independent of production. In this evaluation, optimal average costs $ATC^*$ for Models 3.1 and 3.1A-3.1K against the Cases 1-4 are given in Table 3.6. The unknown parameters , $t_1^*$ and $P^*$ for all models due to Case-1 only are presented in this table as the behaviour of the other expressions and parameters are same as in Exp.-1.

**Experiment-3:** Quality is non-linearly dependent on production $[f(P) = a + bP^2]$ :
In this experiment, f(P) is a non-linear function of P, as $f(P) = 1.25 + 0.000001P^2$. In this evaluation, optimal results for only Model-3.1 for all Cases 1-4 are given in Table 3.7 as the behaviour of optimal results for other models are same as in Exp.-1.

### 3.2.6 Discussion

**From Tables 3.5, 3.6 and 3.7:**

• For the Cases-1, 2, 3 and 4 of all the experiments, EPL Model-3.1G gives the lowest cost. It is the model in which both demand and defective rates are constant and full rework is done. For highest average cost, cost of Model-3.1E is maximum for Cases-1, 2, 3 and 4. Model-3.1E is the model with stock-dependent demand and constant defective rate but no rework. In this case, disposal cost of defective rubbish is highest. These disposal costs play a key role in the case of highest average total costs.

• It is to be noted that the mean-time $(m^*)$ at the beginning of "out-of-control" state is less than the production run time for all Models 3.1 and 3.1A-3.1K in all experiments. This is a necessary condition for the model. But it is not observed in the earlier works by Sana [227] and Khouja and Mehrez [127] because they did not imposed the condition $m^* \leq t_1^*$ through chance constraint (cf. § 3.2.2).

• Table 3.5 reveals that amongst different cases for Exps.-1, 2 and 3, the models' costs with EPC (for Cases-1 and -3) are higher than those of models without EPC (for Cases-2 and -4). Again, when wear and tear costs are proportional to the square of production rate (Cases-3

**Table** 3.5: Exp.-1: Optimal values for Models 3.1 and 3.1A-3.1K with linearly production dependent qualities ($f(P) = 1.25 + 0.005P$)

| | $Models$ | $ATC^*$ | $t_1^*$ | $T^*$ | $P^*$ | $E(N^*)$ | $Q^*$ | $m^*$ | $AHC^*$ | $ASC^*$ | $APC^*$ | $ARC^*$ | $ADC^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Case\,1:$ | 3.1 | 94222 | 0.416 | 0.644 | 546.0 | 2.00 | 79.9 | 0.251 | 80.4 | 1977.0 | 91970.7 | 77.6 | 116.4 |
| | 3.1$A$ | 93536 | 0.592 | 0.843 | 498.9 | 5.20 | 87.9 | 0.267 | 88.0 | 1481.9 | 91349.2 | 616.8 | 00.0 |
| $C_1(P)$ | 3.1$B$ | 94380 | 0.389 | 0.611 | 555.5 | 1.65 | 78.0 | 0.248 | 78.7 | 2088.8 | 92076.9 | 00.0 | 135.7 |
| | 3.1$C$ | 96199 | 0.285 | 0.406 | 506.6 | 2.80 | 42.5 | 0.264 | 42.9 | 3085.8 | 92639.9 | 172.3 | 258.4 |
| $= 250$ | 3.1$D$ | 95059 | 0.284 | 0.412 | 508.2 | 2.80 | 44.9 | 0.264 | 44.9 | 3039.6 | 91295.7 | 679.2 | 00.0 |
| | 3.1$E$ | 96587 | 0.285 | 0.404 | 506.0 | 2.79 | 41.6 | 0.264 | 42.2 | 3101.5 | 93096.8 | 00.0 | 346.3 |
| $+\frac{2500}{P}$ | 3.1$F$ | 94095 | 0.416 | 0.683 | 578.7 | 2.18 | 93.5 | 0.241 | 94.0 | 1886.5 | 91915.1 | 79.9 | 119.8 |
| | 3.1$G$ | 93379 | 0.606 | 0.937 | 540.9 | 6.12 | 115.8 | 0.253 | 115.8 | 1355.2 | 91255.0 | 653.1 | 00.0 |
| $+0.01P$ | 3.1$H$ | 94258 | 0.388 | 0.646 | 586.5 | 1.79 | 90.0 | 0.239 | 90.6 | 2003.0 | 92025.2 | 00.0 | 139.2 |
| | 3.1$I$ | 96135 | 0.279 | 0.411 | 523.5 | 2.82 | 46.3 | 0.258 | 46.7 | 3069.1 | 92589.1 | 172.0 | 258.0 |
| $+0.03P^{1/2}$ | 3.1$J$ | 94995 | 0.278 | 0.418 | 525.4 | 2.83 | 48.8 | 0.258 | 48.8 | 3023.0 | 91245.0 | 678.0 | 00.0 |
| | 3.1$K$ | 96522 | 0.279 | 0.409 | 522.9 | 2.82 | 45.4 | 0.259 | 46.0 | 3084.8 | 93046.0 | 00.0 | 345.7 |
| $Case\,2:$ | 3.1 | 93973 | 0.410 | 0.653 | 562.0 | 2.00 | 85.2 | 0.246 | 85.7 | 1962.4 | 91733.4 | 76.6 | 115.0 |
| | 3.1$A$ | 93299 | 0.582 | 0.853 | 513.7 | 5.16 | 95.0 | 0.262 | 95.0 | 1474.0 | 91124.8 | 605.4 | 00.0 |
| $C_1(P)$ | 3.1$B$ | 94129 | 0.383 | 0.620 | 571.6 | 1.66 | 83.0 | 0.243 | 83.6 | 2072.9 | 91837.9 | 00.0 | 134.1 |
| | 3.1$C$ | 95958 | 0.280 | 0.409 | 519.8 | 2.82 | 45.4 | 0.260 | 45.8 | 3074.3 | 92406.8 | 172.4 | 258.6 |
| $= 250$ | 3.1$D$ | 94821 | 0.280 | 0.416 | 521.5 | 2.83 | 47.9 | 0.259 | 47.9 | 3028.3 | 91065.4 | 679.5 | 00.0 |
| | 3.1$E$ | 96344 | 0.280 | 0.408 | 519.2 | 2.82 | 44.6 | 0.256 | 45.1 | 3090.0 | 92862.2 | 00.0 | 346.4 |
| $+\frac{2500}{P}$ | 3.1$F$ | 93839 | 0.410 | 0.694 | 596.0 | 2.19 | 99.4 | 0.236 | 99.9 | 1869.5 | 91672.0 | 79.2 | 188.8 |
| | 3.1$G$ | 93135 | 0.597 | 0.951 | 557.6 | 6.11 | 123.9 | 0.248 | 123.9 | 1344.9 | 91022.5 | 642.9 | 00.0 |
| $+0.01P$ | 3.1$H$ | 93999 | 0.383 | 0.656 | 603.9 | 1.81 | 95.5 | 0.234 | 96.0 | 1984.9 | 91780.9 | 00.0 | 138.0 |
| | 3.1$I$ | 95889 | 0.274 | 0.414 | 537.2 | 2.85 | 49.2 | 0.254 | 49.6 | 3058.4 | 92351.4 | 172.1 | 258.1 |
| $without\,EPC$ | 3.1$J$ | 94752 | 0.274 | 0.421 | 539.2 | 2.85 | 51.8 | 0.253 | 51.8 | 3012.5 | 91009.9 | 678.3 | 00.0 |
| | 3.1$K$ | 96276 | 0.274 | 0.412 | 536.6 | 2.85 | 48.4 | 0.254 | 48.9 | 3074.0 | 92807.3 | 00.0 | 345.9 |
| $Case\,3:$ | 3.1 | 94544 | 0.499 | 0.550 | 388.6 | 2.01 | 17.7 | 0.313 | 18.5 | 2171.2 | 92125.7 | 91.4 | 137.0 |
| | 3.1$A$ | 93719 | 0.712 | 0.748 | 368.0 | 5.60 | 12.8 | 0.324 | 12.8 | 1582.8 | 91375.6 | 748.3 | 00.0 |
| $C_2(P)$ | 3.1$B$ | 94731 | 0.467 | 0.519 | 392.6 | 1.65 | 18.2 | 0.311 | 19.0 | 2303.1 | 92249.6 | 00.0 | 159.1 |
| | 3.1$C$ | 96434 | 0.340 | 0.364 | 379.3 | 2.49 | 8.1 | 0.318 | 8.6 | 3271.4 | 92725.3 | 171.5 | 257.2 |
| $= 250$ | 3.1$D$ | 95294 | 0.340 | 0.369 | 379.7 | 2.49 | 10.1 | 0.317 | 10.1 | 3223.2 | 91384.7 | 676.0 | 00.0 |
| | 3.1$E$ | 96821 | 0.340 | 0.362 | 379.2 | 2.49 | 7.4 | 0.318 | 08.1 | 3287.8 | 93180.9 | 00.0 | 344.7 |
| $+\frac{2500}{P}$ | 3.1$F$ | 94505 | 0.499 | 0.565 | 399.4 | 2.07 | 23.1 | 0.308 | 23.1 | 2123.5 | 92128.3 | 91.9 | 137.9 |
| | 3.1$G$ | 93671 | 0.722 | 0.791 | 383.1 | 6.04 | 23.9 | 0.316 | 23.9 | 1506.5 | 91376.4 | 763.6 | 00.0 |
| $+0.000027P^2$ | 3.1$H$ | 94694 | 0.466 | 0.532 | 402.7 | 1.70 | 22.9 | 0.306 | 23.7 | 2258.7 | 92251.5 | 00.0 | 159.9 |
| | 3.1$I$ | 96417 | 0.337 | 0.366 | 386.2 | 2.50 | 10.3 | 0.314 | 10.8 | 3256.1 | 92722.7 | 171.2 | 256.8 |
| $+0.03P^{1/2}$ | 3.1$J$ | 95278 | 0.337 | 0.372 | 386.9 | 2.51 | 12.4 | 0.314 | 12.4 | 3208.0 | 91382.3 | 675.0 | 00.0 |
| | 3.1$K$ | 96805 | 0.337 | 0.364 | 386.0 | 2.50 | 9.6 | 0.314 | 10.3 | 3272.5 | 93178.2 | 00.0 | 344.1 |
| $Case\,4:$ | $Models$ | 3.1 | 3.1$A$ | 3.1$B$ | 3.1$C$ | 3.1$D$ | 3.1$E$ | 3.1$F$ | 3.1$G$ | 3.1$H$ | 3.1$I$ | 3.1$J$ | 3.1$K$ |
| $without\,EPC$ | $ATC^*$ | 94343 | 93517 | 94520 | 96226 | 95089 | 96612 | 94293 | 93464 | 94480 | 96207 | 95071 | 96594 |

**Table** 3.6: Exp.-2: Optimal Average costs for Models 3.1 and 3.1A-3.1K with production independent qualities ($f(P) = 1.25$)

| | $Models$ | 3.1 | 3.1$A$ | 3.1$B$ | 3.1$C$ | 3.1$D$ | 3.1$E$ | 3.1$F$ | 3.1$G$ | 3.1$H$ | 3.1$I$ | 3.1$J$ | 3.1$K$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Case\,1:$ | $ATC^*$ | 94153 | 93169 | 94484 | 94314 | 93228 | 94684 | 94024 | 93016 | 94363 | 94145 | 93058 | 94514 |
| | $t_1^*$ | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 | 0.842 |
| | $P^*$ | 443.9 | 462.9 | 437.6 | 478.0 | 476.2 | 478.6 | 489.7 | 514.5 | 481.6 | 532.9 | 531.4 | 533.3 |
| $Case\,2:$ | $ATC^*$ | 93927 | 92942 | 94259 | 94080 | 92997 | 94448 | 93787 | 92776 | 94127 | 93898 | 92814 | 94266 |
| $Case\,3:$ | $ATC^*$ | 94239 | 93304 | 94555 | 94491 | 93398 | 94863 | 94220 | 93275 | 94539 | 94457 | 93362 | 94830 |
| $Case\,4:$ | $ATC^*$ | 94037 | 93103 | 94353 | 94285 | 93195 | 94655 | 94013 | 93068 | 94332 | 94246 | 93154 | 94617 |

**Table** 3.7: Exp.-3: Optimal values for Model 3.1 with non-linearly production dependent qualities ($f(P) = 1.25 + 0.000001P^2$)

| $Model-3.1$ | $ATC^*$ | $t_1^*$ | $T^*$ | $P^*$ | $E(N^*)$ | $Q^*$ | $m^*$ | $AHC^*$ | $ASC^*$ | $APC^*$ | $ARC^*$ | $ADC^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Case\,1:$ | 93965 | 0.701 | 1.000 | 505.3 | 4.77 | 104.9 | 0.664 | 106.3 | 1252.5 | 92308.2 | 119.4 | 179.0 |
| $Case\,2:$ | 93725 | 0.693 | 1.020 | 521.2 | 4.83 | 114.7 | 0.657 | 116.0 | 1235.2 | 92077.5 | 118.4 | 177.6 |
| $Case\,3:$ | 94169 | 0.759 | 0.798 | 372.0 | 3.97 | 13.7 | 0.720 | 15.3 | 1485.8 | 92356.5 | 124.4 | 186.6 |
| $Case\,4:$ | 93963 | 0.757 | 0.806 | 376.5 | 4.00 | 17.0 | 0.718 | 18.5 | 1474.6 | 92159.2 | 124.3 | 186.5 |

and -4), models' costs are higher than those of the models with linearly production dependent wear and tear costs (for Cases-1 and 2). These behaviours are as per expectation.

• From Table 3.6, when inverse mean-time of "out-of-control" state is constant (f(P)=1.25), then the mean-time of defective production commencement ($m^* = 0.8000$) and production run time ($t_1^* = 0.8420 units$) are also constants. It is also seen from Table 3.6 that average total cost is minimum when full reworking is made for all defective units. This is because per unit rework cost (100 units) is less than the UPC (min. 250 units).

•From Table 3.7 (Exp.-3), we observe that average optimal total cost $ATC^*(= 93725 units)$ is minimum for Case-2 of Model-3.1 with maximum inventory level $Q^*(= 114.7 units)$ with respect to other cases. Here average production cost increases due to decrease in production rate and the defective units $E[N^*]$ are larger than those of Model-3.1 in Exps.-1 and 2.

• When UPC is without EPC, the optimal production run time decreases whereas optimal production rate increases and it causes to increase inventory level and average inventory holding cost. But it is seen from the Tables 3.5 and 3.6 that in this case, all other costs decrease. Specially average production cost decreases as UPC is without EPC and the higher production rate decreases labour/energy cost. Finally it is seen from the Tables 3.5 and 3.6 that the average expected total cost $ATC^*$ is less for all models when UPC is free from EPC. The same behaviour is observed for all models in Exp.-3 also.

• From Tables 3.5 and 3.6, for the Models 3.1 and 3.1A-3.1E with stock-dependent demand, the optimal average expected total costs are greater than those of the Models 3.1F-3.1K which are with constant demand. It is observed for all cases of Exps.-1 and 2. Interesting result is that in spite of higher demand (stock-dependent) in market, production rate is lower to minimize the average cost. Here lower production rates (for Models 3.1 and 3.1A-3.1E with stock-dependent demand ) increase the average production costs. The same behaviour is noticed for all models in Exp.-3.

• For all cases in Exps.-1 and 2, the optimal average expected total costs of models with constant defective rate are greater than those of the models with random defective rate. The same behaviour is observed for all these types of models in Exp.-3.

• For all cases in Exps.-1 and 2, as partly mentioned earlier, the optimal average expected total costs of models with no rework are greater than those of the models with full rework. The same behaviour is concluded for all these types of models in Exp.-3.

• It is clear from the Tables 3.5 and 3.6 that the $ATC^*$ is greater for the Exp.-1 i.e., for the models with linearly production dependent qualities than the corresponding results obtained from the similar models with production independent qualities (Exp.-2). But this behaviour can not be concluded in Exp.-3, i.e. when qualities depend non-linearly on production rate (cf. Table 3.7) because the coefficient of non-linearly dependence is different.

**From Fig. 3.3:**

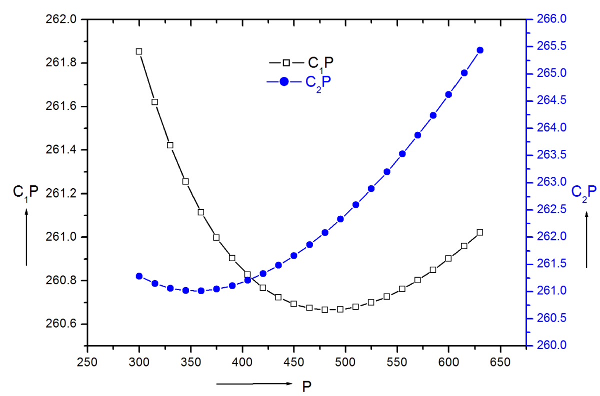



**Figure** 3.3: UPC $C_1(P)$ and $C_2(P)$ with respect to production rate P

UPC $C_1(P)$ for Case-1 has minimum value $260.66$ units for production rate $P = 483.8$ units and when $C_1(P)$ (for Case-2) free from EPC(i.e. $\eta_2 = 0$), it attains minimum value $260.00$ units at P=500.0 units. The corresponding values of $C_2(P)$ are:

$$\text{Min. } C_2(P) = \begin{cases} 261.01 \text{ units for P=354.2 units for Case-3} \\ 260.44 \text{ units for P=359.1 units for Case-4} \end{cases}$$

Now from the Tables 3.6 and 3.7, it is seen that the production rate which minimizes the UPC is quite different from the production rate which minimizes the average expected total cost $ATC$ for a particular model. It is interesting to note that the production rate P which minimizes $ATC$ is higher than the production rate P which minimizes UPC. For example, for Model-3.1 due to Case-1 in Exp.-1, $ATC$ has the minimum value $ATC^* = 94222$ units at $P^* = 546.0$ units, but the corresponding UPC $C_1(P)$ attains minimum value at $P^* = 483.8$ units. For other cases, results are presented in Table 3.8.

### 3.2.7 Sensitivity analysis

The changes in the values of system parameters can take place due to uncertainties and dynamic market conditions in a production-inventory system. In order to examine the implications of these changes in the values of parameters, the sensitivity analysis is of great

**Table** 3.8: Optimum results of $C_i(P)$

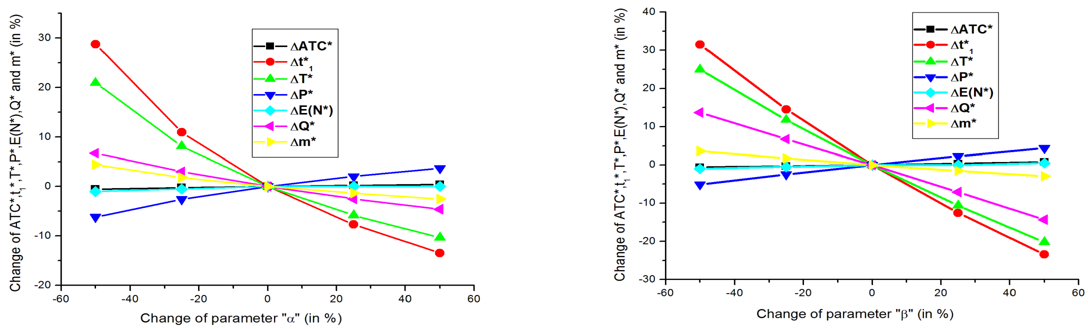
| $Case$ | $Value\,of\,P^*\,for\,Min\,ATC$ | $Value\,of\,P^*\,for\,Min\,C_1(P)\,or\,C_2(P)$ |
|---|---|---|
| 2 | 562.0 | 500.0 |
| 3 | 388.6 | 354.2 |
| 4 | 393.2 | 359.1 |

help in a decision-making process. Using the result of Model-3.1 due to Case-1 in Exp.-1, the sensitivity analyses of various parameters have been carried out. Here the changes of optimal average expected total cost($\Delta ATC^*$), production run time($\Delta t_1^*$), production rate($\Delta P^*$), cycle time($\Delta T^*$), total defective units($\Delta E(N^*)$), inventory level($\Delta Q^*$) and mean-time of beginning of "out-of-control" state($\Delta m^*$) are evaluated in percentages with respect to optimal results of Model-3.1 due to csae-1 in Exp.-1 and depicted for the changes of $\alpha$, $\beta$, $\gamma$, $\theta$, $d_0$, $d_1$, $\eta_1$, $\eta_2$, $r_m$, $g$, $C_r$ and $C_d$ in Figs. 3.4a-3.9b respectively. The behaviour of the above parameters for other models due to other cases are almost same. Here the change (in %) of an optimal value (suppose for $ATC^*$) is defined as $\Delta ATC^* = 100 \times [(ATC^{*old} - ATC^{*new})/ATC^{*old}]\%$, where $ATC^{*old}$ is the optimal result obtained from Model-3.1 due to Case-1 in Exp.-1 and $ATC^{*new}$ is the new optimal result obtained after changing the corresponding parameter (in %) for the same experiment. From the sensitivity analyses the following observation are made:

• ($ATC^*$, $Q^*$, $m^*$) and ( $t_1^*$, $T^*$, $P^*$, $E(N^*)$ )increase and decrease respectively with increase in $\alpha$ (cf. Fig. 3.4a). Increasing in $\alpha$ helps to start the defective products early and makes a lower stock level at the end of production. As the cycle time and inventory level decrease, holding cost decreases. The set-up cost, rework cost, production cost and disposal cost increase due to increase in production rate and defective units. From these figures, it is concluded that changes in $\beta$ give similar behaviour of optimal results as changes in $\alpha$ (cf. Fig. 3.4b).

• From Fig. 3.5a, we see that increase in $\gamma$ increases the values of $t_1^*$, $T^*$, $P^*$, $E(N^*)$ and decreases $ATC^*$, $Q^*$, $m^*$. In this case only holding cost increases with increase in $\gamma$ but all other cost decrease. As a result optimal expected average total cost decreases. Again in Fig. 3.5b, increased $\theta$ gives increased values of $t_1^*$, $T^*$, $E(N^*)$, $Q^*$, $m^*$ and decreased values of $P^*$. Holding and rework costs increase due to increase in $Q^*$ and $E(N^*)$, whereas average set-up, production, disposal cost decrease with increase in $\theta$. Finally $ATC^*$ has a lower value for higher percentage of rework.

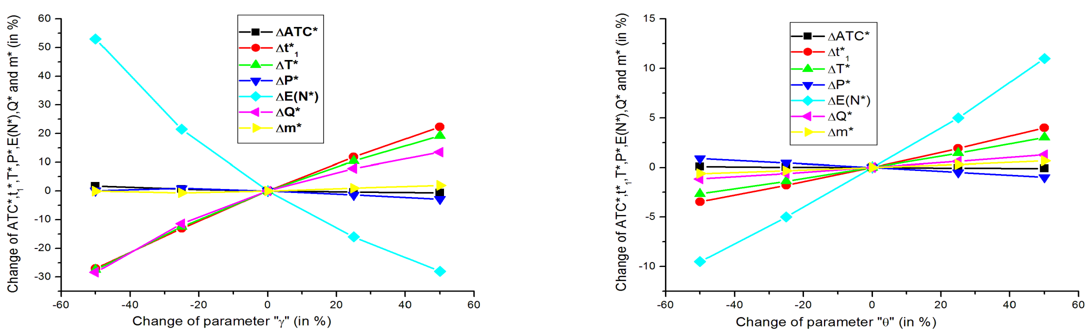
• From Figs. 3.6a and 3.6b, production rate increases due to higher demand in the market in both cases of constant and stock-dependent demands. The defective products increases and stock level at the end of production decreases because of higher production and demands rates respectively. Though production run time increases, cycle time decreases due to lower stock. Here $ATC^*$ increases with increase in both constant and stock-dependent demands.




(a) Sensitivity for key parameter $\alpha$

(b) Sensitivity for key parameter $\beta$

**Figure** 3.4: Changes of optimal results for $\alpha$ and $\beta$




(a) Sensitivity for key parameter $\gamma$

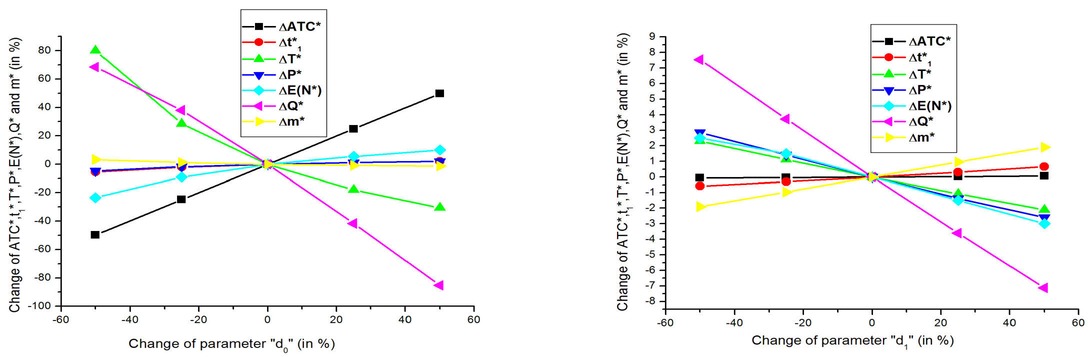
(b) Sensitivity for key parameter $\theta$

**Figure** 3.5: Changes of optimal results for $\gamma$ and $\theta$

• From Figs. 3.7a and 3.7b, $t_1^*$, $m^*$ increase with increase in both wear and tear cost and EPC whereas $P^*$,$T^*$,$Q^*$ decrease. In these cases, holding cost decreases and all other costs increase with increase in $\eta_1$ and $\eta_2$. As a result, average cost $ATC^*$ increases with increase in $\eta_1$ and $\eta_2$.

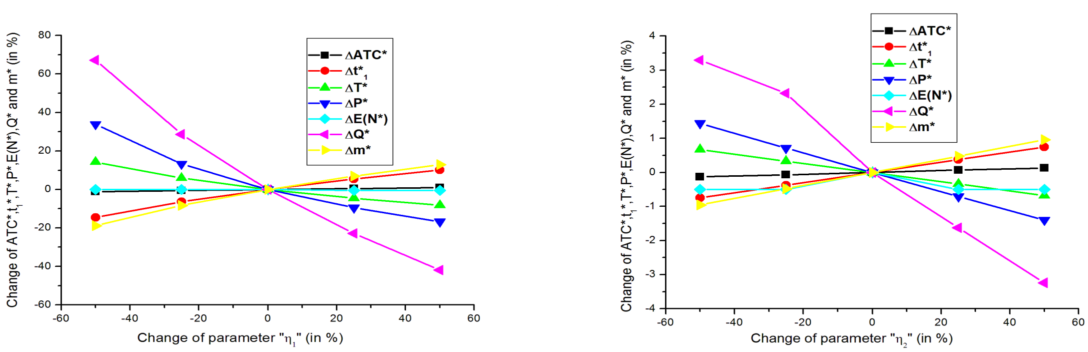
• Raw material cost is highly sensitive to unit production cost. UPC increases with increase in $r_m$ (cf. Fig. 3.8a). In this case, average production cost $APC^*$ increases the average cost $ATC^*$. Again from Fig. 3.8b, it is clear that $P^*$,$Q^*$,$T^*$ increase and $t_1^*$, $m^*$, $E(N^*)$ decrease with increase in labour / energy cost $g$. Higher production cost increases the stock level and due to this, holding cost increases. Average production cost increases with increase in $P^*$. As a result $ATC^*$ increases with increases in $g$.

• With increase in $C_r$ and $C_d$, the values of $t_1^*$, $T^*$, $E(N^*)$, $Q^*$ and $m^*$ increase whereas only $P^*$ decreases (cf. Figs. 3.9a and 3.9b). When $C_r$ increases, average rework cost $ARC^*$ increases and average disposal cost $ADC^*$ decreases. But when $C_d$ increases, $ADC^*$




(a) Sensitivity for key parameter $d_0$ (b) Sensitivity for key parameter $d_1$

**Figure** 3.6: Changes of optimal results for $d_0$ and $d_1$



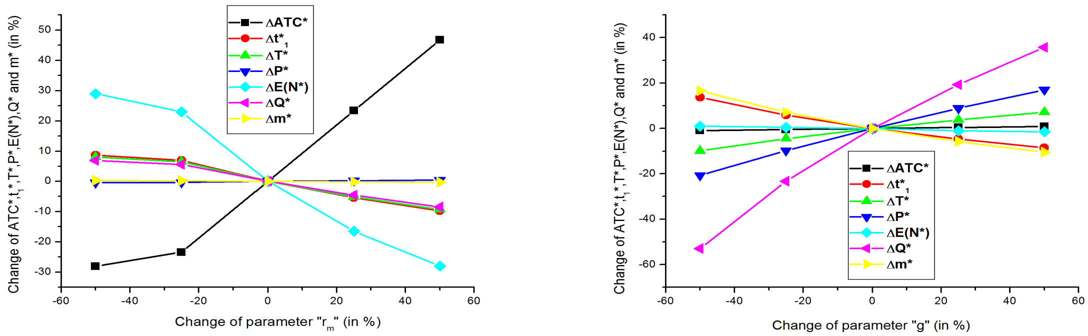

(a) Sensitivity for key parameter $\eta_1$ (b) Sensitivity for key parameter $\eta_2$

**Figure** 3.7: Changes of optimal results for $\eta_1$ and $\eta_2$

increases and $ARC^*$ decreases. These results are as per expectation. Holding, set-up, and production costs increase with increase in $C_r$ and $C_d$.

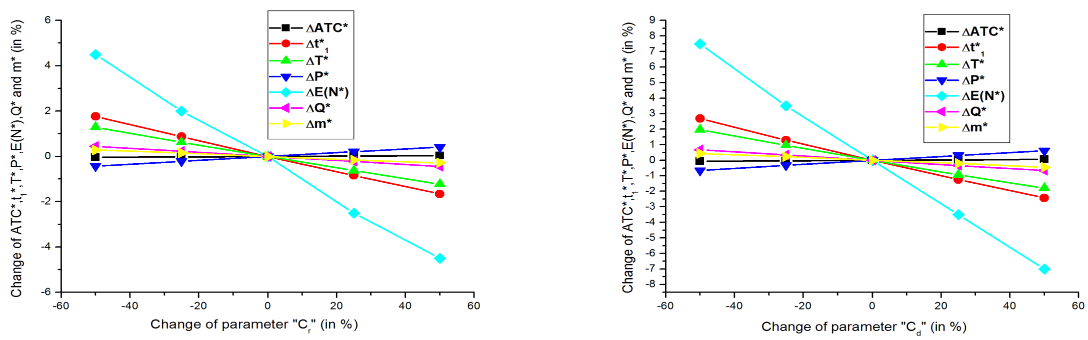
### 3.2.8 A real-life illustration

A toy-manufacturing company at Kolkata, West Bengal, India manufactures different types of toy. For a particular toy-item, different data during the year, 2014 are collected and presented in Table 3.4. Considering that inverse of mean of commence of imperfect production, $f(P)$ is linearly dependent on $P$, from the data of Table 3.4, we express $f(P)$ in the form of $f(P) = 1.25 + 0.005P$ using regression analysis based on Least Square Approximation. For the same year 2014, the raw material cost for a unit production=Rs.250 and the cost against the employed labours=Rs.2500 per day. For the said company, it is assumed that machine repair cost is proportional to rate of production, $P$ and cost for taking




(a) Sensitivity for key parameter $r_m$ (b) Sensitivity for key parameter $g$

**Figure** 3.8: Changes of optimal results for $r_m$ and $g$




(a) Sensitivity for key parameter $C_r$ (b) Sensitivity for key parameter $C_d$

**Figure** 3.9: Changes of optimal results for $C_r$ and $C_d$

some preliminary steps for clean production for EPC as per Kolkata Municipality varies as $\sqrt{P}$. Taking the almost constant unit production costs for different production rates from the Table 3.4, we determine UPC using Least Square Approximation method as $C(P) = 250 + \frac{2500}{P} + 0.01P + 0.03P^{1/2}$. Moreover, for the constant demand rate 350 units per day, defective units are 14 units, out of which 7 units are reworked. Other input data are: $\alpha = 0.25$, $\theta = 0.50$, $\rho = 1$, $z = 0.04$, $\epsilon = 0.02$, $C_h = Rs.2$, $C_{s0} = Rs.1000$, $C_{s1} = 0.5$, $C_r = Rs.75$, $C_d = Rs.50$

For the above input data of the company, the Model-3.1I is formulated and evaluated. The evaluated average total cost is Rs. 101335. This is almost same as the company's cost for the year 2014.

## 3.3 Model-3.2 : An EPL model with reliability dependent randomly imperfect production system over different uncertain finite time horizons [2]

### 3.3.1 Assumptions and Notations

***Assumptions:***

(i) Replenishment rate is finite and taken as a DV.

(ii) Lead time is zero.

(iii) Shortages are not allowed.

(iv) The inventory system considers a single item and the demand rate is stock-dependent.

(v) The time horizon is finite and the production time is taken as a DV.

(vi) The production process shifts from the "In-control" state to an "Out-of-control" state at a time, which is a random variable. Imperfect units are produced in this state.

(vii) Production of defective units commences at a random time after the commencement of production. Defective rate depends on production rate, reliability of the machinery system producing the item and time duration from the starting of defective units' production.

(viii) The system allows immediate partially reworking for the defective units at a certain cost when they are produced in "out-of-control" state and the defective units which are not reworked, are disposed off by a cost.

(ix) UPC is the sum total of per unit material cost, development cost, wear and tear cost and EPC. Here development cost is a function of reliability parameter of the machinery system which is also a DV.

(x) A maintenance cost is considered for the machinery system to bring to its initial position by the maintenance operations during the each time gap between the end of production and beginning of next production. Thus the time for maintenance is shorter than the time gap between the end of the production and beginning of next production.

***Notations:***

The following notation are defined and noted for $i^{th}$ cycle.

[2]This model has been published in **Journal of Intelligent and Fuzzy Systems**, IOS Press, Y. 2016

$q(t)$ Inventory level at time $t$, where $q(t) \geq 0$.

$D[q(t)]$ The demand rate at time $t$ and $D[q(t)] = d_0 + d_1 q(t)$ where $d_0 > 0$, $d_1$ is the stock-dependent consumption rate parameter, $0 \leq d_1 \leq 1$.

$P$ Controllable production rate in units per unit time (DV), where $P - D[q(t)] \geq 0$.

$T$ Cycle time in appropriate unit.

$t_1$ Production run-time in each period (DV).

$C_{s_i}$ Set up cost for $i^{th}$ cycle which is partly constant and partly decreases in each cycle due to learning effect of the employees and is of the form: $Cs_i = C_{s0} + C_{s1}e^{-k_1 i}$, where $k_1 > 0$.

$Cm_i$ Maintenance cost for the machinery system to bring the system to its original position after the end of each production. For the first cycle no maintenance is required, but for the next cycles, it is increased in each cycle due to the reuse of the system for several times. Maintenance cost for $i^{th}$ cycle is taken as: $Cm_i = C_{m0}[1 - e^{-k_2(i-1)}]$.

$Ch$ Holding cost per unit per unit time.

$Cd$ Cost of disposal for an imperfect unit which is not reworked.

$Cr$ Cost for rework of an imperfect unit.

$C(P, r)$ UPC which is considered as $C(P, r) = r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}$, where $\delta_1, \delta_2, \delta_3 > 0$ and $r_m$ is the material cost per unit item, $g$ is the development cost, defined as $g = g_1 + g_2 e^{(1-f)\frac{r - r_{min}}{r_{max} - r}}$, where $r$ is the reliability parameter of machinery system (DV), $r_{max}$ and $r_{min}$ are maximum and minimum value of $r$ respectively, $f$ is the feasibility of increasing reliability, $g_1$ is total labour/energy costs per unit time in a production system which is equally distributed over the unit item and independent of reliability parameter $r$, $g_2$ is technology, resource and design complexity costs for production. So,$(\frac{g}{P^{\delta_1}})$ decreases with increases of $P$. The third term $\eta_1 P^{\delta_2}$ is the wear and tear cost, proportional to the positive power of production rate $P$ and the fourth term $\eta_2 P^{\delta_3}$ is EPC, proportional to the positive power of production rate $P$.

$\frac{1}{f(P)}$ The mean and standard deviation of the random variable $\tau$. Here, $f(P)$ is an increasing function of P and the mean time of failure, $1/f(P)$ is a decreasing function of $P$.

$\theta$ Percentage of rework of defective units.

$N$ Defective units in a production cycle.

$S_q$ Expected production lot size (or inventory) of good units (without defective units) at the end of production period.

$\tau$ An exponential random variable that depends on $P$ and denotes the time at which the process shifts to the "out-of-control" state from "in-control" variable. The distribution function of "out-of-control" state is $G(\tau) = 1-e^{-f(P)\tau}$ such that $\int_0^\infty dG(\tau) = f(P)\int_0^\infty e^{-f(P)\tau} d\tau = 1$. The exponential distribution has often been used to describe the elapsed time to failure of the components of a machinery system.

$\lambda(t', \tau, P, r)$ Percentage of defective units produced at time t when the machine is in the "out-of-control" state. Here $\lambda(t', \tau, P, r)$is defined as $\lambda(t', \tau, P, r) = \alpha P^{\beta} e^{(1-r)t'}(t'-\tau)^{\gamma}$. where $\beta \geq 0$,$\gamma \geq 0$ and $t' \geq \tau$. Generally speaking, the percentage of defective units increases with increase of production rate and production-run time. The formulation of the function $\lambda(t', \tau, P, r)$ shows that it is an increasing function of production rate and production-run time simultaneously and a decreasing function with respect to reliability parameter. Here $t'$ is the time measured from the commencement of production in each cycle and varies between (0, T). This is assumed in this way as the machinery system is brought back to its original condition by its proper maintenance in each cycle after each production run.

$m$ Total number of cycles which is a DV.

$H$ Finite time horizon.

### 3.3.2 Mathematical Model Development

In this production process for the $i^{th}$ cycle, production starts at a rate $P$ from time t=(i-1)T and runs up to time $t = (i-1)T + t_1$. The inventory piles up, during the time span $[0, t_1]$ adjusting demand $D[q(t)]$ in the market and the production process stocks good quality $Sq$ units at time $t = t_1$ and this stock is depleted satisfying the demand in the market and it reaches at zero level at time $iT$ (cf. Fig. 3.10). This production system produces perfect units up to a certain time $\tau$ (i.e., "in-control" state), after that, the production system shifts to an "out-of-control" state. In this "out-of-control" state, some of the produced units are of non-conforming quality (i.e., defective units) with a defective rate $\lambda(t, \tau, P, r) = \alpha P^{\beta} e^{(1-r)[t-(i-1)T]}[t-(i-1)T-\tau]^{\gamma}$ and some of these defective units are in a condition to rework immediately when they are produced. Here we assume that after the end of one production run, the machinery system is maintained against a cost and brought back to its original good condition for the next production. Thus the maintenance time is shorter than the production lay off time, $T - t_1$. The governing differential equations for the $i^{th}$ cycle are

$$\frac{dq(t)}{dt} = \begin{cases} P - D[q(t)], & (i-1)T \leq t \leq (i-1)T + \tau \\ P - D[q(t)] - (1-\theta)\lambda(t, \tau, P, r)P, & (i-1)T + \tau \leq t \leq (i-1)T + t_1 \\ -D[q(t)], & (i-1)T + t_1 \leq t \leq iT \end{cases} \quad (3.52)$$

with the boundary conditions

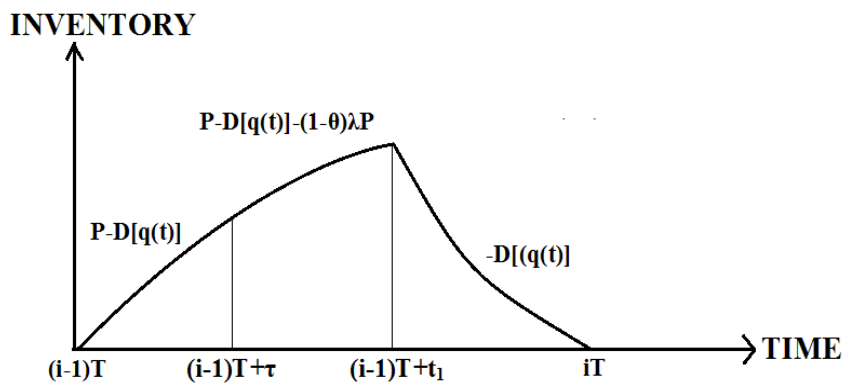



**Figure** 3.10: Inventory versus time for $i^{th}$ cycle.

$$\begin{cases} q(t) = 0, & at \;\; t = (i-1)T \\ q(t) = 0, & at \;\; t = iT \end{cases}$$

and the continuity conditions of $q(t)$ at $t = (i-1)T + \tau$ and $t = (i-1)T + t_1$.

Using the above boundary and continuity conditions, the solutions of the above differential equation are given by,

$$q(t) = \begin{cases} \frac{P-d_0}{d_1}(1 - e^{-d_1[t-(i-1)T]}), (i-1)T \leq t \leq (i-1)T + \tau \\ \frac{P-d_0}{d_1}(1 - e^{-d_1[t-(i-1)T]}) - (1-\theta)\alpha P^{\beta+1}\Big[\sum_{j=0}^{\gamma-1} \frac{(-1)^j\gamma!(t-(i-1)T-\tau)^{\gamma-j}}{(\gamma-j)!u^{j+1}} e^{(u-d_1)[t-(i-1)T]} \\ + \frac{(-1)^\gamma\gamma! e^{-d_1[t-(i-1)T]}}{u^{\gamma+1}}(e^{u[t-(i-1)T]} - e^{u\tau})\Big], \quad (i-1)T + \tau \leq t \leq (i-1)T + t_1, \\ \frac{d_0}{d_1}\left(e^{d_1(iT-t)} - 1\right), (i-1)T + t_1 \leq t \leq iT \end{cases} \tag{3.53}$$

where $u = d_1 + 1 - r$

The total defective units during $[(i-1)T + \tau, (i-1)T + t_1]$ is

$$\begin{aligned} N &= \int_{(i-1)T+\tau}^{(i-1)T+t_1} \lambda P dt = P \int_{(i-1)T+\tau}^{(i-1)T+t_1} \alpha P^\beta [t - (i-1)T - \tau]^\gamma e^{(1-r)[t-(i-1)T]} dt \\ &= \alpha P^{\beta+1}\left[\sum_{j=0}^{\gamma} \frac{(-1)^j\gamma!(t_1-\tau)^{\gamma-j} e^{(1-r)t_1}}{(\gamma-j)!(1-r)^{j+1}} - \frac{(-1)^\gamma\gamma! e^{(1-r)\tau}}{(1-r)^{\gamma+1}}\right] \end{aligned} \tag{3.54}$$

Total expected defective units in a production lot size is

$$E(N) = \int_0^{t_1} N d\Big(G(\tau)\Big) = \alpha P^{\beta+1}\Bigg[\sum_{j=0}^{\gamma-1} \frac{(-1)^j \gamma! e^{(1-r)t_1}}{(\gamma-j)!(1-r)^{j+1}}\Bigg(\sum_{k=0}^{\gamma-j-1} \frac{(-1)^k(\gamma-j)! t_1^{\gamma-j-k}}{(\gamma-j-k)! f^{k+1}(P)} - \frac{(-1)^{\gamma-j+1}(\gamma-j)!}{f^{\gamma-j+1}(P)}[1-e^{-f(P)t_1}]\Bigg) + \frac{(-1)^\gamma \gamma! e^{(1-r)t_1}}{(1-r)^{\gamma+1}} N_2 - \frac{(-1)^\gamma \gamma!}{(1-r)^{\gamma+1}} N_3\Bigg] \tag{3.55}$$

where $N_2 = \int_0^{t_1} e^{-f(P)\tau} d\tau = \frac{1-e^{-f(P)t_1}}{f(P)}$, $N_3 = \int_0^{t_1} e^{-v\tau} d\tau = \frac{1-e^{-vt_1}}{v}$
and $v = f(P) - (1-r)$

Now at time $t = (i-1)T + t_1$, the expected production lot size without defective units is

$$S_q = E[q((i-1)T+t_1)] = \int_0^\infty q[(i-1)T+t_1] d\Big(G(\tau)\Big)$$
$$= \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) - (1-\theta)\alpha P^{\beta+1} f(P) \int_0^{t_1} \Bigg(\sum_{j=0}^{\gamma-1} \frac{(-1)^j \gamma!(t_1-\tau)^{\gamma-j}}{(\gamma-j)! u^{j+1}} e^{(u-d_1)t_1} + \frac{(-1)^\gamma \gamma! e^{-d_1t_1}}{u^{\gamma+1}}(e^{ut_1}-e^{u\tau})\Bigg) e^{-f(P)\tau} d\tau$$

$$= \frac{P-d_0}{d_1}(1-e^{-d_1t_1}) - (1-\theta)\alpha P^{\beta+1} f(P)\Bigg[\sum_{j=0}^{\gamma-1} \frac{(-1)^j \gamma! e^{(u-d_1)t_1}}{(\gamma-j)! u^{j+1}} \Bigg(\sum_{k=0}^{\gamma-j} \frac{(-1)^k(\gamma-j)! t_1^{\gamma-j-k}}{(\gamma-j-k)! f^{k+1}(P)} + \frac{(-1)^{\gamma-j+1}(\gamma-j)!}{f^{\gamma-j+1}(P)} e^{-f(P)t_1}\Bigg) + \frac{(-1)^\gamma \gamma! e^{-d_1t_1}}{u^{\gamma+1}}(e^{ut_1}N_2 - Q_3)\Bigg] \tag{3.56}$$

where, $\qquad Q_3 = \int_0^{t_1} e^{-[f(P)-u]\tau} d\tau = \frac{1-e^{-[f(P)-u]t_1}}{f(P)-u}$

Again from the equation (3.53), we get,

$$S_q = E[q((i-1)T+t_1)] = \frac{d_0}{d_1}\Big(e^{d_1(T-t_1)} - 1\Big)$$
$$\text{or, } T = t_1 + \frac{1}{d_1}\log[1 + \frac{d_1 S_q}{d_0}] \tag{3.57}$$

Now during the period $[(i-1)T, (i-1)T+t_1]$, the inventory which are to be hold, is

$$
\begin{aligned}
Q_{h_1} &= \int_{(i-1)T}^{(i-1)T+t_1} q(t)dt \qquad = \int_{(i-1)T}^{(i-1)T+\tau} q(t)dt + \int_{(i-1)T+\tau}^{(i-1)T+t_1} q(t)dt \\
&= \int_{(i-1)T}^{(i-1)T+\tau} \frac{P-d_0}{d_1}(1-e^{-d_1[t-(i-1)T]})dt + \int_{(i-1)T+\tau}^{(i-1)T+t_1} \Bigg[\frac{P-d_0}{d_1}(1-e^{-d_1[t-(i-1)T]}) \\
&- (1-\theta)\alpha P^{\beta+1}\Bigg(\sum_{j=0}^{\gamma-1}\frac{(-1)^j\gamma![t-(i-1)T-\tau]^{\gamma-j}}{(\gamma-j)!u^{j+1}}e^{(u-d_1)[t-(i-1)T]} \\
&+ \frac{(-1)^\gamma\gamma!e^{-d_1[t-(i-1)T]}}{u^{\gamma+1}}(e^{u[t-(i-1)T]}-e^{u\tau})\Bigg)\Bigg]dt \\
&= \frac{P-d_0}{d_1}\Bigg(t_1-\frac{1}{d_1}(1-e^{-d_1t_1})\Bigg) \\
&- (1-\theta)\alpha P^{\beta+1}\Bigg[\sum_{j=0}^{\gamma-1}\frac{(-1)^j\gamma!}{(\gamma-j)!u^{j+1}}\Bigg(\sum_{k=0}^{\gamma-j}\frac{(-1)^k(\gamma-j)!(t_1-\tau)^{\gamma-j-k}e^{(u-d_1)t_1}}{(\gamma-j-k)!(u-d_1)^{k+1}} \\
&- \frac{(-1)^{\gamma-j}(\gamma-j)!e^{(u-d_1)\tau}}{(u-d_1)^{\gamma-j+1}}\Bigg) + \frac{(-1)^\gamma\gamma!}{u^{\gamma+1}}\Bigg(\frac{e^{(u-d_1)t_1}-e^{(u-d_1)\tau}}{u-d_1}-\frac{e^{u\tau}}{d_1}(e^{-d_1\tau}-e^{-d_1t_1})\Bigg)\Bigg]
\end{aligned}
\tag{3.58}
$$

During the period $[(i-1)T, (i-1)T+t_1]$, the expected quantity of holding units are

$$
\begin{aligned}
E[Q_{h_1}] &= \int_0^\infty Q_{h_1}d\Big(G(\tau)\Big) \\
&= \frac{P-d_0}{d_1}\Bigg(t_1-\frac{1}{d_1}(1-e^{-d_1t_1})\Bigg) \\
&- (1-\theta)\alpha P^{\beta+1}f(P)\Bigg[\sum_{j=0}^{\gamma-1}\frac{(-1)^j\gamma!}{(\gamma-j)!u^{j+1}}\Bigg\{\sum_{k=0}^{\gamma-j}\frac{(-1)^k(\gamma-j)!e^{(u-d_1)t_1}}{(\gamma-j-k)!(u-d_1)^{k+1}} \\
&\Bigg(\sum_{s=0}^{\gamma-j-k}\frac{(-1)^s(\gamma-j-k)!t_1^{\gamma-j-k-s}}{(\gamma-j-k-s)!f^{s+1}(P)}+\frac{(-1)^{\gamma-j-k+1}(\gamma-j-k)!}{f^{\gamma-j-k+1}(P)}e^{-f(P)t_1}\Bigg) \\
&- \frac{(-1)^{\gamma-j}(\gamma-j)!}{(u-d_1)^{\gamma-j+1}}N_3\Bigg\} + \frac{(-1)^\gamma\gamma!}{u^{\gamma+1}}\Bigg(\frac{e^{(u-d_1)t_1}}{u-d_1}N_2-\frac{N_3}{u-d_1}-\frac{N_3}{d_1}+\frac{e^{-d_1t_1}}{d_1}Q_3\Bigg)\Bigg]
\end{aligned}
\tag{3.59}
$$

Now during the period $[(i-1)T+t_1, iT]$, the inventory is

$$
\begin{aligned}
Q_{h_2} &= \int_{(i-1)T+t_1}^{iT} q(t)dt = \int_{t_1}^{T}\frac{d_0}{d_1}\Big(e^{d_1(iT-t)}-1\Big)dt \\
&= \frac{d_0}{d_1}\Bigg[\frac{1}{d_1}\Big(e^{d_1(T-t_1)}-1\Big)-(T-t_1)\Bigg]
\end{aligned}
\tag{3.60}
$$

Thus during the period $[(i-1)T, iT]$, the total expected quantity of holding units is

$$E[Q_h] = E[Q_{h_1}] + Q_{h_2} \tag{3.61}$$

where $E[Q_{h_1}]\, and\, Q_{h_2}$ are given by the equations (3.59) and (3.60) respectively.

In the $i^{th}$ cycle $[(i-1)T, iT]$, the Expected Total cost = Expected holding cost + Rework cost + Disposal cost + Production cost + Set-up cost + Maintenance cost.

$$\text{i.e.}\, TC_i(P, t_1, r) = Ch.E[Q_h] + \theta.Cr.E(N) + (1-\theta).Cd.E(N) + C(P,r)Pt_1 + Cs_i + Cm_i \tag{3.62}$$

Thus the Expected total cost for all cycle is

$$\begin{aligned} TC(P, t_1, r, m) &= \sum_{i=1}^{m} TC_i(P, t_1, r) \\ &= m\Big[Ch.E[Q_h] + \theta.Cr.E(N) + (1-\theta).Cd.E(N) + C(P,r)Pt_1\Big] \\ &+ \left[mC_{s0} + C_{s1}\frac{1-e^{-mk_1}}{e^{k_1}-1}\right] + C_{m0}\left[m - \frac{1-e^{-mk_2}}{1-e^{-k_2}}\right] \end{aligned} \tag{3.63}$$

### 3.3.3 Chance constraint for the “out-of-control” state

In this production system, it is expected to have total production time greater than the beginning time of “out-of-control” state in each cycle . This requirement acts as a constraint and is expressed here as a Chance constraint. Hence, the Chance constraint is

$$\Pr[(t_1 - \tau) \geq \epsilon] \geq x \tag{3.64}$$

where $t_1 \geq 0\ and\ x \in (0,1)$ is a specified permissible probability

Here $m_d[= \frac{1}{f(P)}]$ and $\sigma[= \frac{1}{f(P)}]$ are the mean and standard deviation of the exponential random variable $\tau$. Then the constraint can be written as (cf. Rao [213])

$$Pr(\tfrac{\tau - m_d}{\sigma} \leq \tfrac{t_1 - \epsilon - m_d}{\sigma}) \geq x$$

where $\frac{\tau - m_d}{\sigma}$ is a standard normal variate. Considering $z$, where $\int_0^z \phi(t)dt = x$, $\phi(t)$, being the standard normal density function, we have, $\frac{t_1 - \epsilon - m_d}{\sigma} \geq z$

$$\text{or, } t_1 \geq \frac{1}{f(P)}[1+z] + \epsilon \tag{3.65}$$

where $z$ is obtained from the normal distribution table for a particular value of $x$.

### 3.3.4 Different types of time horizons

#### Crisp time horizon

For the crisp finite time horizon, we consider a constraint as

$$mT = H \tag{3.66}$$

**Random time horizon**

In this consideration, the models remain same as developed above, except the time horizon of the system. Here, $\bar{H}$ is random. For this type of model, we impose the constraint as

$$\text{Pr}(\bar{H} \geq mT) \geq s, \qquad \text{where } s \in (0,1) \text{ is a specified permissible probability.}$$
$$\text{or,} mT \leq m_h + \sigma_h \Phi^{-1}(1-s), \text{(cf. Rao [213])} \tag{3.67}$$

where $m_h$ and $\sigma_h$ are the expectation and standard deviation of normally distributed random variable $\bar{H}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\frac{\bar{H}-m_h}{\sigma_h}$.

**Fuzzy time horizon**

If the time horizon $\tilde{H}$ is fuzzy in nature, it can be expressed by the fuzzy constraint $\tilde{H} \geq mT$ which is interpreted in the setting of possibility and necessity theory (cf. Dubois and Prade, [79]). The above constraint reduces to

$$\text{Pos}(\tilde{H} \geq mT) \geq \rho_1, \qquad \text{and} \qquad \text{Nes}(\tilde{H} \geq mT) \geq \rho_2$$

where $\rho_1$ and $\rho_2$ represent the degree of impreciseness. Let $\tilde{H} = (H_1, H_2, H_3)$ be TFN then, using Lemma 2.1 and 2.2, we get

$$mT \leq \begin{cases} (1-\rho_1)H_3 + \rho_1 H_2, & \text{in possibility sense} \\ (1-\rho_2)H_2 + \rho_2 H_1, & \text{in necessity sense.} \end{cases} \tag{3.68}$$

**Fuzzy-random time horizon**

In this case, the time horizon $\tilde{\bar{H}}$ is fuzzy-random in nature and the fuzzy-random constraint is $\tilde{\bar{H}} \geq mT$. It stands for the relations which are interpreted in the setting of possibility and necessity theories (cf. Dubois and Prade, [79]) along with chance the constraint. The above constraint reduces to

$$\text{Pr}\big[\text{Pos}(\tilde{\bar{H}} \geq mT) \geq \rho_3\big] \geq s_1, \qquad \text{and} \qquad \text{Pr}\big[\text{Nes}(\tilde{\bar{H}} \geq mT) \geq \rho_4\big] \geq s_2$$

where ($\rho_3$ and $\rho_4$) and ($s_1$ and $s_2$) represent the degrees of impreciseness and uncertainty (due to randomness) respectively. Let $\tilde{\bar{H}} = (\bar{H}, H_l, H_r)$ be L-R fuzzy-random variable, then according to Theorem 2.2, we get

$$mT \leq \begin{cases} m_h + \sigma_h \Phi^{-1}(1-s_1) + R^{-1}(\rho_3)H_r, & \text{in possibility sense} \\ m_h + \sigma_h \Phi^{-1}(1-s_2) - L^{-1}(1-\rho_4)H_l, & \text{in necessity sense.} \end{cases} \tag{3.69}$$

where $m_h$ and $\sigma_h$ are the expectation and standard deviation of normally distributed random variable $\bar{H}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\frac{\bar{H}-m_h}{\sigma_h}$.

**Rough time horizon**

If the time horizon $\hat{H}$ is rough in nature, the rough constraint $\hat{H} \geq mT$ is reduced to the crisp form as

$$Tr(\hat{H} \geq mT) \geq tr_1 \ (using\, Theorem\, 2.3).$$

$$\text{i.e. } mT \leq \begin{cases} H_4 - \frac{tr_1(H_4-H_3)}{\xi_1}, & \text{if} \quad H_2 \leq mT \leq H_4 \\ \frac{\xi_1(H_2-H_1)+(1-\xi_1)H_2(H_4-H_3)-tr_1(H_4-H_3)(H_2-H_1)}{\xi_1(H_2-H_1)+(1-\xi_1)(H_4-H_3)}, & \text{if} \ H_1 \leq mT \leq H_2 \\ H_4 + \frac{(1-\xi_1-tr_1)(H_4-H_3)}{\xi_1}, & \text{if} \quad H_3 \leq mT \leq H_1 \\ H_3 \end{cases} \tag{3.70}$$

where $\hat{H} = ([H_1, H_2][H_3, H_4])$, $0 \leq H_3 \leq H_1 \leq H_2 \leq H_4$, is a rough variable and $\xi_1 \in (0, 1)$ and $tr_1 \in [0, 1]$ are the confidence levels.

**Fuzzy-Rough time horizon**

If the time horizon $\tilde{\hat{H}}$ is fuzzy-rough in nature, the fuzzy-rough constraint $\tilde{\hat{H}} \geq mT$ is reduced in the following crisp forms

$$\text{Tr}\big[\text{Pos}(\tilde{\hat{H}} \geq mT) \geq \rho_5\big] \geq tr_2, \qquad \text{and} \qquad \text{Tr}\big[\text{Nes}(\tilde{\hat{H}} \geq mT) \geq \rho_6\big] \geq tr_2$$

According to Theorem 2.4, the above constraints are finally reduced to the following forms.

$$\begin{cases} mT \leq \begin{cases} H_4 - \frac{tr_2(H_4-H_3)}{\xi_2} + (1-\rho_5)H_R, & \text{if} \quad H_2 \leq mT - (1-\rho_5)H_R \leq H_4 \\ \frac{\xi_2(H_2-H_1)+(1-\xi_2)H_2(H_4-H_3)-tr_2(H_4-H_3)(H_2-H_1)}{\xi_2(H_2-H_1)+(1-\xi_2)(H_4-H_3)} + (1-\rho_5)H_R, & \text{if} \ H_1 \leq mT - (1-\rho_5)H_R \leq H_2 \\ H_4 + \frac{(1-\xi_2-tr_2)(H_4-H_3)}{\xi_2} + (1-\rho_5)H_R, & \text{if} \quad H_3 \leq mT - (1-\rho_5)H_R \leq H_1 \\ H_3 + (1-\rho_5)H_R \end{cases} \\ and \\ mT \leq \begin{cases} H_4 - \frac{tr_2(H_4-H_3)}{\xi_2} - \rho_6 H_L, & \text{if} \quad H_2 \leq mT + \rho_6 H_L \leq H_4 \\ \frac{\xi_2(H_2-H_1)+(1-\xi_2)H_2(H_4-H_3)-tr_2(H_4-H_3)(H_2-H_1)}{\xi_2(H_2-H_1)+(1-\xi_2)(H_4-H_3)} - \rho_6 H_L, & \text{if} \quad H_1 \leq mT + \rho_6 H_L \leq H_2 \\ H_4 + \frac{(1-\xi_2-tr_2)(H_4-H_3)}{\xi_2} + (1-\rho_6)H_R, & \text{if} \quad H_3 \leq mT + \rho_6 H_L \leq H_1 \\ H_3 - \rho_6 H_L \end{cases} \end{cases} \tag{3.71}$$

where $\tilde{\hat{H}} = (\hat{H} - H_L, \hat{H}, \hat{H} + H_R)$,$\hat{H} = ([H_1, H_2][H_3, H_4])$, $0 \leq H_3 \leq H_1 \leq H_2 \leq H_4$, is a fuzzy-rough variable and $\xi_2 \in (0, 1)$ and $\rho_5, \rho_6 \in [0, 1]$, $tr_2 \in [0, 1]$ are the possibility and trust confidence levels respectively.

### 3.3.5 Optimization Problem

Therefore, the problem for the imperfect inventory model is finally reduced to the minimization of the expected total cost given by (3.63) subject to the Chance constraint

(3.65) and constraints (3.66)-(3.71) for different time horizons. Thus the problem is

$$\begin{aligned} &\text{Min } TC(P, t_1, r, m) \\ \text{s.t.} \quad & t_1 \geq \frac{1}{f(P)}[1+\Phi(r)]+\epsilon. \\ & \text{and constraints (3.66)-(3.71) for different time horizons.} \end{aligned} \tag{3.72}$$

### 3.3.6 Particular cases

**Model-3.2A: ( Model-3.2 with Constant Demand)**

Letting $d_1 \to 0$ in the above Model-3.2, we have the following reduced necessary expressions:

$$\begin{cases} u = 1-r, \quad Q_3 = N_3, \quad S_q = (P-d_0)t_1 - (1-\theta)E[N], \quad T = t_1 + \frac{S_q}{d_0}, \\ E[Q_h] = \frac{P-d_0}{2}t_1^2 - (1-\theta)\alpha P^{\beta+1} f(P)\Bigg[\sum_{j=0}^{\gamma-1} \frac{(-1)^j \gamma!}{(\gamma-j)!(1-r)^{j+1}}\Bigg\{\sum_{k=0}^{\gamma-j} \frac{(-1)^k(\gamma-j)! e^{(1-r)t_1}}{(\gamma-j-k)!(1-r)^{k+1}} \\ \Bigg(\sum_{s=0}^{\gamma-j-k} \frac{(-1)^s(\gamma-j-k)! t_1^{\gamma-j-k-s}}{(\gamma-j-k-s)! f^{s+1}(P)} + \frac{(-1)^{\gamma-j-k+1}(\gamma-j-k)!}{f^{\gamma-j-k+1}(P)} e^{-f(P)t_1}\Bigg) - \frac{(-1)^{\gamma-j}(\gamma-j)!}{(1-r)^{\gamma-j+1}} N_3\Bigg\} \\ + \frac{(-1)^{\gamma}\gamma!}{(1-r)^{\gamma+1}}\Bigg(\frac{e^{(1-r)t_1}}{1-r}N_2 - \frac{N_3}{1-r} - t_1 N_3\Bigg)\Bigg] + \frac{d_0}{2}(T-t_1)^2. \end{cases} \tag{3.73}$$

Therefore, the problem for the imperfect inventory model with constant demand is finally reduced to the minimization of expected total cost given by (3.63) with (3.73) subject to the Chance constraint (3.65) and constraints (3.66)-(3.71) for different time horizons.

**Model-3.2B: ( Model-3.2 with Single cycle i.e. Infinite time horizon)**

For m=1, the equation (3.63) reduces to

$$\begin{aligned} TC(P,t_1,r) &= Ch.E[Q_h] + \theta.Cr.E(N) + (1-\theta).Cd.E(N) + C(P,r)Pt_1 \\ &+ [C_{s0} + C_{s1}e^{-k_1}] \quad \text{and the expected average total cost is} \\ ATC(P,t_1,r) &= \frac{TC(P,t_1,r)}{T}, \ \text{where T is given by equation (3.57)} \end{aligned} \tag{3.74}$$

Therefore, the production-inventory model for infinite time horizon is finally reduced to the minimization of expected average total cost given by (3.74) subject to the only one chance constraint (3.65). i.e.,

$$\begin{aligned} &\text{Min } ATC(P,t_1,r)\Bigg(=\frac{TC(P,t_1,r)}{T}\Bigg) \\ \text{s.t.} \quad & t_1 \geq \frac{1}{f(P)}[1+z]+\epsilon \end{aligned} \tag{3.75}$$

### 3.3.7 Solution Methodology

The above non-linear optimization problems are solved by a gradient based non-linear optimization method- GRG method (cf. Lasdon *et al.* [136]) using LINGO Solver 11.0 for particular sets of data.

### 3.3.8 Numerical Experiments and Results

**Input Data:** We consider the proposed EPL models (Models-3.2, 3.2A and 3.2B) with following inputs parameters in appropriate units:
$\alpha = 0.25$, $\beta = 0.25$, $\theta = 0.50$, $\gamma = 2$, $z = 0.40$, $\epsilon = 0.10$, $d_0 = 20$, $d_1 = 0.10$, $Ch = 3.0$, $C_{s0} = 200$,$C_{s1} = 150$, $C_{m0} = 100$, $k_1 = 0.80$, $k_2 = 0.80$, $Cr = 5.0$, $Cd = 2.0$ and unit production cost as: $C(P, r) = 20 + \frac{g}{P} + 0.10P + 0.20P^{1/2}$, where $g = 20 + 40e^{(1-0.75)\frac{r-0.10}{0.90-r}}$
For each model, two experiments depending on production quality are performed and the corresponding inputs are presented in Table 3.9. The input parameters for different time horizons are presented in Table 3.10.

**Table** 3.9: Input data of f(P) for two experiments

| **Experiment** | Production dependent quality | f(P) |
|---|---|---|
| 1 | Linear | 1.25+0.05P |
| 2 | Non-linear | $1.25 + 0.0005P^2$ |

**Optimum Results:** With the above parameters and expressions, the Models-3.2 ,3.2A and 3.2B are formulated and optimized using LINGO 11.0 software. The corresponding optimum values of production rate($P^*$), production run time($t_1^*$), number of cycles ($m^*$), reliability ($r^*$) for minimum total cost($TC^*$), holding($CH^*$), rework($CR^*$), disposal($CD^*$), production ($C_p^*$), set-up($C_s^*$) and maintenance costs($C_m^*$) for the total time horizon and the instant of defective production($m_d^*$), total expected defective units($E(N^*)$) and inventory level of good units($S_q^*$) for each production cycle are evaluated for different cases and presented in Tables 3.11-3.16

### 3.3.9 Discussion

**From Tables 3.11, 3.12, 3.13, 3.14, 3.15 and 3.16**

• Table 3.11 represents the optimum results of Exp.-1 (i.e. Experiment-1, when quality is linearly production dependent) for the Model-3.2 with crisp finite time horizon. In this case, minimum total cost is 4001 units for 3 cycles in imperfect production model with $50\%$ rework. This is because, with the increasing of cycle numbers, total holding, rework,

**Table** 3.10: Input values for different time horizons

| Time Horizon | Crisp | Random | Fuzzy (Pos&Nes Sense) | Fuzzy-Random (Pos&Nes Sense) | Rough | Fuzzy-Rough (Pos&Nes Sense) |
|---|---|---|---|---|---|---|
| Related Inputs | H=5.0 | $m_h = 5.0$<br>$\sigma_h = 0.25$<br>s=0.70 | $H_1 = 4.5$<br>$H_2 = 5.0$<br>$H_3 = 5.5$<br>$\rho_1 = 0.80$<br>$\rho_2 = 0.30$ | $m_h = 5.0$,$\sigma_h = 0.25$<br>$H_l = 0.50$,$H_r = 0.50$<br>$\rho_3 = 0.80$,$\rho_4 = 0.30$<br>$s_1 = s_2 = 0.70$<br>$L(x) = R(x) = 1 - x$ | $H_1 = 4.9$,$H_2 = 5.0$<br>$H_3 = 4.8$,$H_4 = 5.1$<br>$\xi_1 = 0.50$<br>$tr_1 = 0.80$ | $H_1 = 4.9$,$H_2 = 5.0$<br>$H_3 = 4.8$,$H_4 = 5.1$<br>$\xi_2 = 0.50$,$tr_2 = 0.80$<br>$H_L = 0.20$,$H_R = 0.50$<br>$\rho_5 = 0.80$,$\rho_6 = 0.30$ |

**Table** 3.11: Model-3.2(Stock-dependent demand with crisp time horizon), f(P)=a+bP

| | $\theta$ | m | $TC^*$ | $CH^*$ | $CR^*$ | $CD^*$ | $C_p^*$ | $C_s^*$ | $C_m^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ | $S_q^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Crisp Time Hori -zon | 0.50 | 1 | 6415 | 717.2 | 86.2 | 34.5 | 5309 | 267.4 | 00.0 | 142.6 | 0.98 | 0.75 | 0.12 | 5.00 | 34.5 | 98.7 |
| | | 2 | 4289 | 253.7 | 46.2 | 18.5 | 3418 | 497.7 | 55.0 | 60.7 | 0.97 | 0.63 | 0.23 | 2.50 | 9.2 | 33.0 |
| | | 3 | 4001 | 123.0 | 30.9 | 12.4 | 2989 | 711.0 | 134.8 | 40.3 | 0.91 | 0.51 | 0.30 | 1.67 | 4.1 | 15.7 |
| | | 4 | 4054 | 65.8 | 22.5 | 8.9 | 2814 | 917.4 | 225.8 | 31.5 | 0.84 | 0.38 | 0.35 | 1.25 | 2.2 | 8.2 |
| | | 5 | 4226 | 36.2 | 16.9 | 6.8 | 2724 | 1120 | 321.7 | 26.8 | 0.78 | 0.25 | 0.38 | 1.00 | 1.4 | 4.5 |
| | 1.00 | $3(m^*)$ | 3714 | 43.4 | 165.2 | 00.0 | 2659 | 711.3 | 134.8 | 24.4 | 1.38 | 0.43 | 0.40 | 1.67 | 11.0 | 5.7 |
| | 0.00 | $4(m^*)$ | 4164 | 86.5 | 00.0 | 12.2 | 2922 | 917.4 | 225.8 | 38.0 | 0.72 | 0.43 | 0.31 | 1.25 | 1.5 | 10.9 |

disposal and production costs decrease but total set up and maintenance costs increase. Up to 3 cycles, total cost decreases as total decrease in costs for holding, rework, etc dominates over the increase in set-up and maintenance costs but when the total no. of cycles is 4, set-up and maintenance costs dominate over the others, hence total cost increases. For fully rework and no rework 3714 and 4164 units are respective minimum total costs for 3 and 4 number of cycles in crisp finite time horizon models.

• Table 3.12 represents the optimum results of Exps.-1 and 2 (i.e. when quality is linearly and non-linearly production dependent)for the Model-3.2 with crisp, fuzzy, fuzzy-random, rough and fuzzy-rough finite time horizons. Here, for same type of time horizon as well as rework, the minimum total cost of Exp.-1 is more than that of Exp.-2. For example, minimum total cost $TC^* = 3522$ units of Exp.-1 is greater than corresponding minimum total cost $TC^* = 3466$ units of Exp.-2 for Model-3.2 with fully rework and fuzzy-rough time horizon for which impreciseness is measured in possibility sense.

• Table 3.13 represents the optimum results of Exp.-1 (i.e. when quality is linearly production dependent)for the Model-3.2A with crisp finite time horizon. In this case, minimum total cost is 3852 units for 3 cycles imperfect production model with $50\%$ rework. In this case, the behaviours of the different costs are the same as in Model-3.2 i.e. total cost initially decreases with cycle numbers and then increases when total cycle no. is 4. This is because, with the increasing of cycle numbers, total holding, rework, disposal and production costs decrease but total set up and maintenance costs increase. For fully rework and no rework, 3584 and 3979 units are the respective minimum total costs for 2 and 3 number of cycles in crisp finite time horizon models.

• Table 3.14 represents the optimum results of Exps.-1 and 2 (i.e. when quality is linearly and non-linearly production dependent)for the Model-3.2A with crisp, fuzzy, fuzzy-random, rough and fuzzy-rough finite time horizons. Here, for same type of time horizon as well as rework, the minimum total cost of Exp.-1 is greater than the minimum total cost of Exp.-2. This behaviour is the same as Model-3.2.

**Table** 3.12: Model-3.2(Stock-dependent demand with different uncertain time horizon)

| Experiment | | Exp-1: Linearly Production dependent quality | | | | | | | | Exp-2:Non-linearly Production dependent quality | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Horizon | $\theta$ | $m^*$ | $TC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ | $m^*$ | $TC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ |
| Crisp | 0.00 | | | | | | | | | 3 | 4082 | 44.5 | 0.84 | 0.53 | 0.44 | 1.67 | 2.6 |
| | 0.50 | | | | | | | | | 3 | 3926 | 36.3 | 1.00 | 0.50 | 0.52 | 1.67 | 3.8 |
| | 1.00 | | | | | | | | | 3 | 3668 | 23.0 | 1.47 | 0.42 | 0.66 | 1.67 | 10.0 |
| Random | 0.00 | 3 | 4056 | 48.9 | 0.74 | 0.54 | 0.27 | 1.62 | 2.6 | 3 | 3971 | 43.4 | 0.84 | 0.53 | 0.45 | 1.62 | 2.5 |
| | 0.50 | 3 | 3897 | 39.4 | 0.90 | 0.50 | 0.31 | 1.62 | 3.9 | 3 | 3825 | 35.5 | 0.99 | 0.49 | 0.53 | 1.62 | 3.6 |
| | 1.00 | 3 | 3627 | 24.0 | 1.37 | 0.42 | 0.40 | 1.62 | 10.4 | 3 | 3586 | 22.6 | 1.44 | 0.40 | 0.66 | 1.62 | 9.3 |
| Fuzzy | 0.00 | 4 | 4242 | 38.7 | 0.72 | 0.44 | 0.31 | 1.28 | 1.6 | 3 | 4172 | 45.4 | 0.84 | 0.54 | 0.43 | 1.70 | 2.7 |
| Pos. | 0.50 | 3 | 4086 | 41.0 | 0.91 | 0.52 | 0.30 | 1.70 | 4.3 | 3 | 4008 | 36.9 | 1.00 | 0.50 | 0.51 | 1.70 | 4.0 |
| Sense | 1.00 | 3 | 3783 | 24.7 | 1.40 | 0.44 | 0.40 | 1.70 | 11.5 | 3 | 3736 | 23.3 | 1.48 | 0.42 | 0.65 | 1.70 | 10.5 |
| Fuzzy | 0.00 | 3 | 4033 | 48.6 | 0.74 | 0.53 | 0.27 | 1.62 | 2.5 | 3 | 3949 | 43.2 | 0.84 | 0.52 | 0.45 | 1.62 | 2.4 |
| Nes. | 0.50 | 3 | 3876 | 39.2 | 0.90 | 0.50 | 0.31 | 1.62 | 3.8 | 3 | 3805 | 35.3 | 0.99 | 0.48 | 0.53 | 1.62 | 3.5 |
| Sense | 1.00 | 3 | 3610 | 23.9 | 1.37 | 0.42 | 0.40 | 1.62 | 10.2 | 3 | 3569 | 22.6 | 1.44 | 0.40 | 0.66 | 1.62 | 9.2 |
| Fuzzy | 0.00 | 4 | 4144 | 37.9 | 0.72 | 0.43 | 0.32 | 1.24 | 1.5 | 3 | 4059 | 44.3 | 0.84 | 0.53 | 0.44 | 1.66 | 2.6 |
| Random | 0.50 | 3 | 3980 | 40.1 | 0.91 | 0.51 | 0.30 | 1.66 | 4.0 | 3 | 3906 | 36.2 | 1.00 | 0.49 | 0.52 | 1.66 | 3.8 |
| Pos.sens | 1.00 | 3 | 3696 | 24.3 | 1.38 | 0.43 | 0.40 | 1.66 | 10.9 | 3 | 3653 | 22.9 | 1.46 | 0.41 | 0.66 | 1.66 | 9.8 |
| Fuzzy | 0.00 | 3 | 3746 | 45.4 | 0.74 | 0.51 | 0.28 | 1.51 | 2.2 | 2 | 3939 | 63.1 | 0.85 | 0.63 | 0.30 | 2.26 | 5.1 |
| Random | 0.50 | 3 | 3614 | 36.9 | 0.89 | 0.48 | 0.32 | 1.51 | 3.4 | 4 | 3690 | 23.5 | 1.01 | 0.37 | 0.65 | 1.13 | 2.2 |
| Nes.sens | 1.00 | 2 | 3364 | 29.8 | 1.57 | 0.54 | 0.36 | 2.26 | 21.9 | 2 | 3309 | 27.7 | 1.68 | 0.53 | 0.61 | 2.26 | 20.2 |
| Rough | 0.00 | 3 | 3828 | 46.3 | 0.74 | 0.52 | 0.28 | 1.54 | 2.3 | 3 | 3752 | 41.4 | 0.83 | 0.51 | 0.47 | 1.54 | 2.2 |
| | 0.50 | 3 | 3689 | 37.5 | 0.89 | 0.48 | 0.32 | 1.54 | 3.3 | 4 | 3750 | 24.2 | 1.00 | 0.37 | 0.64 | 1.15 | 2.2 |
| | 1.00 | 2 | 3441 | 30.2 | 1.58 | 0.55 | 0.36 | 2.31 | 23.0 | 2 | 3384 | 28.1 | 1.69 | 0.54 | 0.61 | 2.31 | 21.2 |
| Fuzzy | 0.00 | 3 | 3916 | 47.3 | 0.74 | 0.53 | 0.28 | 1.57 | 2.4 | 3 | 3836 | 42.0 | 0.83 | 0.51 | 0.46 | 1.57 | 2.3 |
| Rough | 0.50 | 3 | 3770 | 38.3 | 0.90 | 0.49 | 0.31 | 1.57 | 3.6 | 3 | 3702 | 34.5 | 0.98 | 0.47 | 0.54 | 1.57 | 3.3 |
| Pos.sens | 1.00 | 3 | 3522 | 23.6 | 1.35 | 0.41 | 0.41 | 1.57 | 9.6 | 2 | 3466 | 28.6 | 1.71 | 0.55 | 0.60 | 2.36 | 22.3 |
| Fuzzy | 0.00 | 3 | 3776 | 45.7 | 0.74 | 0.51 | 0.28 | 1.52 | 2.2 | 3 | 3699 | 40.6 | 0.82 | 0.50 | 0.48 | 1.52 | 2.1 |
| Rough | 0.50 | 3 | 3641 | 37.1 | 0.89 | 0.47 | 0.32 | 1.52 | 3.4 | 3 | 3578 | 33.7 | 0.98 | 0.46 | 0.54 | 1.52 | 3.2 |
| Nes.sens | 1.00 | 2 | 3392 | 29.9 | 1.58 | 0.54 | 0.36 | 2.28 | 22.3 | 2 | 3336 | 27.9 | 1.68 | 0.53 | 0.61 | 2.28 | 20.6 |

**Table** 3.13: Model-3.2A(Constant demand with crisp time horizon), f(P)=a+bP

| | $\theta$ | m | $TC^*$ | $CH^*$ | $CR^*$ | $CD^*$ | $C_p^*$ | $C_s^*$ | $C_m^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ | $S_q^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 4843 | 617.4 | 55.7 | 22.3 | 3880 | 267.4 | 00.0 | 117.8 | 0.94 | 0.72 | 0.14 | 5.00 | 22.3 | 81.1 |
| Crisp | | 2 | 3929 | 245.9 | 34.3 | 13.7 | 3082 | 497.7 | 55.0 | 59.6 | 0.89 | 0.59 | 0.23 | 2.50 | 6.8 | 32.0 |
| Time | 0.50 | 3 | 3852 | 128.8 | 23.8 | 9.5 | 2844 | 711.3 | 134.8 | 41.9 | 0.83 | 0.46 | 0.30 | 1.67 | 3.2 | 16.7 |
| Hori | | 4 | 3979 | 75.2 | 17.4 | 6.9 | 2736 | 917.4 | 225.8 | 33.8 | 0.76 | 0.32 | 0.34 | 1.25 | 1.7 | 9.7 |
| -zon | | 5 | 4185 | 46.5 | 13.2 | 5.3 | 2678 | 1120 | 321.7 | 29.2 | 0.70 | 0.17 | 0.36 | 1.00 | 1.0 | 5.9 |
| | 1.00 | 2($m^*$) | 3584 | 170.5 | 175.6 | 00.0 | 2685 | 497.7 | 55.0 | 36.7 | 1.36 | 0.51 | 0.32 | 2.50 | 17.6 | 22.7 |
| | 0.00 | 3($m^*$) | 3979 | 148.6 | 00.0 | 13.0 | 2970 | 711.3 | 134.8 | 50.5 | 0.70 | 0.50 | 0.26 | 1.67 | 2.2 | 19.3 |

• Table 3.15 represents the optimum results of Exps.-1 and 2 for the Model-3.2B i.e. imperfect production model with stock-dependent demand and infinite crisp time horizon. In this case, for a same type of rework (say $\theta = 0.50$), the minimum average total cost $ATC^* = 815$ units of Exp.-1 is greater than the minimum average total cost $ATC^* = 801$ units of Exp.-2.

• It is to be noted from the Tables 3.11 to 3.15 that the mean-time ($m_d^*$) of the commencement of "out-of-control" state is less than the production run time for all Models in all experiments. There is a necessary condition for the models imposed by chance constraint as $md^* \leq t_1^*$ (§ 3.3.3).

• For all Experiments-1 and 2, the optimal expected total costs of models Model-3.2 and Model-3.2A with no rework from above tables are more than those of the models corresponding with fully rework. The same behaviour is concluded for all these types experiments for Model-3.2B.

• From Tables 3.11 to 3.14, for the Model-3.2 with stock-dependent demand, the optimal average expected total cost is greater than that of Model-3.2A which is with constant

Table 3.14: Model-3.2A(Constant demand with different uncertain time horizon)

| Experiment | | Exp-1: Linearly Production dependent quality | | | | | | | | Exp-2: Non-linearly Production dependent quality | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Horizon | $\theta$ | $m^*$ | $TC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ | $m^*$ | $TC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $m_d^*$ | $T^*$ | $E(N^*)$ |
| Crisp | 0.00 | | | | | | | | | 3 | 3911 | 45.4 | 0.78 | 0.49 | 0.43 | 1.67 | 2.0 |
| | 0.50 | | | | | | | | | 3 | 3796 | 38.3 | 0.91 | 0.44 | 0.50 | 1.67 | 2.9 |
| | 1.00 | | | | | | | | | 2 | 3540 | 34.4 | 1.45 | 0.50 | 0.54 | 2.50 | 16.2 |
| Random | 0.00 | 3 | 3879 | 49.3 | 0.70 | 0.50 | 0.26 | 1.62 | 2.0 | 3 | 3813 | 44.7 | 0.78 | 0.49 | 0.44 | 1.62 | 1.9 |
| | 0.50 | 3 | 3760 | 41.1 | 0.82 | 0.45 | 0.30 | 1.62 | 3.0 | 2 | 3754 | 53.0 | 0.98 | 0.59 | 0.38 | 2.44 | 6.5 |
| | 1.00 | 2 | 3491 | 36.0 | 1.35 | 0.50 | 0.32 | 2.44 | 16.6 | 2 | 3449 | 33.8 | 1.44 | 0.49 | 0.54 | 2.44 | 15.3 |
| Fuzzy | 0.00 | 3 | 4059 | 51.3 | 0.70 | 0.51 | 0.26 | 1.70 | 2.2 | 3 | 3989 | 46.2 | 0.78 | 0.50 | 0.43 | 1.70 | 2.1 |
| Pos. | 0.50 | 3 | 3927 | 42.7 | 0.83 | 0.47 | 0.29 | 1.70 | 3.3 | 3 | 3869 | 38.9 | 0.91 | 0.45 | 0.49 | 1.70 | 3.0 |
| Sense | 1.00 | 2 | 3659 | 37.2 | 1.37 | 0.52 | 0.32 | 2.55 | 18.3 | 2 | 3614 | 34.8 | 1.46 | 0.50 | 0.53 | 2.55 | 16.9 |
| Fuzzy | 0.00 | 3 | 3859 | 49.1 | 0.69 | 0.49 | 0.26 | 1.61 | 2.0 | 2 | 3937 | 65.8 | 0.80 | 0.62 | 0.29 | 2.42 | 4.5 |
| Nes. | 0.50 | 3 | 3742 | 40.9 | 0.82 | 0.45 | 0.30 | 1.61 | 2.9 | 2 | 3732 | 52.8 | 0.98 | 0.58 | 0.38 | 2.42 | 6.4 |
| Sense | 1.00 | 2 | 3472 | 35.9 | 1.34 | 0.50 | 0.32 | 2.42 | 16.5 | 2 | 3431 | 33.7 | 1.43 | 0.49 | 0.54 | 2.42 | 15.1 |
| Fuzzy | 0.00 | 3 | 3959 | 50.3 | 0.70 | 0.50 | 0.26 | 1.67 | 2.1 | 3 | 3891 | 45.2 | 0.78 | 0.49 | 0.44 | 1.67 | 2.0 |
| Random | 0.50 | 3 | 3834 | 41.8 | 0.83 | 0.46 | 0.29 | 1.67 | 3.1 | 3 | 3778 | 38.1 | 0.91 | 0.44 | 0.50 | 1.67 | 2.8 |
| Pos.sens | 1.00 | 2 | 3565 | 36.6 | 1.36 | 0.51 | 0.32 | 2.49 | 17.4 | 2 | 3521 | 34.2 | 1.45 | 0.50 | 0.54 | 2.49 | 16.0 |
| Fuzzy | 0.00 | 3 | 3606 | 46.2 | 0.69 | 0.46 | 0.28 | 1.51 | 1.8 | 3 | 3549 | 41.7 | 0.76 | 0.45 | 0.47 | 1.51 | 1.6 |
| Random | 0.50 | 3 | 3507 | 38.8 | 0.81 | 0.42 | 0.31 | 1.51 | 2.6 | 2 | 3452 | 49.5 | 0.97 | 0.56 | 0.40 | 2.26 | 5.5 |
| Nes.sens | 1.00 | 2 | 3236 | 34.4 | 1.31 | 0.48 | 0.33 | 2.26 | 14.3 | 2 | 3199 | 32.3 | 1.40 | 0.46 | 0.56 | 2.26 | 13.0 |
| Rough | 0.00 | 3 | 3680 | 47.0 | 0.69 | 0.48 | 0.28 | 1.54 | 1.9 | 3 | 3620 | 42.4 | 0.77 | 0.46 | 0.46 | 1.54 | 1.7 |
| | 0.50 | 3 | 3575 | 39.5 | 0.81 | 0.43 | 0.31 | 1.54 | 2.7 | 2 | 3533 | 50.4 | 0.97 | 0.56 | 0.39 | 2.31 | 5.8 |
| | 1.00 | 2 | 3305 | 34.8 | 1.32 | 0.49 | 0.33 | 2.31 | 14.9 | 2 | 3267 | 32.7 | 1.41 | 0.47 | 0.56 | 2.31 | 13.0 |
| Fuzzy | 0.00 | 3 | 3757 | 47.9 | 0.69 | 0.48 | 0.27 | 1.57 | 1.9 | 2 | 3809 | 63.9 | 0.80 | 0.61 | 0.30 | 2.36 | 4.2 |
| Rough | 0.50 | 3 | 3647 | 40.1 | 0.81 | 0.44 | 0.31 | 1.57 | 2.8 | 2 | 3619 | 51.4 | 0.98 | 0.58 | 0.39 | 2.36 | 6.0 |
| Pos.sens | 1.00 | 2 | 3378 | 35.3 | 1.33 | 0.49 | 0.33 | 2.36 | 15.6 | 2 | 3338 | 33.0 | 1.42 | 0.48 | 0.56 | 2.36 | 14.1 |
| Fuzzy | 0.00 | 3 | 3557 | 45.7 | 0.69 | 0.46 | 0.28 | 1.49 | 1.7 | 3 | 3500 | 40.4 | 0.78 | 0.45 | 0.48 | 1.49 | 1.6 |
| Rough | 0.50 | 2 | 3455 | 53.8 | 0.88 | 0.56 | 0.25 | 2.23 | 5.6 | 2 | 3397 | 48.8 | 0.97 | 0.56 | 0.41 | 2.23 | 5.4 |
| Nes.sens | 1.00 | 2 | 3190 | 34.0 | 1.30 | 0.48 | 0.33 | 2.23 | 13.8 | 2 | 3154 | 31.9 | 1.39 | 0.45 | 0.56 | 2.23 | 12.4 |

Table 3.15: Model-3.2B(Infinite time horizon)

| Rework | f(P)=a+bP | | | | | | | | $f(P) = a + bP^2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | $ATC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $T^*$ | $m_d^*$ | $E(N^*)$ | $S_q^*$ | $ATC^*$ | $P^*$ | $t_1^*$ | $r^*$ | $T^*$ | $m_d^*$ | $E(N^*)$ | $S_q^*$ |
| 0.00 | 815 | 41.5 | 0.73 | 0.48 | 1.37 | 0.30 | 1.8 | 13.3 | 801 | 38.0 | 0.82 | 0.47 | 1.42 | 0.50 | 1.8 | 12.4 |
| 0.50 | 789 | 36.7 | 0.88 | 0.47 | 1.49 | 0.32 | 3.3 | 12.6 | 755 | 34.0 | 0.98 | 0.47 | 1.54 | 0.54 | 3.2 | 11.6 |
| 1.00 | 732 | 26.0 | 1.45 | 0.48 | 1.85 | 0.39 | 14.0 | 8.2 | 722 | 24.8 | 1.56 | 0.47 | 1.90 | 0.64 | 13.7 | 7.0 |

demand. It is observed for all Exps.-1 and 2. Interesting result is that in spite of higher demand (stock-dependent) in the market, production rate is lower to minimize the average cost. Here lower production rates (for Model-3.2 with stock-dependent demand ) increase the average production costs.

• In Tables 3.11 to 3.14, values of $T^*$ are given. As $H = mT^*$ and $H$ is known, $T^*$ changes with $m$ i.e. number of cycle.

• Now from the Table 3.16, it is seen that the production rate and reliability of machinery system which minimizes the UPC is quite different from the production rate which minimizes the expected total cost $TC$ for Exp.-1 of same model with different types of finite time horizon. It is interesting to note that the production rate P and reliability r which minimize $TC$ are higher than those of which minimize unit production cost C(P,r). For example, for Model-3.2 with crisp time horizon due to Exp-1, $C(P, r)$ has the minimum value for $C(P^*, r^*) = 26.88$ units at $P^* = 40.33$ units, $r^* = 0.31$, but the corresponding UPC $C_1(P, r)$ attains minimum value (26.05 units) at $P^* = 28.17$ units and $r^* = 0.18$.

**Table** 3.16: Optimal results of UPC, $C(P, r)$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Result for minimizing $C(P, r)$ : $C(P, r) = 26.05$, P=28.17, r=0.18 | | | | | | | | | | |
| Result for minimizing $TC(P, r, t_1, m)$ on different time horizon for Exp.-1 | | | | | | | | | | |
| Model | Para- | Crisp | Random | Fuzzy | | Fuzzy-Random | | Rough | Fuzzy-Rough | |
| Name | meters | | | Pos | Nes | Pos | Nes | | Pos | Nes |
| Model-3.2 | $C(P, r)$ | 26.88 | 27.01 | 27.16 | 26.99 | 27.07 | 26.80 | 26.85 | 26.91 | 26.80 |
| | $P$ | 40.33 | 39.41 | 41.08 | 39.22 | 40.15 | 36.87 | 37.55 | 38.27 | 37.12 |
| | $r$ | 0.31 | 0.50 | 0.52 | 0.50 | 0.51 | 0.48 | 0.48 | 0.49 | 0.47 |
| Model-3.2A | $C(P, r)$ | 27.14 | 27.07 | 27.20 | 27.05 | 27.13 | 26.86 | 26.92 | 26.98 | 28.26 |
| | $P$ | 41.99 | 41.15 | 42.67 | 40.99 | 41.83 | 38.82 | 39.48 | 40.12 | 53.76 |
| | $r$ | 0.46 | 0.45 | 0.47 | 0.45 | 0.46 | 0.42 | 0.43 | 0.44 | 0.56 |

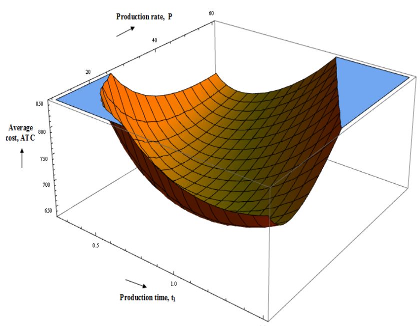


**Figure** 3.11: Average total cost versus production time and production rate when reliability is constant.

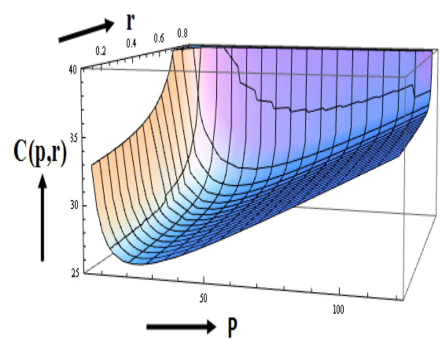



**Figure** 3.12: UPC versus production rate and reliability.

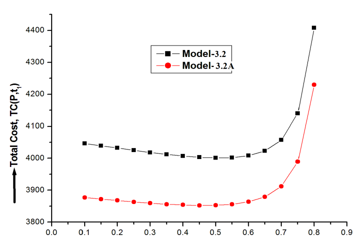



**Figure** 3.13: Total cost $TC(P, t_1, r, 3)$ vs reliability r with variable P, $t_1$.

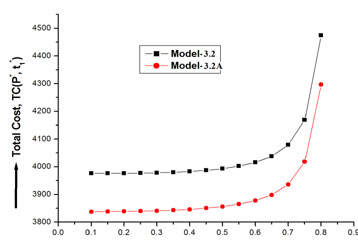



**Figure** 3.14: Total cost $TC(P^*, t_1^*, r, 3)$ vs reliability r.

**From Figs. 3.11, 3.12, 3.13 and 3.14:**

• Considering the optimal value of reliability r as constant, the average total costs for the Model-3.2B due to Exp.-1 are plotted in Fig. 3.11 against the different values of P and $t_1$. This figure shows that the objective function is convex.

• Fig. 3.12 is obtained by plotting the UPC against the different values of production rate and reliability of machinery system. This UPC is a convex function against production rate only.

• Fig. 3.13 represents total cost against the machinery-system reliability for Exp.-1 of Model-3.2 with crisp time horizon when m=3 and P, $t_1$ are treated as variable. In this figure, total cost is a convex function with respect to reliability.

• As Fig. 3.13, Fig.3.14 also represents the total cost against the machinery system reliability for Exp.-1 of Model-3.2 with crisp time horizon when m=3 and P, $t_1$ are the optimal values obtained from Table 3.11. In this figure, total cost increases with reliability of machinery system.

## 3.4 Conclusion

The present investigation portrays an EPL model with random imperfect production with reliability dependent defective rate, stock dependent demand rate and rework (may be partially) of the imperfect products over different imprecise finite time horizons. During the production, defective units are produced from "out-of-control" state. The probability distribution of the beginning of "out-of-control" state follows an exponential distribution with mean and standard deviation $\frac{1}{f(P)}$. Here $f(P) = a,\, a + bP \, or \, a + bP^2$ where $a > 0$ and $b \geq 0$. It may be extended to other types of increasing function. Several sub-cases of it are considered and compared.

Uniqueness of the first model deals with UPC taken in the form: $C(P) = r_m + \frac{g}{P^{\delta_1}} + \eta_1 P^{\delta_2} + \eta_2 P^{\delta_3}$ where,$\delta_1, \delta_2, \delta_3 > 0$. Thus, it investigates the effect of non-instantaneous imperfect production introducing a chance constraint that determines a cost against carbon emission/ production in UPC. The quality of an item always directly varies with the rate of production. Hence different types of variations in quality of produced item are considered and the corresponding costs are evaluated. Finally, the developed model presents different earlier models investigated by different authors as particular cases. Also, the limitation of present study is that though cost against carbon emission/ production has been taken into account, it would have better if the total amount of carbon emission/ production with respect to production rate was derived and incorporated into the analysis. This may be a guide line for future research works.

Meanwhile, Model 3.2 inculcates with the calculation of percentage of defective units produced at time t, $\lambda(t', \tau, P, r) = \alpha P^{\beta} e^{(1-r)t'}(t' - \tau)^{\gamma}$, $\gamma$ is an integer for the convenience of calculation. However it can be any positive value and in that case the integrations connecting $\gamma$ are to be evaluated numerically.

Henceforth, for a EPL model, the necessity of imposition of an "out-of-control" state constraint is laid down and the above chance constraint can be used for other types of imperfect production-inventory models such as inventory models with trade credit, two warehouses inventory system, EPL model with price discount, etc.

# Chapter 4

# Inventory Problems on Complementary and Substitute Products in Random Environment

## 4.1 Introduction

In the last few decades, both academics and practitioners have shown that effective inventory management for a company is one of the key factors for success in challenging business environment. There are several procedures, which can be considered to reflect variations in the structure of the inventory control system [96]. Generally, the operating doctrines or decision rules which guide a company's attempts to maximize profit while satisfying market demands depends upon sound inventory policies and the utilization of models [29]. In spite of the emphasis in quality control, a manufacturing process may be imperfect and results in defective items that are required reworking. Electronics items, glass goods, pharmaceuticals items, etc. are examples of such imperfect product. Most traditional approaches to the problem are described in the literature of the models on imperfect production process (cf. § 1.3.2) and defective manufacturing continues from the beginning of the process. Very few have considered the EPQ model with in control and out-of-control state.

Now-a-days, due to strong competitive market, retailers prefer to do the business/production of several items with the hope that due to dull market, if one item does not fetch profit, the other one will save the situation, provided initial capital for investment permits. Das *et al.* [63] formulated a multi-item inventory model with budgetary and floor-space constraints in a fuzzy environment. In Kar *et al.* [123], they developed an inventory model under the budgetary constraint for deteriorating multi-item with cost-price dependent demand in a fuzzy environment. In marketing of several items, the demand of an item is affected by the other in the case of complementary and substitute items. Demand of each complementary item increases to some extent by the other such items where as in the case of substitute items, the demand of such an item may increase or decrease depending upon the choice of the customers [164, 167].

Generally, there are two consequences when stock-outs happen: one is the use of backorder and other is the lost sales. Backorder means that when stock-outs occur, the unsatisfied customers are willing to wait and so their demands are met in the next replenishment epoch. Lost sales means the customers are impatient and may buy their desired items from other suppliers, and the unmet demand is therefore lost. In practice, it is not always true that all stock-outs are back-ordered or forgoes. But in reality, some of the unsatisfied customers may be willing to met their demands with purchasing substitute items. One item is a substitute for another only if it can be used in exactly the same way and serves the same need. Pasternack and Drezner [199] showed that the optimal inventory level for two items in full substitution could either increase or decrease as transfer revenue increases by inventory pooling. Gerchak and Mossman [86] also showed a possibility that the inventory level can be escalated by pooling under the exponentially distributed demands in full substitution. There have been several studies concerned with an inventory model that allows item substitution [73, 124, 282].

Several extensions to the newsboy model have been reported in the literature [12, 138, 139] for a single item and others are mentioned in the literature review of the models with complementary and substitutable products (cf. § 1.3.3). Early extension for multiple items assumed their independent demands [186]. Parlar and Goyal [198] studied a two product single period inventory model in which substitution occurs in probabilistic sense. More recently, single item newsboy problem is considered with random lot size [240]. Here for the first time, a reduction is given to the selling price of the item when it substitutes other.

Now-a-days managers recognize that effectively managing risks in their business operations imply the successfully managing of their inventories. One of the most common risks is demand uncertainty, a phenomenon that is widely studied in the literature. Single-Period Problem (SPP) under probabilistic demand is reflective of many real life situations and is often used to aid decision making in the fashion and sporting industries, both at the wholesaler and retailer levels [84]. If the order quantity is smaller than the realized demand, then shortages arise [129, 278] and if any inventory remains at the end of the period, a discount is used to sell out or to dispose off [185].

In the last few years, many researchers [163, 170, 268] focused on the inventory control system in which the demand rate is dependent on the displayed stock level. Wee [263] considered a model where the demand rate is a convex decreasing function of the selling price. Jaggi *et al.* [114] and Liang and Zhou [144] solved two warehouse inventory models for deteriorating items with price dependent demand. It is a fact that the demand of an item is influenced by the selling price of that item i.e. whenever the selling price of an item increases, the demand of that decreases and vice-verse. Maiti *et al.* [159] introduced the concept of advanced payment for determining the optimal ordering policy under stochastic lead-time and price dependent demand condition. Again time-to-time promotional policy/advertisement through the modern mass/electronic media (well known media such as TV, Radio, Newspaper, Magazine,etc.) helps to generate more demand in the market [163, 193]. Cárdenas-Barrón and Sana [33, 34] proposed multi-item models in a two-layer supply-chain with promotional effort sensitive demand and found the average profits under channel co-ordination including both collaborative and non-collaborative systems.

Promotional policy and its effects on sales were studied by Blattberg and Neslin [22]. Optimal design of a series of promotions (which might offer gifts, discounts, or special services) periodically mailed to potential customers was addressed in Nair and Tarasewich [187]. Krishnan *et al.* [135] obtained the promotional effort to optimize revenue and stated that promotional strategies include displays of products, free goods, price discounts and advertising. Tsao and Sheen [257] solved the retailer's promotion and replenishment decisions under retailer competition and promotional effort with the sales learning curve. Recently, Maihami and Karimi [156] investigated an appropriate pricing and replenishment policy model for a non-instantaneous deteriorating item with promotional effort and stochastic demand. Cárdenas-Barrón and Sana [34] studied an EOQ inventory model of multi-items in a two-layer supply chain where demand is sensitive to promotional effort. But none has considered the promotional effect on the uniform random demand and apply it a news-vendor problem with two substitute items.

Classical inventory models are usually developed over the infinite planning horizon. According to Chung and Kim [56], the assumption of infinite planning horizon is not realistic due to several reasons such as variation of inventory costs, change in product specifications and designs, technological development, etc. Moreover, for seasonal products like fruits, vegetables, warm garments, fashionable goods, etc., business period is not infinite, rather fluctuates with each season. Hence the planning horizon for these products varies over the years depending upon the environmental effects. Therefore, it is better to estimate this type of products with finite time horizon as in nature. Moon and Yun [181] and Guria *et al.* [94] developed an EOQ model in random planning horizon.

In a production system, better machinery and control systems, expert labours, etc. are required to have the quality of product. So, UPC varies directly with the product's quality. Moreover, in every manufacturing process, it is fact that environment is polluted by the emission of green house gases, specially $CO_2$ in the atmosphere and for that, now-a-days attention is paid not to pollute the environment taking some measures for it. This involves some expenditures and hence UPC increases with this process [13, 88, 246]. So far, these considerations are ignored by the researchers.

Table 4.1: Literature Review for Model-4.1

| Authors with year | Model type | Demand depend on | Complementary or Substitute items | Quality Depend on Production | Constraint | Set up cost |
|---|---|---|---|---|---|---|
| Wee [263], 1999 | EOQ | Selling price | No | No | No | Constant |
| Pal *et al.* [193], 2006 | EOQ | Stock, price & advertisement | No | No | No | Constant |
| Maiti *et al.* [159], 2009 | EOQ | Selling price | No | No | | Constant |
| Maity and Maiti [167], 2009 | EPQ | Stock | Complementary and Substitute | No | Space | Constant |
| Cárdenas-Barrón [32], 2009 | EPQ | Constant | No | No | No | Constant |
| Sana [227], 2010 | EPL | Constant | No | Yes | No | Constant |
| Mandal *et al.* [171], 2010 | EPL | Stock | No | No | Space | No |
| Roy *et al.* [220], 2010 | EPL | Time | No | No | Budget | Constant |
| Cárdenas-Barrón *et al.* [36] 2012 | EPL | No | No | Constant | | |
| Cárdenas-Barrón and Sana [33] 2014 | EPL | Sales teams' initiatives | No | No | No | Constant |
| Cárdenas-Barrón and Sana [34], 2015 | EOQ | Promotional effort | No | No | No | Constant |
| Present model 4.1 | EPl | Price and advertisement | Complementary and Substitute | Yes | Budget | Production dependent |

Due to complex nature of the objective functions, it is difficult to find the optimal strategy of the reduced problems using traditional optimization techniques. GAs are extensively used to face these types of situations during the last decades by several researchers [161, 173, 179, 222, 235, 255]. Here, a GA with rough age based criteria is used to reproduce a new chromosome at crossover level.

Summarizing the above mentioned literature, Table 4.1 presented the systematic chronological developments in the related areas of Model 4.1.

In the context of earlier investigations as follows from Table 4.2, the new features in Model 4.2 are:

- None has considered the production-marketing system for substitutable products under imperfect production process introducing learning effect in the set-up and maintenance costs.
- There is no production systems (inventory) management research and the pricing decisions with product substitution depending on the joint effect of price and quality or on the basis of either price or quality.
- UPC is normally assumed to be dependent on the raw material and labour costs. But, none has considered that quality improvement cost which is a function of quality of an item, is a part of UPC.

**Table** 4.2: Literature Review for Model-4.2

| Authors with year | Model type | Demand | Substitution for | Learning effect on | UPC | Time Horizon |
|---|---|---|---|---|---|---|
| Chand [41], 1989 | EOQ | Constant | No | Set up | No | Crisp and Infinite |
| Moon and Yun [181], 1993 | EOQ | Constant | No | No | No | Random and finite |
| Khouja and Mehrez [127], 1994 | EPL | Constant | No | No | Production dependent | Crisp and Infinite |
| Das and Maiti [60], 2007 | News-boy | Random | Shortage | No | No | Infinite |
| Pal *et al.* [196], 2009 | EPQ | Time and price dependent | No | Production and set-up cost | Dynamic | Fuzzy and finite |
| Roy *et al.* [221], 2009 | EPQ | Stock dependent | No | Production and set-up cost | No | Random and finite |
| Hu *et al.* [106], 2010 | EPQ | Constant | No | No | No | Crisp and Infinite |
| Zhao *et al.* [291], 2012 | Supply chain | Price dependent | Price | No | Constant | Crisp and Infinite |
| Guria *et al.* [94], 2013 | EOQ | Inflation and selling price dependent | No | No | No | Random and finite |
| Rad *et al.* [207], 2014 | Supply chain | Price dependent | Price | No | Constant | Crisp and Infinite |
| Present model 4.2 | EPL | Price and quality dependent | Price and quality | Set up and Maintenance | Production and quality dependent | Random and finite |

- Several authors [1, 50, 88, 118, 246, 287] have studied the environmental effect on the production inventory/inventory management systems, mainly considering the carbon emission or product greening improvement. But, none introduces the EPC for EPL models, which is again varies with the rate of production.

- In the literature, there is no model for substitutable products formulated over a random planning horizon.

Summarizing the above mentioned literature, the systematic chronological developments in the related areas of Model 4.3 are presented in Table 4.3.

Therefore, there is a strong motivation for further research in this area. Hence, in this chapter, we consider all the above lacunas and formulate three models. In the first model, we develop a randomly imperfect multi-item (complementary or substitute) production inventory model with production dependent set-up cost, advertisement /promotional cost and selling price dependent demand, partially reworked, disposal of defective units, chance-constraint for commencement of imperfect production and variable production cost including EPC. In real life EPL models, a production system remains in control at the beginning and after some time, it goes to out-of-control state. Thus, the occurrence of production of imperfect units is random after the lapse of certain time and is imposed here

Table 4.3: Literature Review for Model-4.3

| Authors with year present in | No of items & their type | Type of demand | Effect on demand | Constraints used on | Solved by |
|---|---|---|---|---|---|
| Parlar and Goyal [198], 1984 | Two, Substitute | Random | - | - | Numerical Optimization Scheme |
| Pasternack and Drezne [199], 1991 | Two, Substitute | Random | - | - | Analytic method |
| Gallego and Moon [84], 1993 | Single | Random | - | - | Analytic method |
| Khouja [128], 1996 | Two, Substitute | Random | - | - | Monte Carlo simulation |
| Mishra and Raghunathan [175], 2004 | Two, Substitute | Random | - | - | Classical Optimization |
| Das and Maiti [60], 2007 | Two, Substitute | Random | - | Storage Space | Genetic Algorithm |
| Huang *et al.* [108], 2011 | Multi, Substitute | Random | - | - | Iterative Algorithm |
| Liu *et al.* [151], 2013 | Two, Substitute | Random | - | - | Nash Game Theory |
| Ding [72], 2013 | Multi-product | Rough | - | Storage Space | Classic Integer Programming |
| Wang *et al.* [261], 2015 | Multi-item | Uncertain-Random | - | - | Analytic method |
| Present Model 4.3 | Two, Substitute | Uniform random | Promotional effort | Promotional & Purchasing cost | Rough Age based Genetic Algorithm (RAGA) |

through a chance constraint. The set-up cost, UPC and defective rate are production dependent and part of UPC is taken as EPC. The problem is formulated as a profit maximization problem and solved using GRG through LINGO11.0. Several special cases are derived. Numerical experiments are performed to illustrate the general and particular models. Some sensitivity analyses are presented against few model parameters.

In the second model, we formulate an imperfect substitutable multi-item production-inventory model with selling price and quality dependent demand, partially reworked, disposal of not reworkable defective units incorporating environmental protection cost over a finite time horizon. In real life EPL models, a production system remains in control at the beginning and after some time, it goes to out-of-control state and then defective units are produced. The UPC has four components- the raw material cost to produce an unit, labour cost per unit production and quality improvement and EPC. Here demands of the substitute products are defined as linear functions of the products' selling prices and qualities. The demand of a merchandise has downward slopping in its own price and increasing with respect to the competitor's price. It is reversed with respect to quality e.g. increases in its own quality and decreases for other's quality. There may be different relations amongst the coefficients of demand functions. The models are formulated as profit maximization problems in which number of cycles, selling prices, production rates and qualities are DVs. With the different relations in demand functions, it is solved by using FAGA. The models are illustrated with numerical examples and some results are presented graphically.

In the last model, a newsboy type inventory control problem is considered for two substitute items. The model is formulated for the uniform random demands of the items. Here, promotional effort is shared to increase the demand of the items. The objective of the problem is to find the optimal quantities and respective promotional effort to maximize the profit. For the solution, an imprecise GA is proposed with rough age based probability of crossover.

## 4.2 Model-4.1 : EPL models for complementary and substitute items under imperfect production process with promotional cost and selling price dependent demands [1]

### 4.2.1 Assumptions and Notations

*The following assumptions are used to develop the proposed model:*

(i) Production rate is finite and taken as a DV.

(ii) Lead time is zero.

(iii) No shortages are allowed.

(iv) The inventory system is developed for complementary or substitute items under a budget limitation and the demand rate is advertisement/promotional cost and selling price dependent.

(v) The time horizon is infinite and the production time taken as a DV.

(vi) The production process shifts from "In-control" state to "Out-of-control" state at a time, which is a random variable. Imperfect units are produced in this state.

(vii) Production of defective units commences at a random time after the commencement of production. Defective rate depends on production rate and time duration from the starting of defective units' production.

(viii) There are partially reworking for the defective units at a cost immediately when they are produced in 'out-of-control' state and the defective units which are not reworked, are disposed off by a cost.

(ix) UPC is production dependent and one part of it is EPC.

(x) Set up cost is considered as partly production dependent.

*The following notations are used for $i^{th}$ item to develop the proposed model:*

$P_i$ Production rate (tons/time unit) (DV).
$q_i(t)$ Inventory level at time t (tons).
$T_i$ Cycle period (years).
$t_{1i}$ Production run-time in one cycle (years) (DV).
$C_{si}$ Set up cost (\$/cycle), $C_{si} = C_{s0i} + C_{s1i}P_i^{\rho_i}$, where $C_{s0i}$, $C_{s1i}$ and $\rho_i$ >0.
$C_{hi}$ Holding cost (\$/unit/time unit).
$C_{di}$ Cost of disposal for an imperfect unit (\$).

[1]This model has been published in **OPSEARCH**, SPRINGER, Y. 2015

$C_{ri}$ Cost for rework of an imperfect unit (\$).

$C_{ai}$ Advertisement cost (\$/time unit).

$C_i(P_i)$ UPC (\$/unit quantity) which is considered as $C_i(P_i) = r_{mi} + \frac{g_i}{P_i^{\delta_{1i}}} + \eta_{1i}P_i^{\delta_{2i}} + \eta_{2i}P_i^{\delta_{3i}}$ where, $\delta_{1i}, \delta_{2i}, \delta_{3i} > 0$ and $r_{mi}$ is the raw material cost (\$) per unit quantity, $g_i$ is the total energy costs (\$) per unit time in a production system which is equally distributed over the unit quantity. So, $(\frac{g_i}{P_i^{\delta_{1i}}})$ decreases with increasing $P_i$. The third term $\eta_{1i}P_i^{\delta_{2i}}$ is the wear and tear cost (\$), proportional to the positive power of production rate $P_i$ and the fourth term $\eta_{2i}P_i^{\delta_{3i}}$ is EPC assuming that cost due to the measures taken for the environment protection is proportional to a positive power of production rate $P_i$, where the power term varies with the nature of production firms.

$\tau_i$ An exponential random variable that depends on $P_i$ and denotes the time (years) at which the process shifts to the "out-of-control" state. The distribution function of "out-of-control" state is $G_i(\tau_i) = 1 - e^{-f_i(P_i)\tau_i}$ such that $\int_0^\infty dG_i(\tau_i) = f_i(P_i)\int_0^\infty e^{-f_i(P_i)\tau_i}d\tau_i = 1$. The exponential distribution has often been used to describe the elapsed time to failure of many components of the machinery system.

$\frac{1}{f_i(P_i)}$ The mean and standard deviation of the random variable $\tau_i$ . Here, $f_i(P_i)$ is an increasing function of $P_i$ and the mean time of failure, $1/f_i(P_i)$ is a decreasing function of $P_i$.

$\lambda_i(t, \tau_i, P_i)$ Rate of defective units (tons/time unit) produced at time t when the machine is in the 'out-of-control' state. Here $\lambda_i(t, \tau_i, P_i)$ is defined as $\lambda_i(t, \tau_i, P_i) = \alpha_i P_i^{\beta_i}(t - \tau_i)^{\gamma_i}$. Where $\beta_i \geq 0$,$\gamma_i \geq 0$ and $t \geq \tau_i$. Generally speaking, the percentage of defective units increases with increase of production rate and production-run time. The formulation of the function $\lambda_i(t, \tau_i, P_i)$ shows that it is an increasing function of production rate and production-run time simultaneously.

$\theta_i$ Percentage of rework of defective units.

$N_i$ Defective units in a production cycle (tons)

$Q_i$ Expected lot size (tons) without defective units at the end of production period.

$S_{pi}$ Selling price (\$/unit quantity) of $i^{th}$ item which is a mark-up $(m_i)$ of UPC $C_i(P_i)$. i.e., $S_{pi} = m_iC_i(P_i)$.

$D_i$ The demand rate of independent item (tons/time unit), one part of it is constant and other is advertisement cost and selling price dependent. Generally, demand of an item is proportional to advertisement cost and inversely proportional to selling price. Thus we consider it as: $D_i = u_{1i} + u_{2i}\frac{(C_{ai})^{\mu_{1i}}}{(S_{pi})^{\mu_{2i}}}$, where $u_{1i} \geq 0$, $u_{2i} \geq 0$, $\mu_{1i} \geq 0$ and $0 \leq \mu_{2i} \leq 1$.

$F_i(D_1, .., D_n)$ Demand rate of $i^{th}$ item for complementary or substitute items (tons/time unit)

$ATF_i(P_i, t_{1i})$ Average expected total profit for $i^{th}$ item (\$/time unit).

$B$ Available maximum budget (\$).

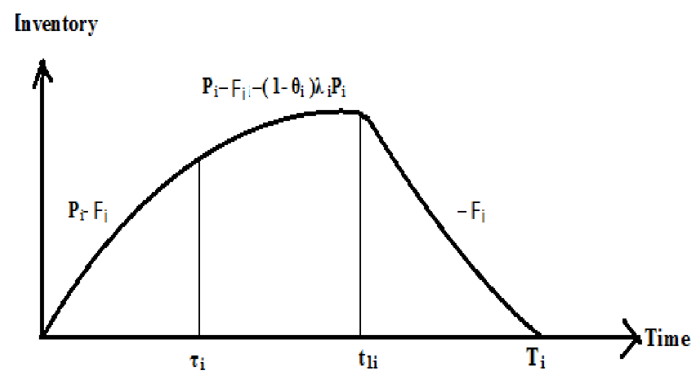



**Figure** 4.1: Inventory versus time for $i^{th}$ item.

### 4.2.2 Mathematical Model Formulation

In this production process for the $i^{th}$ item, production starts with a rate $P_i$ at time t=0 and runs up to time $t = t_{1i}$. The inventory piles up, during the time interval $[0, t_{1i}]$ adjusting demand $F_i(D_1, D_2, ..., D_n)$ in the market and the production process stocks good quality $Q_i$ units at time $t = t_{1i}$ and this stock is depleted satisfying the demand in the market and it reaches at zero level at time $T_i$ (cf. Fig. 4.1). The production system produces perfect quality up to a certain time $\tau_i$ (i.e., in-control state), after the lapse of this certain time the production system shifts to an "out-of-control" state. In this "out-of-control" state, some of the produced units are of non-conforming quality (i.e., defective units) and some of these defective units are reworked immediately when they are produced. The rest units are disposed at a cost. Thus governing differential equations for the $i^{th}$ item of this model are:

$$\frac{dq_i(t)}{dt} = \begin{cases} P_i - F_i, & 0 \leq t \leq \tau_i \\ P_i - F_i - (1-\theta_i)\lambda_i P_i, & \tau_i \leq t \leq t_{1i} \\ -F_i, & t_{1i} \leq t \leq T_i \end{cases} \tag{4.1}$$

with the boundary conditions

$$\begin{cases} q_i(t) = 0, & at \ \ t = 0 \\ q_i(t) = q_i(\tau_i), & at \ \ t = \tau_i \\ q_i(t) = 0, & at \ \ t = T_i \end{cases}$$

The solutions of the above differential equation are :

$$q_i(t) = \begin{cases} (P_i - F_i)t, & 0 \leq t \leq \tau_i \\ (P_i - F_i)t - \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)}(t-\tau_i)^{\gamma_i+1}, & \tau_i \leq t \leq t_{1i}, \\ F_i(T_i - t), & t_{1i} \leq t \leq T_i \end{cases} \tag{4.2}$$

The total defective units during $[\tau_i, t_{1i}]$ are

$$N_i = P_i \int_{\tau_i}^{t_{1i}} \alpha_i P_i^{\beta_i} (t-\tau_i)^{\gamma_i} dt = \frac{\alpha_i}{\gamma_i+1} P_i^{\beta_i+1} (t_{1i}-\tau_i)^{\gamma_i+1} \tag{4.3}$$

Therefore, the total expected defective units in a production lot size are

$$\begin{aligned} E(N_i) &= \int_0^\infty N_i d\Big(G_i(\tau_i)\Big) = \frac{\alpha_i}{\gamma_i+1} P_i^{\beta_i+1} \int_0^{t_{1i}} (t_{1i}-\tau_i)^{\gamma_i+1} d\Big(1-e^{-f_i(P_i)\tau_i}\Big) \\ &= \frac{\alpha_i}{\gamma_i+1} P_i^{\beta_i+1} f_i(P_i) e^{-f_i(P_i)t_{1i}} \int_0^{t_{1i}} (t_{1i}-\tau_i)^{\gamma_i+1} e^{f_i(P_i)(t_{1i}-\tau_i)} d\tau_i \\ &= \frac{\alpha_i}{\gamma_i+1} P_i^{\beta_i+1} f_i(P_i) e^{-f_i(P)t_{1i}} \psi_i(P_i, t_{1i}) \end{aligned} \tag{4.4}$$

$$\text{where, } \psi_i = \frac{t_{1i}^{\gamma_i+2}}{\gamma_i+2} + \frac{f_i(P_i) t_{1i}^{\gamma_i+3}}{1!(\gamma_i+3)} + \frac{[f_i(P_i)]^2 t_{1i}^{\gamma_i+4}}{2!(\gamma_i+4)} + \cdots = \sum_{j=1}^{\infty} \frac{[f_i(P_i)]^{j-1} t_{1i}^{\gamma_i+j+1}}{(j-1)!(\gamma_i+j+1)} \tag{4.5}$$

Now at time $t = t_{1i}$ the expected production lot size without defective units are

$$\begin{aligned} Q_i &= E[q(t_{1i})] = \int_0^\infty q_i(t_{1i}) d\Big(G_i(\tau_i)\Big) \\ &= (P_i - F_i)t_{1i} \int_0^\infty d\Big(G_i(\tau_i)\Big) \\ &\quad - \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)} f_i(P_i) e^{-f_i(P_i)t_{1i}} \int_0^{t_{1i}} (t_{1i}-\tau_i)^{\gamma_i+1} e^{f_i(P_i)(t_{1i}-\tau_i)} d\tau_i \\ &= (P_i - F_i)t_{1i} - \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)} f_i(P_i) e^{-f_i(P_i)t_{1i}} \psi_i(P_i, t_{1i}) \\ &= (P_i - F_i)t_{1i} - (1-\theta_i)E(N_i) \end{aligned} \tag{4.6}$$

Again from the equation (4.2) we get,

$$\begin{aligned} & Q_i = F_i(T_i - t_{1i}) \\ & \text{or,} (P_i - F_i)t_{1i} - (1-\theta_i)E(N_i) = F_i(T_i - t_{1i}) \\ & \text{or,} T_i = \frac{1}{F_i}[P_i t_{1i} - (1-\theta_i)E(N_i)] \end{aligned} \tag{4.7}$$

Now during the period $(0, t_{1i})$ the inventory which are to be hold are

$$\begin{aligned} Qh_{1i} &= \int_0^{t_{1i}} q_i(t) dt \\ &= \int_0^{\tau_i} (P_i - F_i) t dt + \int_{\tau_i}^{t_{1i}} [(P_i - F_i)t - \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)} (t-\tau_i)^{\gamma_i+1}] dt \\ &= \frac{P_i - F_i}{2} t_{1i}^2 - \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)(\gamma_i+2)} (t_{1i}-\tau_i)^{\gamma_i+2} \end{aligned}$$

During the period $(0, t_{1i})$ the expected inventory which are to be hold are

$$\begin{aligned}
E[Qh_{1i}] \quad &= \int_0^\infty Q_{h_{1i}} d\Big(G_i(\tau_i)\Big) = \int_0^\infty [\frac{P_i - F_i}{2} t_{1i}^2] d\Big(G_i(\tau_i)\Big) \\
&- \frac{(1-\theta_i)\alpha_i P_i^{\beta_i+1}}{(\gamma_i+1)(\gamma_i+2)} \int_0^{t_{1i}} (t_{1i} - \tau_i)^{\gamma_i+2} d\Big(1 - e^{-f_i(P_i)\tau_i}\Big) \\
&= \frac{P_i - F_i}{2} t_{1i}^2 - \frac{(1-\theta_i)\alpha_i f_i(P_i) P_i^{\beta_i+1} e^{-f_i(P_i)t_{1i}}}{(\gamma_i+1)(\gamma_i+2)} \int_0^{t_{1i}} (t_{1i} - \tau_i)^{\gamma_i+2} e^{f_i(P_i)(t_{1i}-\tau_i)} d\tau_i \\
&= \frac{P_i - F_i}{2} t_{1i}^2 - \frac{(1-\theta_i)\alpha_i f_i(P_i) P_i^{\beta_i+1} e^{-f_i(P_i)t_{1i}}}{(\gamma_i+1)(\gamma_i+2)} \xi_i(P_i, t_{1i}) \qquad (4.8)
\end{aligned}$$

where $\xi_i(P_i, t_{1i})$ are given by the equations as

$$\xi_i(P_i, t_{1i}) = \frac{t_{1i}^{\gamma_i+3}}{(\gamma_i+3)} + \frac{f_i(P_i) t_{1i}^{\gamma_i+4}}{1!(\gamma_i+4)} + \frac{[f_i(P_i)]^2 t_{1i}^{\gamma_i+5}}{2!(\gamma_i+5)} + \cdots = \sum_{r=1}^{\infty} \frac{[f_i(P_i)]^{r-1} t_{1i}^{\gamma_i+r+2}}{(r-1)!(\gamma_i+r+2)}$$

During the period $(t_{1i}, T_i)$ the inventory which are to be hold are

$$Qh_{2i} = \int_{t_{1i}}^{T_i} q_i(t)dt = \int_{t_{1i}}^{T_i} F_i(T_i - t)dt = \frac{F_i}{2}(T_i - t_{1i})^2 \qquad (4.9)$$

Therefore, during the period $(0, T_i)$ the total expected holding units are,

$$E[Q_{hi}] = E[Qh_{1i}] + Qh_{2i} \qquad (4.10)$$

where $E[Qh_{1i}]\, and\, Qh_{2i}$ are given by the $Eqs.$ (4.8) and (4.9) respectively.

In a cycle $(0, T_i)$, for $i^{th}$ item the expected total cost = Expected holding cost + Rework cost + Disposal cost + Set-up cost + Production cost + Advertisement cost.

$$\begin{aligned}
&\text{i.e.}\, TC_i(P_i, t_{1i}) = C_{hi}E(Q_{hi}) + \theta_i C_{ri} E(N_i) + (1-\theta_i) C_{di} E(N_i) + C_{si} \\
&+ C_i(P_i) P_i t_{1i} + C_{ai} T_i \\
&= C_{hi}\left[\frac{P_i - F_i}{2} t_{1i}^2 - \frac{(1-\theta_i)\alpha_i f_i(P_i) P_i^{\beta_i+1} e^{-f_i(P_i)t_{1i}}}{(\gamma_i+1)(\gamma_i+2)} \xi_i(P_i, t_{1i}) + \frac{F_i}{2}(T_i - t_{1i})^2\right] \\
&+ [\theta_i C_{ri} + (1-\theta_i) C_{di}] \frac{\alpha_i}{\gamma_i+1} P_i^{\beta_i+1} f_i(P_i) e^{-f_i(P)t_{1i}} \psi_i(P_i, t_{1i}) \\
&+ C_{s0i} + C_{s1i} P_i^{\rho_i} + [r_{mi} + \frac{g_i}{P_i^{\delta_{1i}}} + \eta_{1i} P_i^{\delta_{2i}} + \eta_{2i} P_i^{\delta_{3i}}] P_i t_{1i} + C_{ai} T_i \qquad (4.11)
\end{aligned}$$

In a cycle $(0, T_i)$ the expected average total cost for $i^{th}$ item is

$$ATC_i(P_i, t_{1i}) = \frac{TC_i(P_i, t_{1i})}{T_i}, \qquad \text{where } T_i \text{ is obtained by } Eq.\,(4.7)$$

and the expected average total selling revenue for $i^{th}$ item is $F_i S_{pi}$. Thus the average expected total profit for $i^{th}$ item is

$$ATF_i(P_i, t_{1i}) = F_i S_{pi} - ATC_i(P_i, t_{1i}) \qquad (4.12)$$

### Chance constraints for "out-of-control" state

In this production system, it is expected to have total production time greater than the time of occurrence of "out-of-control" state. This requirement acts as a constraint and expressed here as a chance constraint. Hence, the chance constraint is

$$\text{Prob}\Big(t_{1i} - \tau_i \geq \epsilon_i\Big) \geq r_i \tag{4.13}$$

where $t_{1i} \geq 0$ *and* $r_i \in (0,1)$ is a specified permissible probability.

Here $m_i\Big( = \frac{1}{f_i(P_i)}\Big)$ and $\sigma_i\Big( = \frac{1}{f_i(P_i)}\Big)$ are the mean and standard deviation of the exponential random variable $\tau_i$. Then the constraint can be written as

$Prob(\frac{\tau_i - m_i}{\sigma_i} \leq \frac{t_{1i} - \epsilon_i - m_i}{\sigma_i}) \geq r_i$

where $\frac{\tau_i - m_i}{\sigma_i}$ is a random normal variate. Considering $z_i$, where $\int_0^{z_i} \phi_i(t)dt = r_i$, $\phi_i(t)$, being the standard normal density function, we have

$$\frac{t_{1i} - \epsilon_i - m_i}{\sigma_i} \geq z_i$$

$$\text{or, } t_{1i} \geq \frac{1}{f_i(P_i)}[1 + z_i] + \epsilon_i \tag{4.14}$$

where $z_i$ is obtained from the normal distribution table for a particular value of $r_i$.

### Budget Constraint

For this multi-item imperfect production system, we take a pre-assigned budget B such that

$$\sum_{i=1}^{n} C_i(P_i)P_i t_{1i} \leq B \tag{4.15}$$

### Optimization Problem

Therefore, the problem for n number of multi-items inventory model is finally reduced to the maximization of expected average total profit subject to the chance constraints (4.14) and budget constraint (4.15). Hence the problem is reduced to

$$\begin{aligned}
&\text{Max} \sum_{i=1}^{n} ATF_i(P_i, t_{1i}) \\
\text{s.t.} \quad & t_{1i} \geq \frac{1}{f_i(P_i)}[1 + z_i] + \epsilon_i, \qquad \text{for all i=1,2,3,.....,n.} \\
\text{and,} \quad & \sum_{i=1}^{n} c_i(P_i)P_i t_{1i} \leq B
\end{aligned} \tag{4.16}$$

**Model 4.1A: Complementary items**

In the case of complementary items, the demand of an item is marginally increased by other. Hence, the demands of the $i^{th}$ complementary item is

$$F_i(D_1, D_2, ...., D_n) = D_i + \sum_{j=1, j\neq i}^{n} k_j D_j,$$
$$where, k_j\text{'s}\, are\, dependency\, levels\, and\ 0 \leq k_j < 1\, for\, all\, j.$$

**Model 4.1A1: Model for complementary items with linearly production dependent quality.**

In this formulation, the inverse of mean (= standard deviation ) of the elapsed time to failure increases linearly with the production rate. i.e., $f_i(P_i) = a_i + b_i P_i$, where $a_i$ , $b_i \geq 0$.

**Model 4.1A2: Model for complementary items with non-linearly production dependent quality.**

In this case, the inverse of mean (= standard deviation ) of the elapsed time to failure depends quadratically with the production rate. i.e., $f_i(P_i) = a_i + b_i P_i^2$, where $a_i$ , $b_i \geq 0$.

**Model 4.1B: Substitute items**

In the case of substitute items, original demand of an item decreases and at same time, it gets some additional customers due to other substitute items. Thus, the demands of the $i^{th}$ substitute item can be expressed as

$$F_i(D_1, D_2, ...., D_n) = k_i D_i + \sum_{j=1, j\neq i}^{n} (1 - k_j) D_j,$$
$$where\, k_j\text{'s}\, are\, dependency\, levels\, and\ \ 0.5 < k_j < 1\, for\, all\, j.$$

As in earlier cases, **Model-4.1B1** and **Model-4.1B2** are formulated with $f_i(P_i) = a_i + b_i P_i$ and $f_i(P_i) = a_i + b_i P_i^2$ respectively.

### 4.2.3 Particular Cases

**Model 4.1C: Items are independent to each other**

In this case, the demand of the $i^{th}$ item is not influenced by the other items and as a result,

$$F_i(D_1, D_2, ...., D_n) = D_i.$$

As before, **Model-4.1C1** and **Model-4.1C2** are developed with $f_i(P_i) = a_i + b_i P_i$ and $f_i(P_i) = a_i + b_i P_i^2$ respectively.

**Model 4.1D: ( Single item Model)**

In this case, there is no budget constraint and i=1. As usual, models **Model-4.1D1** and **Model-4.1D2** are formulated with the appropriate form of $f_i(P_i)$.

### 4.2.4 Solution Methodology

The above non-linear optimization problems of Models 4.1A1, 4.1A2, 4.1B1, 4.1B2, 4.1C1, 4.1C2, 4.1D1 and 4.1D2 are solved by a gradient based non-linear optimization method-GRG method (cf. Lasdon *et al.* [136] using LINGO Solver 11.0 for particular sets of data.

### 4.2.5 Numerical Experiments and Results

**Experiment-1: For multi-item**
**Input Data:** Let n=2. We illustrate the above inventory models (4.1A, 4.1B and 4.1C) numerically, for two complementary, substitute and independent items. Here the data for the parameters are taken in appropriate units as mentioned in § 4.2.1. Let available maximum budget is B=5000. The unit production costs for these two items are considered as:
$C_1(P_1) = 25 + \frac{100}{P_1} + 0.10P_1 + 0.10P_1^{1/2}$ and $C_2(P_2) = 20 + \frac{90}{P_2} + 0.07P_2 + 0.09P_2^{1/2}$.
The input data for two types of production dependent quality is taken as:
$f_i(P_i) = 1.25 + 0.0001P_i$ for i=1, 2 when quality is linearly production dependent and $f_i(P_i) = 1.25 + 0.00004P_i^2$ for i=1, 2 when quality is non-linearly production dependent. The other relevant input data are given in Table 4.4. For the above different models, the demand functions $F_i(D_1, D_2)$ (for i=1,2) are shown in Table 4.5, where $D_1 = u_{11} + u_{21}\frac{(C_{a1})^{\mu_{11}}}{(S_{p1})^{\mu_{21}}}$ and $D_2 = u_{12} + u_{22}\frac{(C_{a2})^{\mu_{12}}}{(S_{p2})^{\mu_{22}}}$ and the values of $k_1$ and $k_2$ for complementary (Model 4.1A) and substitute (Model 4.1B) items are considered as prescribed in Table 4.6. Here, all the parameters are presented in appropriate units.

**Table** 4.4: Input Data for Models 4.1A, 4.1B and 4.1C

| $i$ | $\alpha_i$ | $\beta_i$ | $\gamma_i$ | $z_i$ | $\epsilon_i$ | $\theta_i$ | $C_{hi}$ | $C_{di}$ | $C_{ri}$ | $C_{s0i}$ | $C_{s1i}$ | $C_{ai}$ | $u_{1i}$ | $u_{2i}$ | $\mu_{1i}$ | $\mu_{2i}$ | $m_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.25 | 1.5 | 0.5 | 0.10 | 0.75 | 3.0 | 2.0 | 8.0 | 550 | 0.5 | 120 | 20 | 4.75 | 0.25 | 0.30 | 2.50 |
| 2 | 0.25 | 0.30 | 1.5 | 0.5 | 0.10 | 0.25 | 2.0 | 2.0 | 6.0 | 500 | 0.5 | 100 | 15 | 4.50 | 0.25 | 0.20 | 2.25 |

**Optimal Results:** With the above parameters and expressions, the above Models 4.1A1, 4.1A2, 4.1B1, 4.1B2, 4.1C1 and 4.1C2 are formulated and optimized using LINGO 11.0 solver. The corresponding optimum values of production rate ($P_i^*$), production run time ($t_{1i}^*$) for maximum profit ($ATF^*$) and the related cycle time ($T_i^*$), demand ($F_i^*(D_1, D_2, ....D_n)$), total expected defective units ($E(N_i^*)$), inventory level of good units ($Q_i^*$), instant of defective production ($\frac{1}{f_i^*(P_i^*)}$), average selling price ($ASP_i^*$), average total cost ($ATC_i^*$) and average total profit ($ATF_i^*$) are evaluated and presented in Table 4.7.

**Table** 4.5: Demand function for proposed models

| $Models$ | $Demand\ functions$ |
|---|---|
| $Model4.1A$ | $F_1 = D_1 + k_2 D_2\ and\ F_2 = D_2 + k_1 D_1$ |
| $Model4.1B$ | $F_1 = k_1 D_1 + (1-k_2) D_2\ and\ F_2 = k_2 D_2 + (1-k_1) D_1$ |
| $Model4.1C$ | $F_1 = D_1\ and\ F_2 = D_2$ |

**Table** 4.6: Dependency levels of items

| $Items$ | $values\,of\ (k_1\ ,\ k_2)$ |
|---|---|
| $Complementary$ | $(0.20\ ,\ 0.25)$ |
| $Substitute$ | $(0.80\ ,\ 0.75)\ and\ (0.70\ ,\ 0.80)$ |

**Table** 4.7: Optimal results for multi-item models

| $Models$ | $i$ | $P_i^*$ | $t_{1i}^*$ | $T_i^*$ | $\frac{1}{f_i^*(P_i^*)}$ | $F_i^*$ | $Q_i^*$ | $E(N_i^*)$ | $ATC_i^*$ | $ASP_i^*$ | $ATF_i^*$ | $ATF^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Complementary | items: | with | $k_1$= | 0.20 | and | $k_2$= | 0.25 | | | | | |
| 4.1A1 | 1 | 69.4 | 1.29 | 2.94 | 0.79 | 29.5 | 48.5 | 12.6 | 1466 | 2520 | 1054 | 1454 |
| | 2 | 56.8 | 1.29 | 2.47 | 0.79 | 26.1 | 30.6 | 12.0 | 1142 | 1542 | 400 | |
| 4.1A2 | 1 | 86.7 | 1.07 | 3.06 | 0.64 | 29.4 | 58.5 | 10.4 | 1507 | 2629 | 1122 | 1513 |
| | 2 | 54.0 | 1.20 | 2.41 | 0.73 | 26.1 | 26.5 | 09.4 | 1143 | 1534 | 391 | |
| Substitute | items: | with | $k_1$= | 0.80 | and | $k_2$= | 0.75 | | | | | |
| 4.1B1 | 1 | 70.2 | 1.29 | 3.56 | 0.79 | 24.6 | 55.7 | 12.8 | 1268 | 2111 | 843 | 1125 |
| | 2 | 55.7 | 1.29 | 3.04 | 0.79 | 20.8 | 36.3 | 11.7 | 943 | 1225 | 282 | |
| 4.1B2 | 1 | 89.7 | 1.05 | 3.73 | 0.64 | 24.6 | 66.0 | 10.6 | 1308 | 2214 | 906 | 1176 |
| | 2 | 50.8 | 1.20 | 2.63 | 0.74 | 20.8 | 29.6 | 8.8 | 945 | 1215 | 270 | |
| Substitute | items: | with | $k_1$= | 0.70 | and | $k_2$= | 0.80 | | | | | |
| 4.1B1 | 1 | 62.6 | 1.29 | 3.70 | 0.79 | 21.2 | 50.9 | 11.1 | 1111 | 1780 | 669 | 1051 |
| | 2 | 65.8 | 1.29 | 3.06 | 0.79 | 24.3 | 42.8 | 14.6 | 1076 | 1458 | 382 | |
| 4.1B2 | 1 | 50.8 | 1.21 | 2.81 | 0.74 | 21.2 | 34.0 | 7.2 | 1107 | 1733 | 626 | 1101 |
| | 2 | 103.5 | 0.99 | 3.82 | 0.60 | 24.2 | 68.5 | 13.8 | 1105 | 1580 | 475 | |
| Independent | items: | | | | | | | | | | | |
| 4.1C1 | 1 | 69.0 | 1.29 | 3.56 | 0.79 | 24.1 | 55.0 | 12.5 | 1245 | 2063 | 818 | 1114 |
| | 2 | 57.3 | 1.29 | 3.05 | 0.79 | 21.3 | 37.5 | 12.2 | 962 | 1258 | 296 | |
| 4.1C2 | 1 | 86.5 | 1.06 | 3.72 | 0.64 | 24.0 | 64.0 | 10.4 | 1279 | 2151 | 872 | 1162 |
| | 2 | 54.3 | 1.19 | 2.72 | 0.73 | 21.3 | 32.5 | 9.4 | 962 | 1252 | 290 | |

The non-linear UPCs with respect to production rate are also evaluated for each item for Model 4.1A1 and their values are plotted in Fig. 4.2 and optimum results are presented in Table 4.8. These unit costs are separately minimized and their optimum values are compared with the corresponding values of the whole model, which maximizes the average total profit (cf. Table 4.8).

It is believed that minimum production cost fetches the maximum profit. But, from the

Table 4.8: Produced units for minimum UPC and maximum ATP

| | | Minimization of | UPC | Maximization of | ATP |
|---|---|---|---|---|---|
| Model | Item i | $C_i(P_i)$ | $P_i$ | $C_i(P_i)$ | $P_i$ |
| 4.1A1 | 1 | 31.88 | 30.0 | 34.21 | 69.4 |
| | 2 | 25.55 | 34.0 | 26.24 | 56.8 |

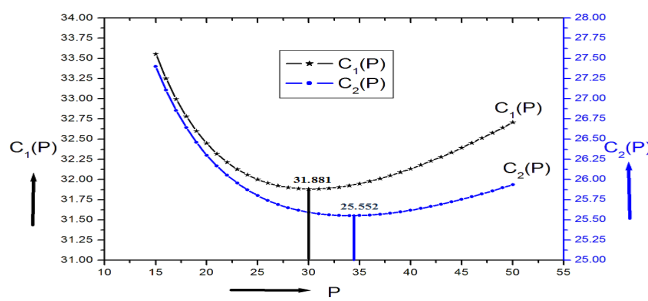



Figure 4.2: Production rate versus UPC for different items.

Table 4.8 and Fig. 4.2, it is seen that the production rates for which the production costs are minimum, are not the same for maximum average profit.

**Experiment-2: For single item**

Consider the proposed EPL models 4.1D1 and 4.1D2 with following input parameters in appropriate units as mentioned in § 4.2.1.

$\alpha_1 = 0.25$, $\beta_1 = 01.00$, $\gamma_1 = 1.5$, $z_1 = 0.75$, $\epsilon = 0.10$, $\theta_1 = 0.75$, $u_{11} = 20$, $u_{21} = 10$, $\mu_{11} = 0.25$, $\mu_{21} = 0.25$, $C_{h1} = 3.0$, $C_{s01} = 200$, $C_{s11} = 0.50$, $C_{d1} = 2.0$, $C_{r1} = 8.0$, $C_{a1} = 120.0$, $m_1 = 1.45$, $C_1(P_1) = 25+\frac{100}{P_1}+0.10P_1+0.10P_1^{1/2}$ and $f_1(P_1) = 1.25+0.0001P_1$ and $f_1(P_) = 1.25 + 0.00004P_1^2$ respectively for Model 4.1D1 and 4.1D2. Demands expressions are same as Experiment-1.

For these inputs the expected average total profits for Model 4.1D1 are plotted in Fig. 4.3 against the different values of $t_{11}$ and $P_1$.

### 4.2.6 Discussion

• Table 4.7 reveals that profits of the models with non-linear production quality (i.e., Model 4.1A2, 4.1B2 and 4.1C2) are greater than those of the models with linear production dependent quality (i.e., Model 4.1A1, 4.1B1 and 4.1C1). Moreover, among the linear and non-linear production dependent quality models, the models 4.1A1 and 4.1A2 give the highest profit respectively. This is because, in complementary item models, demand of an item increases by the influence of the other.

On the other hand, in the substitute item models, the demand of an item is altered by the influence of the corresponding substitute item and in that process, the resulting demand of each item may be more or less than the individual item's demand depending upon the values of dependency levels $(k_1, k_2)$. In Table 4.7, for two sets of values of $k_1$ and $k_2$, optimal profits are given. In one case, the profit is more and in other case, it is less than the corresponding profit for independent items i.e. when the demand of an item is not influenced by the other.

• It is to be noted that the mean-time ($\frac{1}{f_i^*(P_i^*)}$) of occurrence of out-of-control state is less than the production run time for all models. This is a necessary condition for the models. But it is not observed in the earlier works presented by Sana [227] and Khouja and Mehrez [127] because they did not impose the condition $\frac{1}{f_i^*(P_i^*)} \leq t_1^*$ through a constraint.

• From Fig. 4.3, it is seen that the optimal expected average profit for the model 4.1D1(i.e., for 1st item only ) is concave in nature as per the expectation. It is similar for the second item also.

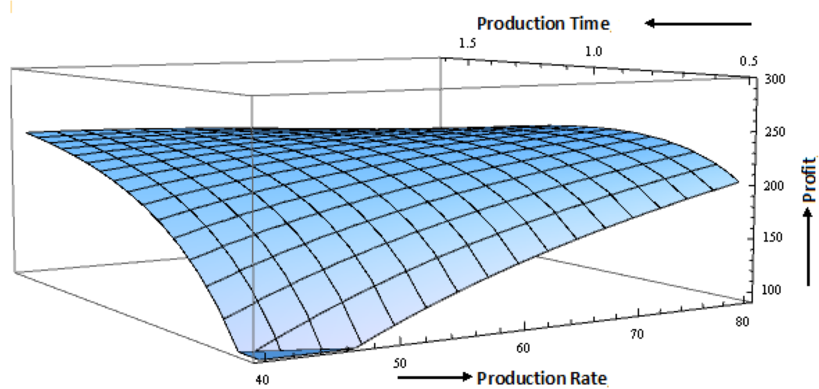



**Figure** 4.3: $ATF_1$ for a single item w. r. to Production rate $P_1$ and production run-time $t_{11}$.

### 4.2.7 Sensitivity Analysis

The changes in the values of system parameters can take place due to uncertainties and dynamic market conditions in a production-inventory system. In order to examine the implications of these changes in the values of parameters, the sensitivity analysis is of great help in a decision-making process. Using the result of Model 4.1A1 and Model 4.1B1 the sensitivity analyses due to the changes in the parameters $k_1$ and $k_2$ have been carried out. Here the changes of optimal average expected total profit ($\Delta ATF^*$) is evaluated in percentages with respect to optimal results of Model 4.1A1 and Model 4.1B1 and depicted in Figs. 4.4 and 4.5 for the changes of $k_1$ and $k_2$ respectively. The behaviour of the above parameters for other models are almost same. Here the change (in %) of an optimal value (suppose for $ATF^*$) is defined as $\Delta ATF^* = 100 \times [(ATF^{*old} - ATF^{*new})/ATF^{*old}]\%$,

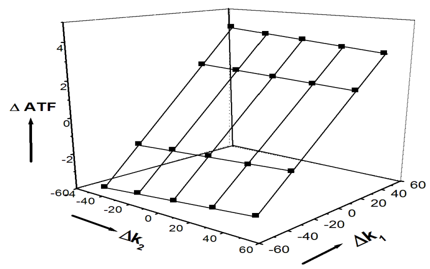



**Figure** 4.4: Change in % of ATF with respect to changes of $k_1$ and $k_2$ (in %) for complementary items.

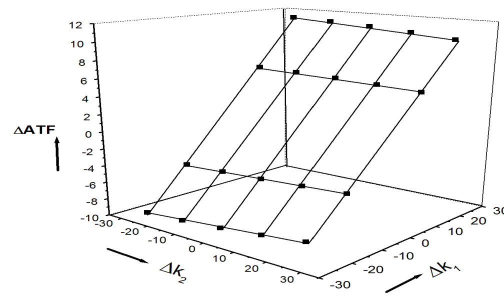



**Figure** 4.5: Change in % of ATF with respect to changes $k_1$ and $k_2$ (in %) for substitute items.

where $ATF^{*old}$ is the optimal result obtained from Model 4.1A1 or Model 4.1B1 and $ATF^{*new}$ is the new optimal result obtained after changing the corresponding parameter (in %) for the same models. From the sensitivity analyses the following observations are made.:

• For complementary items, the expected average total profit ATF is directly proportional to the level of dependencies (i.e., $k_1$ or $k_2$) of demand of other item. In this case demand of an item is increased by the demand of other.

• For substitute items, the expected total profit for both items may be increased or decreased for different pair set of dependency levels.

## 4.3 Model-4.2 : Quality and pricing decisions for substitutable items under imperfect production process over a random planning horizon[2]

### 4.3.1 Assumptions and Notations

*The following notations are used for $i^{th}$ product to develop the proposed models:*

**Decision variables:**

| | |
|---|---|
| $m_i$ | Number of cycles in a planning horizon |
| $M_i$ | Mark-up for a perfect unit |
| $P_i$ | Production rate in units per unit time |
| $q_i$ | Level of quality of a product, $\beta_i \leq q_i \leq 1$ where $\beta_i$ is the minimum quality level of $i^{th}$ product, which manufacturer intends to maintain |

[2]This model has been accepted for publication in **Hacettepe Journal of Mathematics and Statistics**, Hacettepe University, Y. 2016

**Parameters:**

| | |
|---|---|
| $\bar{H}$ | The length of the finite planning horizon which is random with a normal distribution $(m_h,\ \sigma_h)$ |
| $T_i$ | Cycle time in appropriate unit |
| $\tau_i$ | Time (measured from the commencement of production), at which defective unit production begins. i.e., the beginning time of the "out-of-control" state |
| $c_{hi}$ | Holding cost per unit per unit time |
| $c_{di}$ | Price of disposal for an imperfect unit |
| $c_{ri}$ | Cost to rework an imperfect unit |
| $\lambda_i$ | Constant production rate of defective units per unit production. The machine produces imperfect units at this rate when the machinery system is in "out-of-control" state |
| $\theta_i$ | Percentage of rework of defective units |
| $r_{mi}$ | Cost of raw materials required to produce an unit |
| $d_{i0}$ | Market based original / prime demand, not taking effects of its own and substitute product's prices and qualities |
| $d_{i1},\ d_{i2}$ | Measures of responsiveness of each product's consumer demand to its own price and competitor's price respectively |
| $d_{i3},\ d_{i4}$ | Measures of responsiveness of each product's consumer demand to its own quality and competitor's quality respectively |

**Dependent variables:**

| | |
|---|---|
| $I_i(t)$ | Inventory level at time t |
| $t_i$ | Production run-time in one cycle |
| $C_i(P_i, q_i)$ | UPC |
| $Cs_{ij}$ | The set up cost for $j^{th}$ cycle |
| $Cm_{ij}$ | The maintenance cost for $j^{th}$ cycle |
| $N_i$ | Defective units in a production cycle |
| $s_i$ | Selling price per unit perfect product. It is mark-up of raw material cost. i.e., $s_i = M_i r_{mi}$ |
| $D_i$ | Resultant demand in the market. This is the demand of a product after taking influence of prices and qualities of its own and substitute product |
| $Q_i$ | Total inventory unit for a single production |

$HC_i$, $PC_i$, $RC_i$, $SC_i$, $MC_i$ and $TC_i$ are the total holding, production, reworking, set-up, maintenance and relevant total costs during (0, $\bar{H}$) respectively.

$PSR_i$, $DSR_i$, $TSR_i$ and $TP_i$ are the sales revenue for perfect units, sales revenue for imperfect units which are not reworked, total sales revenue and total profit during (0, $\bar{H}$) respectively.

$ds_{p_i}(=\ d_{i1} - d_{i2}), ds_{q_i}(=\ d_{i4} - d_{i3})$ are proportional to Inverse Of Degree Of Substitutability (IODOS) due to price and quality respectively.

$Dp_i(= -d_{i1}s_i + d_{i2}s_j)$, $Dq_i(= d_{i3}q_i - d_{i4}q_j)$, where $j = 1, 2,\ j \neq i$ are amount of substitution demand rates due to price and quality. Here the above variables and parameters are taken in appropriate units.

*The following assumptions are used to develop the proposed models:*

(i) Multi-product imperfect production inventory models are considered. Products are substitutable depending on their prices and qualities jointly or either of these two. Here prices and qualities are assumed to be independent to each other.

(ii) Finite time planning horizon (random) is considered.

(iii) Production rate is finite and taken as a DV.

(iv) Lead time is zero and no shortages are allowed.

(v) The inventory system considers price and quality dependent demand rate.

(vi) The production process shifts from "In-control" state to "Out-of-control" state after a certain time. Imperfect units are produced at a constant rate per unit production in the "Out-of-control" state only.

(vii) There is immediate partially reworking for the defective units at a cost and the defective units which are not reworked, are sold at a lower price.

(viii) UPC is dependent on raw material, labour and quality improvement cost and one part of it is also spent for environment protection.

(ix) A maintenance cost is considered for the production system of each product to bring back the system to its initial condition by some maintenance operations (these may be mechanical, electrical, technical, replacement of parts, etc.) during the each time gap between the end of production and beginning of next production.

(x) "Fully substitution" means the loss of customers for a product is equal to the gain of its competitor product.

(xi) The sum of resultant demands of all substitutable products after substitution does not exceed the total market based (i.e. prime) demands of the products.

(xii) For any type of multi-product substitution, there is either loss of customers or fully substitution (i.e. no loss of sales) for the system if and only if the sum of resultant demands is either strictly less or equal to the total market based demand respectively.

(xiii) During substitution, demand of a product is more or equally sensitive to the changes due to its own price than the changes due to its competitor's price.

(xiv) During substitution, loss of customers of product -1 due to its own price is more or equal than the gain of customers of product-2 due to the price of product-1.

(xv) During substitution, demand of a product is less or equally sensitive to the changes due to its own quality than the changes due to its competitor's quality.

(xvi) During substitution, loss of customers of product -1 due to its own quality is more or equal than the gain of customers of product -2 due to the quality of product -1.

### 4.3.2 Demands based on price dependent substitution

In the case of only price dependent substitutable products, original demand of a product decreases for the increase of its own price and at the same time, it gets some additional customers due to its competitor's price. Thus, the resultant demands of the two substitutable merchandises can be expressed as

$$D_i(s_i, s_j) = d_{i0} - d_{i1}s_i + d_{i2}s_j,\ i, j = 1, 2,\ j \neq i.$$

where $D_i$ is the Resultant Demand (RD) for $i^{th}$ product at price $s_i$ given that the price of the other product j is $s_j$. Here, the range of selling price of $i^{th}$ product is assumed as $r_{mi} \leq s_i \leq d_{i0}/d_{i1}$.

$$\begin{aligned} i.e. \quad & D_1(s_1, s_2) = d_{10} - d_{11}s_1 + d_{12}s_2, \qquad r_{m1} \leq s_1 \leq d_{10}/d_{11}, \\ & D_2(s_2, s_1) = d_{20} - d_{21}s_2 + d_{22}s_1, \qquad r_{m2} \leq s_2 \leq d_{20}/d_{21}. \end{aligned} \tag{4.17}$$

where $d_{i0}s\,(> \ 0)$, i=1, 2; represent the market based prime demand of product i. $d_{i1}, d_{i2}\,(> \ 0)$, i=1, 2; denote the measures of the responsiveness of each product's consumer demand to its own price and to its competitor's price respectively. These parameters $d_{i0}$, $d_{i1}$ and $d_{i2}$ are mutually independent and non negative. According to assumptions (xiii) and (xiv), they satisfied the conditions $d_{11} \geq d_{12}$, $d_{21} \geq d_{22}$, $d_{11} \geq d_{22}$ and $d_{21} \geq d_{12}$. The difference $d_{11} - d_{12}(= ds_{p_1})$ is inversely related to the degree of substitutability (IODOS) of the 1st product with respect to the 2nd product. If this difference is smaller, then the product-1 is more substitutable with the 2nd product. i.e. product-1 is less differentiable. Hence the price of the product is higher. Same is true for the 2nd product with the difference $d_{21} - d_{22}(= ds_{p_2})$.

The ranges of limit of selling prices of $i^{th}$ merchandise are determined on the basis of two realistic requirements- (i) It should be more than the raw material cost per unit product and (ii) less than $\frac{d_{i0}}{d_{i1}}$ as loss of customers due to $i^{th}$ product's price should be less than or equal to its original demand ($d_{i0} - d_{i1}s_i \geq 0$).

**Proposition 4.1.** *For two products substitutable under price with demands (4.17), there is loss of sales (i.e. customers) or no loss in the system if and only if*

$$s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) > 0 \textit{ or } = 0 \ \textit{ respectively.} \tag{4.18}$$

*Proof.* Necessary part: Let us assume that for any two substitutable products, only loss of sales or fully substitution case can arise. Therefore, from assumption (xii) we have,
Sum of resultant demands $\leq$ Total market based demand.

$$\begin{aligned}
&\text{i.e., } D_1 + D_2 \leq d_{10} + d_{20}\\
&\text{or, } d_{10} - d_{11}s_1 + d_{12}s_2 + d_{20} - d_{21}s_2 + d_{22}s_1 \leq d_{10} + d_{20}\\
&\text{or, } s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0.
\end{aligned}$$

Therefore, the necessary part is complete.
Sufficient part: Let $s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0$.

$$\begin{aligned}
\text{Sum of resultant demands} &= D_1 + D_2\\
&= d_{10} - d_{11}s_1 + d_{12}s_2 + d_{20} - d_{21}s_2 + d_{22}s_1\\
&= d_{10} + d_{20} - [s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12})]\\
&\leq d_{10} + d_{20}, \text{ since } s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0.\\
\text{i.e., Sum of resultant demands} &\leq \text{ Total market based demand.}
\end{aligned}$$

Thus it is concluded that $s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0$ is the condition to be satisfied for the above assumption. Thus the sufficient part is complete.

Hence the Proposition. □

**Possible relations amongst the responsivenesses:**

Under the restrictions (4.18), the possible relations amongst the measures of responsivenesses due to product's prices are:

$$\begin{aligned}
&Case - 1P : d_{11} \geq d_{22} \;\; and \;\; d_{21} \geq d_{12}\\
&Case - 2P : d_{11} \geq d_{22} \;\; and \;\; d_{21} < d_{12} \;\;\; satisfying \text{ (4.18)}.\\
&Case - 3P : d_{11} < d_{22} \;\; and \;\; d_{21} \leq d_{12} \;\;\; satisfying \text{ (4.18)}
\end{aligned}$$

Here the cases -2P and -3P are not feasible due to assumption (xiv). Dissecting the case -1P, we have

case -1P1: $d_{11} > d_{22}$ and $d_{21} > d_{12}$ case -1P3: $d_{11} = d_{22}$ and $d_{21} > d_{12}$
case -1P2: $d_{11} > d_{22}$ and $d_{21} = d_{12}$ case -1P4: $d_{11} = d_{22}$ and $d_{21} = d_{12}$.

Again, satisfying the assumptions (xiii) and (xiv), we dissect the cases -1P1, -1P2, -1P3 and -1P4 and presented in Table 4.9.

### 4.3.3 Demands based on quality dependent substitution

In the case of only quality dependent substitution, demand of an product increases due to increase of its own quality and at same time, it looses some customers due to its competitor's

**Table** 4.9: Relations amongst the responsivenesses due to prices

| Cases | Subcases | Relations | Subcases | Relations |
|---|---|---|---|---|
| $1P1$ : | 1P11: | $d_{11} > d_{21} > d_{12} > d_{22}$ | 1P17: | $d_{21} > d_{11} > d_{12} > d_{22}$ |
| | 1P12: | $d_{11} > d_{21} > d_{12} = d_{22}$ | 1P18: | $d_{21} > d_{11} > d_{12} = d_{22}$ |
| | 1P13: | $d_{11} > d_{21} > d_{22} > d_{12}$ | 1P19: | $d_{21} > d_{11} > d_{22} > d_{12}$ |
| | 1P14: | $d_{11} = d_{21} > d_{12} > d_{22}$ | 1P110: | $d_{11} > d_{22} = d_{21} > d_{12}$ |
| | 1P15: | $d_{11} = d_{21} > d_{12} = d_{22}$ | 1P111: | $d_{21} > d_{11} = d_{12} > d_{22}$ |
| | 1P16: | $d_{11} = d_{21} > d_{22} > d_{12}$ | | |
| $1P2$ : | 1P21: | $d_{11} = d_{21} = d_{12} > d_{22}$ | 1P23: | $d_{11} > d_{21} = d_{12} = d_{22}$ |
| | 1P22: | $d_{11} > d_{21} = d_{12} > d_{22}$ | | |
| $1P3$ : | 1P31: | $d_{21} = d_{11} = d_{22} > d_{12}$ | 1P33: | $d_{21} > d_{11} = d_{22} = d_{12}$ |
| | 1P32: | $d_{21} > d_{11} = d_{22} > d_{12}$ | | |
| $1P4$ : | 1P41: | $d_{11} = d_{12} = d_{21} = d_{22}$ | | |

quality. Thus, the RD functions for the two substitutable products are expressed as

$$D_1(q_1, q_2) = d_{10} + d_{13}q_1 - d_{14}q_2,$$
$$D_2(q_2, q_1) = d_{20} + d_{23}q_2 - d_{24}q_1; \qquad \beta_i \le q_i \le 1 \ for\, i = 1,\, 2. \tag{4.19}$$

where $d_{i3}$, $d_{i4}\,(> 0)$, i=1, 2; denote the measures of the responsiveness of each product's consumer demand to its own quality and to its competitor's quality respectively. These parameters $d_{i0}$, $d_{i3}$ and $d_{i4}$ are mutually independent and non negative. According to assumptions (xv) and (xvi), they satisfied the conditions $d_{13} \le d_{14}$ , $d_{23} \le d_{24}$, $d_{13} \le d_{24}$ and $d_{23} \le d_{14}$. The difference $d_{14} - d_{13}(= ds_{q_1})$ is inversely related to the degree of substitutability (IODOS) of the 1st product with the 2nd product. If this difference is smaller, the the product -1 is more substitutable with the 2nd product. i.e. product -1 is less differentiable. Same is true for the 2nd product with the difference $d_{24} - d_{23}(= ds_{q_2})$. Here it is assumed that qualities $q_1$ and $q_2$ lies within [$\beta_i$, 1.0].

**Proposition 4.2.** *For two substitutable products under quality with demands* (4.19)*, there is loss of sales (i.e. customers) or no loss in the system if and only if*

$$q_1(d_{13} - d_{24}) + q_2(d_{23} - d_{14}) < 0 \; or \; = 0 \;\; \textit{respectively.} \tag{4.20}$$

*Proof.* Proceeding as Proposition 4.1, this proposition can be proved. □

**Possible relations amongst the responsivenesses:**

Under the restrictions (4.20), the possible relations amongst measures of responsiveness due to product's qualities are:

case -1q: $d_{13} \le d_{24}$ and $d_{23} \le d_{14}$ case -3q: $d_{13} > d_{24}$ and $d_{23} \le d_{14}$
case -2q: $d_{13} \le d_{24}$ and $d_{23} > d_{14}$ .

**Table** 4.10: Relations amongst the responsivenesses due to qualities

| Cases | Subcases | Relations | Subcases | Relations |
|---|---|---|---|---|
| $1q1$ : | 1q11: | $d_{14} > d_{24} > d_{13} > d_{23}$ | 1q17: | $d_{24} > d_{14} > d_{13} > d_{23}$ |
| | 1q12: | $d_{14} > d_{24} > d_{13} = d_{23}$ | 1q18: | $d_{24} > d_{14} > d_{13} = d_{23}$ |
| | 1q13: | $d_{14} > d_{24} > d_{23} > d_{13}$ | 1q19: | $d_{24} > d_{14} > d_{23} > d_{13}$ |
| | 1q14: | $d_{14} = d_{24} > d_{13} > d_{23}$ | 1q110: | $d_{14} > d_{23} = d_{24} > d_{13}$ |
| | 1q15: | $d_{14} = d_{24} > d_{13} = d_{23}$ | 1q111: | $d_{24} > d_{14} = d_{13} > d_{23}$ |
| | 1q16: | $d_{14} = d_{24} > d_{23} > d_{13}$ | | |
| $1q2$ : | 1q21: | $d_{14} = d_{24} = d_{13} > d_{23}$ | 1q23: | $d_{14} > d_{24} = d_{13} = d_{23}$ |
| | 1q22: | $d_{14} > d_{24} = d_{13} > d_{23}$ | | |
| $1q3$ : | 1q31: | $d_{24} = d_{14} = d_{23} > d_{13}$ | 1q33: | $d_{24} > d_{14} = d_{23} = d_{13}$ |
| | 1q32: | $d_{24} > d_{14} = d_{23} > d_{13}$ | | |
| $1q4$ : | 1q41: | $d_{14} = d_{13} = d_{24} = d_{23}$ | | |

Here the cases -2q and -3q are not feasible due to assumption (xvi). Dissecting the case -1q, we have

case -1q1: $d_{13} < d_{24}$ and $d_{23} < d_{14}$ case -1q3: $d_{13} < d_{24}$ and $d_{23} = d_{14}$

case -1q2: $d_{13} = d_{24}$ and $d_{23} < d_{14}$ case -1q4: $d_{13} = d_{24}$ and $d_{23} = d_{14}$.

Again, dissecting the case -1q1, -1q2, -1q3 and -1q4 satisfying the assumptions (xv) and (xvi), we present the sub-cases in Table 4.10.

### 4.3.4 Demands based on both price and quality dependent substitution

Here, we assume that price and quality of a product are independent to each other. Then, in the case of both price and quality dependent substitutable items, the original demand of an item is downward slopping in its own price and at same time, it gets some additional customers due to its competitor's price. It is reversed with respect to quality e.g. increases in its own quality and decreases for other's quality. Thus, RDs of the substitutable items on joint effect of price and quality can be expressed as

$$\begin{aligned} &D_1(s_1, s_2, q_1, q_2) = d_{10} - d_{11}s_1 + d_{12}s_2 + d_{13}q_1 - d_{14}q_2, \\ &D_2(s_1, s_2, q_1, q_2) = d_{20} - d_{21}s_2 + d_{22}s_1 + d_{23}q_2 - d_{24}q_1 \\ &\text{with } r_{mi} \leq s_i \leq \frac{d_{i0}}{d_{i1}} \text{ and } \beta_i \leq q_i \leq 1. \end{aligned} \tag{4.21}$$

where $d_{i0}$, $d_{i1}$, $d_{i2}$, $d_{i3}$, $d_{i4}$ for i=1, 2 have the meanings as earlier. These parameters are mutually independent.

**Proposition 4.3.** *For two substitutable products under both price and quality with demands (4.21), there is loss of sales (i.e. customers) or no loss in the system if and only if*

$$[-s_1(d_{11} - d_{22}) - s_2(d_{21} - d_{12}) + q_1(d_{13} - d_{24}) + q_2(d_{23} - d_{14})] < 0 \text{ or } = 0 \tag{4.22}$$

*respectively.*

**Table** 4.11: Relations amongst the responsivenesses due to prices and qualities (case-A)

| Cases | Relations | Cases | Relations |
|---|---|---|---|
| A1: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} > d_{24} > d_{13} > d_{23}$ | A11: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} = d_{23} > d_{13}$ |
| A2: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} > d_{24} > d_{23} > d_{13}$ | A12: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} = d_{13} > d_{23}$ |
| A3: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{24} > d_{14} > d_{13} > d_{23}$ | A13: | $d_{21} = d_{11} = d_{22} > d_{12}$ and $d_{14} > d_{24} > d_{13} > d_{23}$ |
| A4: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{24} > d_{14} > d_{23} > d_{13}$ | A14: | $d_{11} = d_{21} = d_{12} > d_{22}$ and $d_{14} > d_{24} > d_{13} > d_{23}$ |
| A5: | $d_{11} = d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} > d_{13} > d_{23}$ | A15: | $d_{21} = d_{11} = d_{22} > d_{12}$ and $d_{14} = d_{24} > d_{13} > d_{23}$ |
| A6: | $d_{11} = d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} > d_{23} > d_{13}$ | A16: | $d_{11} = d_{21} = d_{12} > d_{22}$ and $d_{14} = d_{24} > d_{13} > d_{23}$ |
| A7: | $d_{11} = d_{21} > d_{22} > d_{12}$ and $d_{14} > d_{24} > d_{13} > d_{23}$ | A17: | $d_{11} = d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} = d_{23} > d_{13}$ |
| A8: | $d_{11} = d_{21} > d_{12} > d_{22}$ and $d_{24} > d_{14} > d_{23} > d_{13}$ | A18: | $d_{11} = d_{21} > d_{22} > d_{12}$ and $d_{14} = d_{24} = d_{23} > d_{13}$ |
| A9: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} > d_{23} > d_{13}$ | A19: | $d_{21} = d_{11} = d_{22} > d_{12}$ and $d_{14} = d_{24} = d_{23} > d_{13}$ |
| A10: | $d_{11} > d_{21} > d_{12} > d_{22}$ and $d_{14} = d_{24} > d_{13} > d_{23}$ | A20: | $d_{11} = d_{21} = d_{12} > d_{22}$ and $d_{14} = d_{24} = d_{23} > d_{13}$ |

**Table** 4.12: Relations amongst the responsivenesses due to prices and qualities (case-B)

| Cases | Relations | Cases | Relations |
|---|---|---|---|
| B1: | $d_{11} > d_{21} > d_{22} > d_{12}$ and $d_{23} = d_{24} > d_{13} > d_{14}$ | B8: | $d_{22} > d_{11} > d_{21} > d_{12}$ and $d_{24} > d_{14} = d_{13} > d_{23}$ |
| B2: | $d_{11} > d_{21} > d_{22} > d_{12}$ and $d_{13} > d_{24} = d_{14} > d_{23}$ | B9: | $d_{22} > d_{11} > d_{21} > d_{12}$ and $d_{23} = d_{24} > d_{13} > d_{14}$ |
| B3: | $d_{11} > d_{21} > d_{22} > d_{12}$ and $d_{13} > d_{23} > d_{24} > d_{14}$ | B10: | $d_{22} > d_{11} > d_{21} > d_{12}$ and $d_{13} > d_{24} = d_{14} > d_{23}$ |
| B4: | $d_{11} = d_{12} > d_{21} > d_{22}$ and $d_{24} > d_{14} = d_{13} > d_{23}$ | B11: | $d_{22} > d_{11} > d_{21} > d_{12}$ and $d_{13} > d_{23} > d_{24} > d_{14}$ |
| B5: | $d_{11} = d_{12} > d_{21} > d_{22}$ and $d_{23} = d_{24} > d_{13} > d_{14}$ | B12: | $d_{22} > d_{11} = d_{12} > d_{21}$ and $d_{24} > d_{14} = d_{13} > d_{23}$ |
| B6: | $d_{11} = d_{12} > d_{21} > d_{22}$ and $d_{13} > d_{24} = d_{14} > d_{23}$ | B13: | $d_{22} > d_{11} = d_{12} > d_{21}$ and $d_{23} = d_{24} > d_{13} > d_{14}$ |
| B7: | $d_{11} = d_{12} > d_{21} > d_{22}$ and $d_{13} > d_{23} > d_{24} > d_{14}$ | B14: | $d_{22} > d_{11} = d_{12} > d_{21}$ and $d_{13} > d_{24} = d_{14} > d_{23}$ |

*Proof.* The proof is similar as Propositions 4.1 and 4.2. □

**Possible relations amongst the responsiveness:**

Possible relations amongst measures of responsivenesses due to both prices and qualities satisfying the condition (4.35) are:

**Case-A :** Let us assume that the effects in the changes of the demands due to prices and qualities are independent. In this case, $d_{ij}s$ (i=1,2; j=1,2,3,4) satisfy the Proposition 4.3 and assumptions (xiii) to (xvi) together. Dissecting this case we have 324 cases combining the cases of prices and qualities dependent substitution. Some of these cases are followed in Table-4.11.

**Case-B :** Here, the effects in the changes of the demands due to prices and qualities are not independent. i.e. the effect of changes of demand due to price (quality) may influence the changes of demand due to quality (price). These cases are observed in reality due to the joint effect of prices and qualities on the demand substitutions but they do not satisfy all the Proposition 4.3 and assumptions (xiii) to (xvi) at a time. Some of these cases are given in Table 4.12.

### 4.3.5 Mathematical Model Development

In this investigation, an imperfect EPL model for $i^{th}$ item is assumed over a finite random planning horizon of length $\bar{H}$ in which time $m_i$ number of full cycles are completed. In this production process, for $j^{th}$ cycle, the production starts with a rate $P_i$ at time $t = (j-1)T_i$ and runs up to time $t = (j-1)T_i + t_i$. The system produces perfect quality units up to a certain time $(j-1)T_i + \tau_i$ (i.e., in-control state), after that, the production system shifts to an "out-of-control" state $[(j-1)T_i + \tau_i, (j-1)T_i + t_i]$. In this "out-of-control" state, some of the produced units are of non-conforming quality (i.e., defective units) and some of these defective units are reworked immediately. The inventory piles up, during the time interval $[(j-1)T_i, (j-1)T_i + t_i]$ adjusting demand $D_i$ in the market and the production and reworking processes produce perfect product $Q_i$ units upto time $t = (j-1)T_i + t_i$, i.e., when the system stops the production. The stock at $t = (j-1)T_i + t_i$ is depleted satisfying the demand $D_i$ in the market and it reaches zero level at time $jT_i$ (cf. Fig. 4.6). After the end of one production run, we assume that the machinery system is maintained against a cost and brought back to its original good condition before the next production.

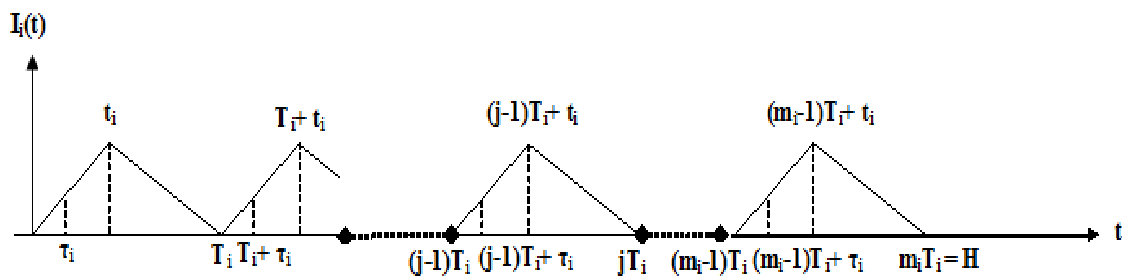



**Figure** 4.6: Inventory versus time for $i^{th}$ item.

For the multi-item imperfect production process with different demand functions, the governing differential equations for the $j^{th}$ cycle of $i^{th}$ (i=1,2) item are:

$$\frac{dI_i(t)}{dt} = \begin{cases} P_i - D_i, & (j-1)T_i \leq t \leq (j-1)T_i + \tau_i \\ P_i - D_i - (1-\theta_i)\lambda_i P_i, & (j-1)T_i + \tau_i \leq t \leq (j-1)T_i + t_i \\ -D_i, & (j-1)T_i + t_i \leq t \leq jT_i \end{cases} \quad (4.23)$$

with the boundary conditions

$$\begin{cases} I_i(t) = 0, & at \ \ t = (j-1)T_i \\ I_i(t) = 0, & at \ \ t = jT_i \end{cases}$$

The solutions of the above differential equations are :

$$I_i(t) = \begin{cases} (P_i - D_i)\{t-(j-1)T_i\}, & (j-1)T_i \leq t \leq (j-1)T_i + \tau_i \\ (P_i - D_i)\{t-(j-1)T_i\} \\ -(1-\theta_i)\lambda_i P_i\{t-(j-1)T_i - \tau_i\}, & (j-1)T_i + \tau_i \leq t \leq (j-1)T_i + t_i, \\ D_i(jT_i - t,) & (j-1)T_i + t_i \leq t \leq jT_i \end{cases}$$

where $t_i = \frac{D_i T_i - (1-\theta_i)\lambda_i P_i \tau_i}{P_i\{1-(1-\theta_i)\lambda_i\}}$ and $Q_i = P_i t_i - (1-\theta_i)\lambda_i P_i (t_i - \tau_i)$

### Holding cost

The total holding cost in the time horizon H is $HC_i = \sum_{j=1}^{m_i} c_{hi}(Ih_{1i} + Ih_{2i} + Ih_{3i})$ where,

$$\begin{aligned}
Ih_{1i} &= \int_{(j-1)T_i}^{(j-1)T_i+\tau_i} I_i(t)dt = \int_{(j-1)T_i}^{(j-1)T_i+\tau_i} (P_i - D_i)\{t-(j-1)T_i\}dt = \frac{P_i - D_i}{2}\tau_i^2. \\
Ih_{2i} &= \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} I_i(t)dt \\
&= \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} [(P_i - D_i)\{t-(j-1)T_i\} - (1-\theta_i)\lambda_i P_i\{t-(j-1)T_i-\tau_i\}]dt \\
&= \frac{P_i - D_i}{2}(t_i^2 - \tau_i^2) - \frac{(1-\theta_i)\lambda_i P_i}{2}(t_i - \tau_i)^2. \\
Ih_{3i} &= \int_{(j-1)T_i+t_i}^{jT_i} I_i(t)dt = \int_{(j-1)T_i+t_i}^{jT_i} D_i(jT_i - t,)dt = \frac{D_i}{2}(T_i - t_i)^2.
\end{aligned}$$

### Rework cost

The total rework cost $(RC_i)$ in the time horizon H is $RC_i = \sum_{j=1}^{m_i} c_{ri}\theta_i N_i,$ where $N_i$ are the defective units during $[(j-1)T_i + \tau_i, (j-1)T_i + t_i]$ for i=1,2 and expressed as

$$N_i = \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} \lambda_i P_i dt = \lambda_i P_i(t_i - \tau_i).$$

### Production cost

UPC is considered for $i^{th}$ item (i=1,2) as

$$C_i(P_i, q_i) = r_{mi} + \frac{g_{1i}}{P_i} + \frac{g_{2i}q_i}{1-a_i q_i} + g_{3i}P_i^{\frac{1}{2}},$$

where $r_{mi}$ is the raw material cost per unit item, $g_{1i}$ is the total labour/energy costs per unit time in a production system which is equally distributed over the unit item. So,$(\frac{g_i}{P_i})$ decreases with increases of $P_i$. The third term $\frac{g_{2i}q_i}{1-a_i q_i}$ is quality improvement cost, proportional to the positive power of quality of a product and the fourth term $g_{3i}P_i^{\frac{1}{2}}$ is EPC assuming that the cost due to the measures taken for the environment protection is proportional to square root of production rate $P_i$, where the power term varies with the nature of production firms.

Therefore, the total production cost for $i^{th}$ item is

$$PC_i = \sum_{j=1}^{m_i} C_i(P_i, q_i)P_i t_i.$$

**Setup cost**

Some researchers [41, 53, 196] considered the learning effect modelling into the set up cost in different forms. Here, the set up cost for $j^{th}$ cycle $(j = 1, 2, ..., m_i)$ of $i^{th}$ item (i=1,2) is partly constant and partly decreases in each cycle due to learning effect of the employees and is of the form: $Cs_{ij} = C_{s0i} + C_{s1i}e^{-jc_i}$, where $c_i > 0$. Therefore total set up cost for $m_i$ number of cycle is

$$SC_i = \sum_{j=1}^{m_i} Cs_{ij} = m_iC_{s0i} + C_{s1i}\frac{1 - e^{-m_ic_i}}{e^{c_i} - 1}.$$

**Maintenance cost**

Maintenance cost for the machinery system is used to bring the system to its original position after the end of each production. In Tarakci *et al.* [250], a manufacturer contracts to an external contractor who is responsible for scheduling and performing preventive maintenance and carrying out minimal repairs when the process fails. Here, learning occurs in both cost and time of preventive maintenance. For the first cycle no maintenance is required, but for the next cycles on wards, it is increased in each cycle due to the reuse of the system for several times. Maintenance cost for $j^{th}$ cycle of the $i^{th}$ item is taken as: $Cm_{ij} = C_{m0i}[1 - e^{-(j-1)c'_i}]$, where $c'_i > 0$. Therefore total maintenance cost for $m_i$ number of cycle is

$$MC_i = \sum_{j=1}^{m_i} Cm_{ij} = C_{m0i}[m_i - \frac{1 - e^{-m_ic'_i}}{1 - e^{-c'_i}}].$$

**Total Relevant Model cost**

As a result, the total model cost $=$ Holding cost + Rework cost + Production cost + Set-up cost + Maintenance cost.

$$\text{i.e.}\, TC_i = HC_i + RC_i + PC_i + SC_i + MC_i. \tag{4.24}$$

**Total Sale Revenue**

**Revenue for perfect units:** Total sales revenue of perfect products for $m_i$ number of cycles is

$$PSR_i = \sum_{j=1}^{m_i} s_i \int_{(j-1)T_i}^{jT_i} D_i dt = \sum_{j=1}^{m_i} s_iD_iT_i,$$

where, $s_i = M_ir_{mi}$ is the selling price of each product which is mark-up of raw material cost $r_{mi}$

**Sales Revenue for imperfect units:** The defective products which are not to be reworked is disposed by a lower price and total sales revenue for $m_i$ number of cycles is

$DSR_i = \sum_{j=1}^{m_i} c_{di}(1-\theta_i)N_i$, where $c_{di} = x_i s_i$, $0 < x_i < 1$.

Therefore, total sales revenue for this model is

$$TSR_i = PSR_i + DSR_i. \tag{4.25}$$

**Total Profit**

Total profit during the whole planning horizon for $i^{th}$ item is

$$\begin{aligned} TP_i &= TSR_i - TC_i = \sum_{j=1}^{m_i} M_i r_{mi}[D_i T_i + x_i(1-\theta_i)\lambda_i P_i(t_i-\tau_i)] \\ &- \sum_{j=1}^{m_i} c_{hi}[\tfrac{P_i-D_i}{2}t_i^2 - \tfrac{(1-\theta_i)\lambda_i P_i}{2}(t_i-\tau_i)^2 + \tfrac{D_i}{2}(T_i-t_i)^2] - \sum_{j=1}^{m_i} c_{ri}\theta_i\lambda_i P_i(t_i-\tau_i) \\ &- \sum_{j=1}^{m_i}[r_{mi} + \tfrac{g_{1i}}{P_i} + \tfrac{g_{2i}q_i}{1-a_i q_i} + g_{3i}P_i^{\frac{1}{2}}]P_i t_i - [m_i C_{s0i} + C_{s1i}\tfrac{1-e^{-m_i c_i}}{e^{c_i}-1}] - C_{m0i}[m_i - \tfrac{1-e^{-m_i c_i'}}{1-e^{-c_i'}}] \end{aligned} \tag{4.26}$$

## 4.3.6 Model Constraints

**Chance Constraint for Random Time Horizon**

For the random time horizon, we consider two constraints as $\bar{H} \geq m_i T_i$ for i=1, 2. In this consideration, constraints are expressed as Chance constraints which are

$$\begin{aligned} &\Pr(\bar{H} \geq m_i T_i) \geq r, \text{ for i=1, 2; where } r \in (0,1) \text{ is a specified probability.} \\ &\text{or, } m_i T_i \leq m_h + \sigma_h \Phi^{-1}(1-r), \text{ for i=1, 2 (cf. Rao, [213])} \end{aligned} \tag{4.27}$$

where $m_h$ and $\sigma_h$ are the expectation and standard deviation of normally distributed random variable $\bar{H}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\frac{\bar{H}-m_h}{\sigma_h}$.

**Demand function Constraints**

In reality, the consumer demands $D_i(s_i, q_i)$ are non negative. Sum of RDs of all substitutable items under any type substitution does not exceed the total market based demand of those items. Thus,

$$\begin{aligned} &D_i(s_i, q_i) > 0, \ \ for \ i = 1, 2; \\ &and \ \sum_{i=1}^{2} D_i(s_i, q_i) \leq d_{10} + d_{20}. \end{aligned} \tag{4.28}$$

**Ranges of mark-up and quality**

According to Yao and Wu [281], we have ranges of the best prices for i=1, 2 as

$$r_{mi} \leq s_i \leq d_{i0}/d_{i1} \qquad \text{or, } r_{mi} \leq M_i r_{mi} \leq d_{i0}/d_{i1} \qquad \text{or, } 1 \leq M_i \leq \frac{d_{i0}}{d_{i1} r_{mi}} \tag{4.29}$$

From our earlier assumption, we take the ranges of quality as

$$\beta_i \leq q_i \leq 1 \text{ for i=1, 2} \tag{4.30}$$

### 4.3.7 Optimization Problems

**Model 4.2A**

Considering the demand is measured only on selling price, the problem for multi-items inventory model is finally reduced to the maximization of total profit subject to Chance constraints on the Random Time Horizon and Demand constraints. Hence the problem is reduced to

$$\begin{cases} \text{Maximize } Z_1 = \sum_{i=1}^{2} TP_i(m_1, m_2, M_1, M_2, P_1, P_2) \\ \quad \text{with constraints (4.27), (4.28) and (4.29).} \end{cases} \tag{4.31}$$

where $D_i(s_i)$ is given by the equation (4.17) and

$$\begin{aligned} TP_i &= TSR_i - TC_i = \sum_{j=1}^{m_i} M_i r_{mi}[D_i T_i + x_i(1-\theta_i)\lambda_i P_i(t_i - \tau_i)] \\ &- \sum_{j=1}^{m_i} c_{hi}[\tfrac{P_i - D_i}{2} t_i^2 - \tfrac{(1-\theta_i)\lambda_i P_i}{2}(t_i - \tau_i)^2 + \tfrac{D_i}{2}(T_i - t_i)^2] - \sum_{j=1}^{m_i} c_{ri}\theta_i\lambda_i P_i(t_i - \tau_i) \\ &- \sum_{j=1}^{m_i}[r_{mi} + \tfrac{g_{1i}}{P_i} + g_{3i}P_i^{\frac{1}{2}}]P_i t_i - [m_i C_{s0i} + C_{s1i}\tfrac{1-e^{-m_i c_i}}{e^{c_i}-1}] - C_{m0i}[m_i - \tfrac{1-e^{-m_i c_i'}}{1-e^{-c_i'}}] \end{aligned} \tag{4.32}$$

**Model 4.2B**

Similarly, considering the demand is measured only on quality, the problem is reduced to

$$\begin{cases} \text{Maximize } Z_2 = \sum_{i=1}^{2} TP_i(m_1, m_2, P_1, P_2, q_1, q_2) \\ \quad \text{with constraints (4.27), (4.28) and (4.30).} \end{cases} \tag{4.33}$$

$D_i(q_i)$ and $TP_i$ are given by the equations (4.19) and (4.26) respectively.

**Model 4.2C**

Considering the demand is measured on the joint effect of selling price and quality, the problem for multi-items inventory model is finally reduced to

$$\begin{cases} \text{Maximize } Z_3 = \sum_{i=1}^{2} TP_i(m_1, m_2, M_1, M_2, P_1, P_2, q_1, q_2) \\ \quad \text{with constraints (4.27), (4.28), (4.29) and (4.30).} \end{cases} \tag{4.34}$$

where $D_i(s_i, q_i)$ and $TP_i$ are given by the equations (4.21) and (4.26) respectively.

### 4.3.8 Solution Methodology

To solve above maximization problems, we have applied the Fuzzy Age based Genetic Algorithm (FAGA) as described in § 2.2.3.2.

### 4.3.9 Numerical Experiments and Results

**Input Data:** We consider the proposed EPL models(Model 4.2A, 4.2B and 4.2C) with following inputs parameters in appropriate units:
$m_h = 25$, $\sigma_h = 2.0$, $r = 0.70$;
$C_{s01} = 1000$, $C_{s11} = 200$, $c_1 = 0.70$, $C_{m01} = 210$, $c'_1 = 0.75$, $c_{h1} = 1.80$, $c_{r1} = 2.50$, $\theta_1 = 0.75$, $\lambda_1 = 0.35$, $x_1 = 0.50$, $\tau_1 = 0.75$, $d_{10} = 55$, $\beta_1 = 0.50$;
$C_{s02} = 1150$, $C_{s12} = 225$, $c_2 = 0.75$, $C_{m02} = 220$, $c'_2 = 0.80$, $c_{h2} = 1.75$, $c_{r2} = 2.75$, $\theta_2 = 0.70$, $\lambda_2 = 0.30$, $x_2 = 0.45$, $\tau_2 = 0.80$, $d_{20} = 60$, $\beta_2 = 0.50$ for Models 4.2A, 4.2B and 4.2C.
$C_1(P_1, q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20P_1^{\frac{1}{2}}$, $C_2(P_2, q_2) = 22 + \frac{460}{P_2} + \frac{8.50q_2}{1-0.55q_2} + 0.18P_2^{\frac{1}{2}}$ for Models 4.2B and 4.2C and $C_1(P_1) = 20 + \frac{450}{P_1} + 0.20P_1^{\frac{1}{2}}$, $C_2(P_2) = 22 + \frac{460}{P_2} + 0.18P_2^{\frac{1}{2}}$ for Model 4.2A. The bounds of DVs $M_i$ and $q_i$ are considered using constraints (4.29) and (4.30) and the bounds of other DVs are considered as $P_i \in [50, 250]$ and $m_i \in [1, 8]$.
**Optimum results:** With the above parameters and expressions, the Models 4.2A, 4.2B and 4.2C are formulated and optimized using FAGA. The corresponding $i^{th}$ item's optimum values - number of cycles ($m_i^*$), Mark-ups ($M_i$), production rates ($P_i^*$), qualities ($q_i$), selling prices $s_i^*$ per unit perfect product, amount of substitution demand rates due to price ($Dp_i^* = -d_{i1}s_i^* + d_{i2}s_{3-i}^*$) and quality ($Dq_i^* = d_{i3}q_i^* - d_{i4}q_{3-i}^*$), resultant demand ($D_i$) and production run time ($t_i^*$), defective units ($N_i^*$), total produced good inventories ($Q_i^*$) for each production cycle and maximum total profit ($Z_1^*$, $Z_2^*$ and $Z_3^*$) for whole time horizon are evaluated for the different set values of $d_{i1}, d_{i2}, d_{i3}\, and\, d_{i4}$ which are satisfied the assumptions (xiii) to (xvi) and Propositions 4.1 to 4.3. For every set of these parameter, we treat it as a case of the corresponding model. The obtained results are presented in Tables 4.13, 4.14, 4.15, 4.16, 4.17, 4.18 and 4.19.

### 4.3.10 Discussion

**Effect of IODOSs (with respect to price) on profit for Model 4.2A**

(i) We perform some experiments with Model 4.2A in which substitutability occurs due price only and the results with different marks-up for the sale of the items are presented in Table 4.13. Here, the different mark-ups for the products -1 and -2 are bounded by the expression (4.29) i.e. $1 \leq M_i \leq d_{i0}/(d_{i1}r_{mi})$, i=1,2. As $d_{i0}$ and $r_{mi}$ are constants, mark-up changes with $d_{i1}$, i=1,2, i.e. the measure of responsiveness of the products to their own prices. Here, the responsivenesses have been assumed to be less than 1 (i.e. $0 < d_{i1} < 1$) and therefore, **smaller the resposiveness, larger the mark-up**. The optimum mark-ups change the respective prices and as a consequence, alter the demands, production rates and finally the maximum profits. Here, for the cases -1P17, -1P18, -1P111, -1P32 and -1P33, mark-ups are almost same. Similarly, cases -1P11 and -1P12 have nearly same mark-up. Comparing these two cases, the IODOS of the 2nd product is reduced from 0.25 to 0.20 where IODOS of 1st product

Table 4.13: Results (optimum quantities) for Model 4.2A with different mark-ups

| Case | Responsiveness | | IODOS | Optimum results | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{11}$ | $d_{12}$ | $ds_{p_1}$ | $m_1^*$ | $M_1^*$ | $P_1^*$ | $Z_1^*$ | $s_1^*$ | $Dp_1^*$ | $D_1^*$ | $t_1^*$ | $N_1^*$ | $Q_1^*$ |
| | $d_{21}$ | $d_{22}$ | $ds_{p_2}$ | $m_2^*$ | $M_2^*$ | $P_2^*$ | | $s_2^*$ | $Dp_2^*$ | $D_2^*$ | $t_2^*$ | $N_2^*$ | $Q_2^*$ |
| 1P11 | 0.50 | 0.25 | 0.25 | 3 | 5.50 | 120 | 110382 | 110 | -22 | 33 | 2.32 | 66 | 261 |
| | 0.45 | 0.20 | 0.25 | 2 | 5.94 | 121 | | 131 | -37 | 23 | 2.46 | 60 | 279 |
| 1P12 | 0.50 | 0.25 | 0.25 | 3 | 5.49 | 118 | 123869 | 110 | -22 | 33 | 2.40 | 68 | 267 |
| | 0.45 | 0.25 | 0.20 | 3 | 6.05 | 125 | | 133 | -32 | 28 | 1.86 | 40 | 220 |
| 1P13 | 0.50 | 0.15 | 0.35 | 3 | 5.26 | 133 | 86029 | 105 | -35 | 20 | 1.26 | 24 | 161 |
| | 0.45 | 0.20 | 0.25 | 3 | 5.38 | 136 | | 118 | -32 | 28 | 1.72 | 38 | 223 |
| 1P14 | 0.50 | 0.25 | 0.25 | 3 | 5.50 | 122 | 96429 | 110 | -25 | 30 | 2.05 | 55 | 236 |
| | 0.50 | 0.20 | 0.30 | 3 | 5.37 | 137 | | 118 | -37 | 23 | 1.39 | 24 | 184 |
| 1P15 | 0.50 | 0.25 | 0.25 | 3 | 5.49 | 124 | 108216 | 110 | -25 | 30 | 2.06 | 57 | 241 |
| | 0.50 | 0.25 | 0.25 | 3 | 5.45 | 135 | | 120 | -33 | 27 | 1.71 | 37 | 219 |
| 1P16 | 0.50 | 0.15 | 0.35 | 3 | 5.05 | 137 | 74920 | 101 | -35 | 20 | 1.23 | 23 | 164 |
| | 0.50 | 0.20 | 0.30 | 3 | 4.85 | 147 | | 107 | -33 | 27 | 1.53 | 32 | 215 |
| 1P17 | 0.40 | 0.25 | 0.15 | 3 | 6.85 | 106 | 146451 | 137 | -22 | 33 | 2.70 | 72 | 268 |
| | 0.45 | 0.20 | 0.25 | 3 | 6.04 | 124 | | 133 | -32 | 28 | 1.87 | 40 | 221 |
| 1P18 | 0.40 | 0.25 | 0.15 | 3 | 6.87 | 104 | 163852 | 137 | -22 | 33 | 2.73 | 72 | 267 |
| | 0.45 | 0.25 | 0.20 | 3 | 6.05 | 117 | | 133 | -26 | 34 | 2.51 | 60 | 275 |
| 1P19 | 0.40 | 0.15 | 0.25 | 3 | 6.60 | 115 | 111396 | 132 | -34 | 21 | 1.57 | 33 | 172 |
| | 0.45 | 0.20 | 0.25 | 3 | 5.84 | 127 | | 128 | -31 | 29 | 1.90 | 42 | 229 |
| 1P110 | 0.50 | 0.25 | 0.25 | 3 | 5.49 | 111 | 192123 | 110 | -17 | 38 | 2.90 | 83 | 300 |
| | 0.40 | 0.40 | 0.00 | 3 | 6.81 | 98 | | 150 | -16 | 44 | 3.87 | 90 | 352 |
| 1P111 | 0.40 | 0.40 | 0.00 | 1 | 6.85 | 59 | 203829 | 137 | -2 | 53 | 23.93 | 475 | 1282 |
| | 0.45 | 0.20 | 0.25 | 3 | 6.05 | 124 | | 133 | -32 | 28 | 1.87 | 40 | 220 |
| 1P21 | 0.50 | 0.50 | 0.00 | 1 | 5.50 | 66 | 160715 | 110 | 5 | 60 | 23.88 | 533 | 1439 |
| | 0.50 | 0.20 | 0.30 | 2 | 5.45 | 129 | | 120 | -38 | 22 | 2.18 | 53 | 265 |
| 1P22 | 0.50 | 0.45 | 0.05 | 1 | 5.50 | 66 | 167830 | 110 | 5 | 60 | 23.84 | 532 | 1437 |
| | 0.45 | 0.20 | 0.25 | 2 | 6.06 | 120 | | 133 | -39 | 22 | 2.36 | 56 | 266 |
| 1P23 | 0.50 | 0.45 | 0.05 | 1 | 5.47 | 66 | 236598 | 109 | 5 | 60 | 23.99 | 534 | 1441 |
| | 0.45 | 0.45 | 0.00 | 3 | 6.04 | 88 | | 133 | -11 | 49 | 4.87 | 107 | 396 |
| 1P31 | 0.50 | 0.25 | 0.25 | 3 | 5.49 | 123 | 171945 | 110 | -25 | 30 | 2.06 | 57 | 240 |
| | 0.50 | 0.50 | 0.00 | 1 | 5.45 | 61 | | 120 | -5 | 55 | 23.80 | 419 | 1320 |
| 1P32 | 0.40 | 0.25 | 0.15 | 3 | 6.86 | 111 | 219616 | 137 | -22 | 33 | 2.57 | 71 | 267 |
| | 0.45 | 0.40 | 0.05 | 1 | 6.05 | 60 | | 133 | -5 | 55 | 23.93 | 419 | 1319 |
| 1P33 | 0.40 | 0.40 | 0.00 | 1 | 6.87 | 59 | 276713 | 137 | -2 | 53 | 23.88 | 475 | 1281 |
| | 0.45 | 0.40 | 0.05 | 1 | 6.06 | 62 | | 133 | -5 | 55 | 23.12 | 418 | 1319 |
| 1P41 | 0.50 | 0.50 | 0.00 | 1 | 5.46 | 56 | 206095 | 109 | -4 | 51 | 23.90 | 451 | 1217 |
| | 0.50 | 0.50 | 0.00 | 1 | 4.57 | 72 | | 101 | 4 | 64 | 23.61 | 490 | 1543 |

remains same(i.e. 0.25) and as a consequence, RD of the 2nd product increases from 23 to 28 units whereas the RD of 1st item remains unaltered at 33 units. It can be seen from the values of $d_{11}s_1, d_{12}s_2, d_{21}s_2, d_{22}s_1$ as (=55, 33, 59, 22) and (=55, 33, 60, 27) for the cases -1P11 and -1P12 respectively that more customers of the 1st product adapt for the 2nd product i.e. **2nd product is more substitutable**. Same observation can be made from the cases -(1P14 and 1P15) and the cases -(1P17 and 1P18). The opposite observations are observed in cases -(1P32 and 1P33). Here, from the values of $d_{11}s_1, d_{12}s_2, d_{21}s_2, d_{22}s_1$ as (=55, 33, 60,55) and (=55, 53, 60, 55) for the cases -1P32 and -1P33 respectively and it can be said as before that **the 1st product is more substitutable**. Thus it can be concluded that **lower IODOS increase the corresponding RD and vice versa**. i.e. it makes the products more substitutable. If the mark-ups remain same, **lower IODOS fetches more profit**. This observation is

**Table** 4.14: Results (optimum quantities) for Model 4.2A with same mark-ups in each case

| Case | Responsiveness | | IODOS | Optimum results | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{11}$ / $d_{21}$ | $d_{12}$ / $d_{22}$ | $ds_{p_1}$ / $ds_{p_2}$ | $m_1^*$ / $m_2^*$ | $M^*$ | $P_1^*$ / $P_2^*$ | $Z_1^*$ | $s_1^*$ / $s_2^*$ | $Dp_1^*$ / $Dp_2^*$ | $D_1^*$ / $D_2^*$ | $t_1^*$ / $t_2^*$ | $N_1^*$ / $N_2^*$ | $Q_1^*$ / $Q_2^*$ |
| 1P11 | 0.50 | 0.25 | 0.25 | 3 | 5.50 | 122 | 109546 | 110 | -25 | 30 | 2.10 | 58 | 242 |
| | 0.45 | 0.20 | 0.25 | 3 | | 134 | | 121 | -32 | 28 | 1.73 | 37 | 221 |
| 1P12 | 0.50 | 0.25 | 0.25 | 3 | 5.50 | 122 | 121704 | 110 | -25 | 30 | 2.10 | 58 | 242 |
| | 0.45 | 0.25 | 0.20 | 3 | | 128 | | 121 | -27 | 33 | 2.20 | 53 | 264 |
| 1P13 | 0.50 | 0.15 | 0.35 | 3 | 5.32 | 132 | 85984 | 106 | -36 | 19 | 1.21 | 21 | 155 |
| | 0.45 | 0.20 | 0.25 | 3 | | 135 | | 117 | -31 | 29 | 1.78 | 40 | 229 |
| 1P14 | 0.50 | 0.25 | 0.25 | 3 | 5.45 | 123 | 96277 | 109 | -25 | 30 | 2.10 | 58 | 244 |
| | 0.50 | 0.20 | 0.30 | 2 | | 129 | | 120 | -38 | 22 | 2.15 | 52 | 262 |
| 1P15 | 0.50 | 0.25 | 0.25 | 3 | 5.45 | 123 | 108011 | 109 | -25 | 30 | 2.10 | 58 | 244 |
| | 0.50 | 0.25 | 0.25 | 3 | | 135 | | 120 | -33 | 27 | 1.70 | 36 | 218 |
| 1P16 | 0.50 | 0.15 | 0.35 | 3 | 4.94 | 139 | 74768 | 99 | -33 | 22 | 1.31 | 27 | 175 |
| | 0.50 | 0.20 | 0.30 | 3 | | 146 | | 109 | -35 | 25 | 1.46 | 29 | 203 |
| 1P17 | 0.40 | 0.25 | 0.15 | 3 | 6.06 | 100 | 139991 | 121 | -15 | 40 | 3.42 | 94 | 319 |
| | 0.45 | 0.20 | 0.25 | 3 | | 126 | | 133 | -36 | 24 | 1.62 | 31 | 194 |
| 1P18 | 0.40 | 0.25 | 0.15 | 3 | 6.06 | 101 | 155147 | 121 | -15 | 40 | 3.40 | 93 | 319 |
| | 0.45 | 0.25 | 0.20 | 3 | | 124 | | 133 | -30 | 30 | 2.07 | 47 | 243 |
| 1P19 | 0.40 | 0.15 | 0.25 | 3 | 6.05 | 119 | 109585 | 121 | -28 | 27 | 1.88 | 47 | 212 |
| | 0.45 | 0.20 | 0.25 | 3 | | 125 | | 133 | -36 | 24 | 1.63 | 31 | 194 |
| 1P110 | 0.50 | 0.25 | 0.25 | 3 | 5.49 | 123 | 173838 | 110 | -25 | 30 | 2.09 | 58 | 242 |
| | 0.40 | 0.40 | 0.00 | 1 | | 61 | | 121 | -5 | 55 | 23.75 | 420 | 1321 |
| 1P111 | 0.40 | 0.40 | 0.00 | 1 | 6.05 | 66 | 189661 | 121 | 5 | 60 | 23.85 | 532 | 1436 |
| | 0.45 | 0.20 | 0.25 | 3 | | 126 | | 133 | -36 | 24 | 1.61 | 31 | 195 |
| 1P21 | 0.50 | 0.50 | 0.00 | 1 | 5.45 | 67 | 159833 | 109 | 5 | 60 | 23.65 | 537 | 1451 |
| | 0.50 | 0.20 | 0.30 | 2 | | 129 | | 120 | -38 | 22 | 2.16 | 52 | 262 |
| 1P22 | 0.50 | 0.45 | 0.05 | 1 | 5.50 | 60 | 161339 | 110 | -1 | 54 | 23.68 | 484 | 1307 |
| | 0.45 | 0.20 | 0.25 | 3 | | 135 | | 121 | -32 | 28 | 1.72 | 37 | 221 |
| 1P23 | 0.50 | 0.45 | 0.05 | 1 | 5.49 | 62 | 225059 | 110 | -1 | 54 | 23.16 | 484 | 1307 |
| | 0.45 | 0.45 | 0.00 | 1 | | 60 | | 121 | -5 | 55 | 23.98 | 420 | 1321 |
| 1P31 | 0.50 | 0.25 | 0.25 | 3 | 5.45 | 124 | 171231 | 109 | -25 | 30 | 2.09 | 58 | 244 |
| | 0.50 | 0.50 | 0.00 | 1 | | 60 | | 120 | -5 | 55 | 23.93 | 416 | 1309 |
| 1P32 | 0.40 | 0.25 | 0.15 | 3 | 6.05 | 100 | 203436 | 121 | -15 | 40 | 3.41 | 93 | 319 |
| | 0.45 | 0.40 | 0.05 | 1 | | 54 | | 133 | -11 | 49 | 23.78 | 370 | 1164 |
| 1P33 | 0.40 | 0.40 | 0.00 | 1 | 6.05 | 66 | 253083 | 121 | 5 | 60 | 23.95 | 532 | 1436 |
| | 0.45 | 0.40 | 0.05 | 1 | | 55 | | 133 | -11 | 49 | 23.14 | 369 | 1164 |
| 1P41 | 0.50 | 0.50 | 0.00 | 1 | 5.45 | 67 | 234912 | 109 | 5 | 60 | 23.66 | 537 | 1451 |
| | 0.50 | 0.50 | 0.00 | 1 | | 60 | | 120 | -5 | 55 | 23.75 | 416 | 1309 |

also substantiated from the following cases. The cases -1P31 and -1P21 with respective IODOSs (0.25, 0.00) and (0.00, 0.30) furnish that the RDs of 1st and 2nd products in the case -1P21 respectively increase and decrease than those of the case -1P31. This is because the values of $D_i$'s change from 30 to 60 units for the 1st product and from 55 to 22 units for the 2nd product. But when the mark-ups are different in two cases, it is difficult to predict the behaviour of RDs. This can be seen from the cases -1P15 and -1P19. In these cases, both IODOSs are (0.25, 0.25), but the RDs are different and as a result, profits are different. This is because, in these cases, mark-ups are different. From this table, it is also observed that when IODOSs are high, the production rates for the products are high, but the production times are much small (cases -1P13,-1P14,-1P15 and -1P16). On the other hand, for the cases with low IODOSs, the product rates are much small but the production time are very high (cases -1P23, -1P33, -1P41, etc).

Total defective products are more for the cases with higher profits (cases -1P33, -1P41, -1P23, etc) and in these cases, salvage amounts contribute more than the rework costs.

(ii) We evaluate the profits of Model 4.2A with the same marks-up for the sale of the units and the optimum results are presented in Table 4.14. Here, the expression (4.29) is modified as $1 \leq M \leq Min[d_{10}/(d_{11}r_{m1}),\, d_{20}/(d_{21}r_{m2})]$. The observation made from Table 4.13 are also true with respect to Table 4.14. In addition it is observed in this table that lower IODOSs are related to less number of cycles required for the system. For example, the cases -1P23, -1P33 and -1P41 with corresponding IODOSs (0.05, 0.00), (0.00, 0.05) and (0.00, 0.00) have the single time cycle for both products. i.e. $m_1 = 1 = m_2$. The cases -1P22, -1P32, -1P110, -1P31, -1P111 and -1P21 with corresponding IODOSs (0.05, 0.25), (0.15, 0.05), (0.25, 0.00), (0.25, 0.00), (0.00, 0.25) and (0.00, 0.30)have the cycles for 1st and 2nd products as (1,3), (3, 1), (3, 1), (3, 1), (1, 3) and (1, 2) respectively. The other cases in Table 4.14 with higher IODOSs are having no. of cycles as (3, 3) for both products. Though in the cases -1P110 and -1P31, $d_{22}$ and $d_{21}$ are different, their IODOSs are same (0.25, 0.00) and all optimum parameters are almost same. Here the cases (-1P11, -1P12, -1P14, -1P15, -1P22, -1P110, -1P23, -1P31, -1P41) with same or almost same mark-ups have the different optimum parametric values with different IODOSs. Comparing the Tables 4.13 and 4.14, case -1P110 have the same IODOS (0.25, 0.00), but all other results are different including the cycle numbers as (3, 3) and (3, 1). The Table 4.14 with same mark-up fetches the lower profits in all cases than the corresponding profits in Table 4.13 with different mark-ups. The main reasons for this are that the mark-ups in Table 4.14 are selected following modified (4.29) as mentioned above.

**Effect of IODOSs (with respect to quality) on profit for Model 4.2B**

For the Model 4.2B, which is developed substitutability due to qualities, some experiments like Model 4.2A are performed and the optimum results are presented in Table 4.15. Here, mark-ups are same (5.00, 5.00), because mark-ups are related to selling prices only. For all cases, number of cycles are less, most of the cases are having only one cycle. With these values of $d_{i3}, d_{i4},\ i = 1, 2$; profits are more than those in Tables 4.13 and 4.14 except few cases. In all cases, quality level goes down to the lowest value as a cost is involved for the improvement of quality of the products. This is a part of UPC. Here, losses of sales are minimum, rates of production are moderate and durations of production are high in most of all the cases. For the cases -1q22 and -1q32 with IODOSs (5, 10) and (10, 5) respectively, there is a single number of cycle in both cases but the losses of sales due to qualities are just reversed as expected. All other observations made for the IODOSes with respect to prices in Tables 4.13 and 4.14 are also true in this case.

**Effect of IODOSs (with respect to both prices and qualities) on profits for Model 4.2C and its comparison with other models**

**Table** 4.15: Results (optimum quantities) for Model 4.2B with same mark-ups (5.0, 5.0)

| Case | Responsiveness | | IODOS | Optimum results | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{13}$ | $d_{14}$ | $ds_{q_1}$ | $m_1^*$ | $P_1^*$ | $q_1^*$ | $Z_2^*$ | $Dq_1^*$ | $D_1^*$ | $t_1^*$ | $N_1^*$ | $Q_1^*$ |
| | $d_{23}$ | $d_{24}$ | $ds_{q_2}$ | $m_2^*$ | $P_2^*$ | $q_2^*$ | | $Dq_2^*$ | $D_2^*$ | $t_2^*$ | $N_2^*$ | $Q_2^*$ |
| 1q11 | 20 | 30 | 10 | 1 | 55 | 0.50 | 173260 | -5 | 50 | 23.82 | 444 | 1199 |
| | 15 | 25 | 10 | 1 | 61 | 0.50 | | -5 | 55 | 23.93 | 419 | 1320 |
| 1q12 | 20 | 30 | 10 | 1 | 56 | 0.50 | 177847 | -5 | 50 | 23.40 | 444 | 1201 |
| | 20 | 25 | 05 | 1 | 63 | 0.50 | | -3 | 57 | 2386 | 438 | 1379 |
| 1q13 | 10 | 30 | 20 | 3 | 102 | 0.50 | 162794 | -10 | 45 | 3.82 | 109 | 360 |
| | 15 | 25 | 10 | 1 | 60 | 0.50 | | -5 | 55 | 23.95 | 419 | 1319 |
| 1q14 | 20 | 30 | 10 | 3 | 83 | 0.50 | 166088 | -5 | 50 | 5.20 | 130 | 400 |
| | 15 | 30 | 15 | 1 | 58 | 0.50 | | -8 | 52 | 23.72 | 400 | 1259 |
| 1q15 | 20 | 30 | 10 | 1 | 55 | 0.51 | 173201 | -5 | 50 | 23.91 | 445 | 1202 |
| | 20 | 30 | 10 | 1 | 60 | 0.50 | | -5 | 55 | 23.95 | 418 | 1316 |
| 1q16 | 10 | 30 | 20 | 3 | 95 | 0.50 | 157471 | -10 | 45 | 4.10 | 111 | 360 |
| | 15 | 30 | 15 | 1 | 58 | 0.50 | | -8 | 52 | 23.64 | 399 | 1257 |
| 1q17 | 20 | 30 | 10 | 1 | 55 | 0.50 | 161139 | -5 | 50 | 23.96 | 445 | 1202 |
| | 15 | 35 | 20 | 3 | 100 | 0.50 | | -10 | 50 | 4.29 | 105 | 398 |
| 1q18 | 20 | 30 | 10 | 1 | 55 | 0.50 | 168190 | -5 | 50 | 23.78 | 444 | 1200 |
| | 20 | 35 | 15 | 1 | 58 | 0.50 | | -8 | 52 | 23.89 | 400 | 1259 |
| 1q19 | 10 | 30 | 20 | 1 | 52 | 0.50 | 151841 | -10 | 45 | 22.88 | 399 | 1078 |
| | 15 | 35 | 20 | 3 | 99 | 0.50 | | -10 | 50 | 4.36 | 106 | 400 |
| 1q110 | 20 | 30 | 10 | 3 | 84 | 0.50 | 180863 | -5 | 50 | 5.14 | 129 | 400 |
| | 25 | 25 | 00 | 1 | 66 | 0.50 | | 0 | 60 | 23.85 | 457 | 1440 |
| 1q111 | 30 | 30 | 00 | 1 | 60 | 0.50 | 170117 | 0 | 55 | 23.94 | 489 | 1321 |
| | 15 | 35 | 20 | 3 | 99 | 0.50 | | -10 | 50 | 4.34 | 105 | 400 |
| 1q21 | 30 | 30 | 00 | 1 | 60 | 0.50 | 174700 | 0 | 55 | 23.88 | 489 | 1321 |
| | 15 | 30 | 15 | 3 | 90 | 0.50 | | -8 | 52 | 5.02 | 114 | 419 |
| 1q22 | 25 | 30 | 05 | 1 | 58 | 0.50 | 177396 | -2 | 53 | 23.90 | 467 | 1262 |
| | 15 | 25 | 10 | 1 | 61 | 0.50 | | -5 | 55 | 23.50 | 418 | 1318 |
| 1q23 | 25 | 30 | 05 | 1 | 57 | 0.50 | 187521 | -2 | 53 | 24.00 | 467 | 1261 |
| | 25 | 25 | 00 | 1 | 66 | 0.50 | | 0 | 60 | 23.86 | 457 | 1439 |
| 1q31 | 20 | 30 | 10 | 1 | 55 | 0.50 | 183048 | -5 | 50 | 23.98 | 443 | 1196 |
| | 30 | 30 | 00 | 1 | 66 | 0.50 | | 0 | 60 | 23.99 | 458 | 1443 |
| 1q32 | 20 | 30 | 10 | 1 | 55 | 0.50 | 177999 | -5 | 50 | 23.95 | 445 | 1201 |
| | 30 | 35 | 05 | 1 | 64 | 0.50 | | -3 | 57 | 23.69 | 438 | 1379 |
| 1q33 | 30 | 30 | 00 | 1 | 60 | 0.50 | 186827 | 0 | 55 | 23.99 | 486 | 1313 |
| | 30 | 35 | 05 | 1 | 64 | 0.51 | | -2 | 58 | 23.75 | 441 | 1387 |
| 1q41 | 30 | 30 | 00 | 1 | 56 | 0.50 | 188047 | -4 | 51 | 23.97 | 453 | 1222 |
| | 30 | 30 | 00 | 1 | 72 | 0.64 | | 4 | 64 | 23.49 | 488 | 1538 |

(i) In Model 4.2C, the substitutability among the items are due to both prices and qualities. By changing both these parameters, optimum parameters of the Model 4.2C are evaluated and presented in Table 4.16. Here, it is assumed that the customers who adopt for substitution on the basis of prices are not influenced by the quality and vice versa. Due to this assumption, $Dp_1$, $Dp_2$, $Dq_1$ and $Dq_2$ all are not positive and in such cases, there is loss of sales. Depending on the relations amongst the responsivenesses due to prices and qualities jointly, there will be in total 324 cases. Here results of some cases are presented in Table 4.16. In this table, profit of all cases are less than those of the corresponding cases in Table 4.13 in which only prices have been considered for substitution. This is because in the combined (both price and quality) effect on substitution, the effect of quality reduces the profit, whereas in Table 4.13, this effect is not considered. But, against the profit values in Table 4.15, no conclusion can be made as in this case(Table 4.15), mark-ups, instead of being calculated, have been assumed and taken as 5.00. For this reason, in some cases, profits in Table 4.16 are less than the corresponding profits in Table 4.15 and in few cases (cases -A13, -A14), it does not hold.

**Table** 4.16: Results (optimum quantities) for Model 4.2C (case-A)

| Case | Responsiveness | | | | Optimum results | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{11}$ / $d_{21}$ | $d_{12}$ / $d_{22}$ | $d_{13}$ / $d_{23}$ | $d_{14}$ / $d_{24}$ | $m_1^*$ / $m_2^*$ | $M_1^*$ / $M_2^*$ | $P_1^*$ / $P_2^*$ | $q_1^*$ / $q_2^*$ | $Z_3^*$ | $s_1^*$ / $s_2^*$ | $Dp_1^*$ / $Dp_2^*$ | $Dq_1^*$ / $Dq_2^*$ | $D_1^*$ / $D_2^*$ | $t_1^*$ / $t_2^*$ | $N_1^*$ / $N_2^*$ | $Q_1^*$ / $Q_2^*$ |
| A1 | 0.50 | 0.25 | 20 | 30 | 3 | 5.49 | 135 | 0.50 | 82174 | 110 | -23 | -5 | 27 | 1.66 | 43 | 214 |
| | 0.45 | 0.20 | 15 | 25 | 2 | 5.73 | 137 | 0.50 | | 126 | -35 | -5 | 20 | 1.85 | 43 | 241 |
| A2 | 0.50 | 0.25 | 10 | 30 | 3 | 5.45 | 140 | 0.50 | 73146 | 109 | -23 | -10 | 22 | 1.29 | 26 | 174 |
| | 0.45 | 0.20 | 15 | 25 | 2 | 5.68 | 137 | 0.50 | | 125 | -34 | -5 | 21 | 1.89 | 45 | 246 |
| A3 | 0.50 | 0.25 | 20 | 30 | 3 | 5.49 | 137 | 0.50 | 72018 | 110 | -25 | -5 | 25 | 1.55 | 38 | 203 |
| | 0.45 | 0.20 | 15 | 35 | 2 | 5.49 | 147 | 0.50 | | 121 | -32 | -10 | 17 | 1.49 | 30 | 210 |
| A4 | 050 | 0.25 | 10 | 30 | 3 | 5.43 | 139 | 0.50 | 62973 | 109 | -24 | -10 | 21 | 1.24 | 24 | 167 |
| | 0.45 | 0.20 | 15 | 35 | 2 | 5.50 | 146 | 0.50 | | 121 | -33 | -10 | 17 | 1.48 | 30 | 207 |
| A5 | 0.50 | 0.25 | 20 | 30 | 3 | 5.46 | 139 | 0.50 | 65808 | 109 | -26 | -5 | 24 | 1.42 | 33 | 189 |
| | 0.50 | 0.20 | 15 | 30 | 2 | 5.14 | 156 | 0.50 | | 113 | -35 | -7 | 18 | 1.43 | 29 | 213 |
| A6 | 0.50 | 0.25 | 10 | 30 | 3 | 5.21 | 146 | 0.50 | 56938 | 104 | -24 | -10 | 21 | 1.17 | 21 | 165 |
| | 0.50 | 0.20 | 15 | 30 | 2 | 5.05 | 158 | 0.50 | | 111 | -35 | -7 | 18 | 1.40 | 29 | 213 |
| A7 | 0.50 | 0.15 | 20 | 30 | 2 | 4.90 | 150 | 0.50 | 52315 | 98 | -34 | -5 | 16 | 1.36 | 32 | 197 |
| | 0.50 | 0.20 | 15 | 25 | 3 | 4.67 | 174 | 0.50 | | 103 | -32 | -5 | 23 | 1.09 | 15 | 186 |
| A8 | 0.50 | 0.25 | 10 | 30 | 3 | 5.11 | 148 | 0.50 | 52869 | 102 | -24 | -10 | 21 | 1.16 | 21 | 166 |
| | 0.50 | 0.20 | 15 | 35 | 2 | 4.90 | 162 | 0.50 | | 108 | -33 | -10 | 17 | 1.27 | 23 | 199 |
| A9 | 0.50 | 0.25 | 10 | 30 | 3 | 5.49 | 137 | 0.50 | 68026 | 110 | -24 | -10 | 21 | 1.27 | 25 | 167 |
| | 0.45 | 0.20 | 15 | 30 | 2 | 5.60 | 144 | 0.50 | | 123 | -34 | -7 | 19 | 1.66 | 37 | 228 |
| A10 | 0.50 | 0.25 | 20 | 30 | 3 | 5.50 | 136 | 0.50 | 77108 | 110 | -24 | -5 | 26 | 1.60 | 41 | 208 |
| | 0.45 | 0.20 | 15 | 30 | 2 | 5.64 | 140 | 0.50 | | 124 | -34 | -7 | 19 | 1.68 | 37 | 224 |
| A11 | 0.50 | 0.25 | 20 | 30 | 3 | 5.48 | 140 | 0.50 | 93572 | 110 | -22 | -9 | 24 | 1.46 | 35 | 196 |
| | 0.45 | 0.20 | 30 | 30 | 3 | 6.02 | 144 | 0.63 | | 132 | -38 | 4 | 26 | 1.52 | 31 | 209 |
| A12 | 0.50 | 0.25 | 30 | 30 | 2 | 5.49 | 135 | 0.50 | 86239 | 110 | -24 | 0 | 31 | 1.94 | 56 | 247 |
| | 0.45 | 0.20 | 15 | 30 | 2 | 5.60 | 144 | 0.50 | | 123 | -34 | -7 | 19 | 1.66 | 37 | 228 |
| A13 | 0.50 | 0.25 | 20 | 30 | 3 | 5.49 | 137 | 0.50 | 136769 | 110 | -25 | -5 | 25 | 1.53 | 37 | 200 |
| | 0.50 | 0.50 | 15 | 25 | 3 | 5.44 | 96 | 0.50 | | 120 | -5 | -5 | 50 | 4.48 | 106 | 400 |
| A14 | 0.50 | 0.50 | 20 | 30 | 1 | 5.49 | 61 | 0.50 | 128684 | 110 | 5 | -5 | 55 | 23.56 | 487 | 1316 |
| | 0.50 | 0.20 | 15 | 25 | 2 | 5.43 | 150 | 0.50 | | 119 | -38 | -5 | 17 | 1.43 | 28 | 206 |
| A15 | 0.50 | 0.25 | 20 | 30 | 3 | 5.50 | 136 | 0.50 | 133443 | 110 | -25 | -5 | 25 | 1.52 | 37 | 198 |
| | 0.50 | 0.50 | 15 | 30 | 1 | 5.42 | 53 | 0.50 | | 119 | -5 | -7 | 48 | 23.94 | 365 | 1149 |
| A16 | 0.50 | 0.50 | 20 | 30 | 1 | 5.49 | 59 | 0.50 | 123473 | 110 | 4 | -5 | 54 | 23.89 | 480 | 1295 |
| | 0.50 | 0.20 | 15 | 30 | 2 | 5.35 | 115 | 0.50 | | 118 | -37 | -7 | 16 | 1.25 | 21 | 187 |
| A17 | 0.50 | 0.25 | 20 | 30 | 3 | 5.43 | 140 | 0.50 | 79743 | 109 | -25 | -5 | 25 | 1.52 | 38 | 204 |
| | 0.50 | 0.20 | 30 | 30 | 3 | 5.42 | 149 | 0.50 | | 119 | -38 | 0 | 22 | 1.22 | 19 | 177 |
| A18 | 0.50 | 0.15 | 20 | 30 | 2 | 4.98 | 148 | 0.50 | 60485 | 100 | -34 | -5 | 16 | 1.37 | 32 | 196 |
| | 0.50 | 0.20 | 30 | 30 | 3 | 4.90 | 163 | 0.50 | | 108 | -34 | 0 | 26 | 1.32 | 26 | 208 |
| A19 | 0.50 | 0.25 | 20 | 30 | 3 | 5.47 | 140 | 0.50 | 148763 | 109 | -25 | -5 | 25 | 1.50 | 37 | 200 |
| | 0.50 | 0.50 | 30 | 30 | 1 | 5.41 | 63 | 0.50 | | 119 | -5 | 0 | 55 | 23.17 | 420 | 1325 |
| A20 | 0.50 | 0.50 | 20 | 30 | 1 | 5.49 | 61 | 0.50 | 138471 | 110 | 5 | -5 | 55 | 23.62 | 487 | 1315 |
| | 0.50 | 0.20 | 30 | 30 | 3 | 5.43 | 149 | 0.50 | | 119 | -38 | 0 | 22 | 1.23 | 19 | 178 |

(ii) The customers attracted or buck away due to prices may be attracted by the qualities of the products also. In this case, some of $Dp_1$, $Dp_2$, $Dq_1$ and $Dq_2$ may be positive. On the basis of this assumption, optimum results are presented in Table 4.17 taking both prices and qualities into consideration for substitution. For this reason, in all cases, the quality levels do not reach to the bottom level of their values (0.50).

**Table** 4.17: Results (optimum quantities) for Model 4.2C (case-B)

| Case | Responsiveness | | | | Optimum results | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ | $m_1^*$ | $M_1^*$ | $P_1^*$ | $q_1^*$ | $Z_3^*$ | $s_1^*$ | $Dp_1^*$ | $Dq_1^*$ | $D_1^*$ | $t_1^*$ | $N_1^*$ | $Q_1^*$ |
| | $d_{21}$ | $d_{22}$ | $d_{23}$ | $d_{24}$ | $m_2^*$ | $M_2^*$ | $P_2^*$ | $q_2^*$ | | $s_2^*$ | $Dp_2^*$ | $Dq_2^*$ | $D_2^*$ | $t_2^*$ | $N_2^*$ | $Q_2^*$ |
| B1 | 0.50 | 0.15 | 20 | 15 | 3 | 5.33 | 143 | 0.50 | 87721 | 107 | -33 | -4 | 18 | 1.04 | 15 | 145 |
| | 0.45 | 0.20 | 25 | 25 | 3 | 6.06 | 148 | 0.91 | | 133 | -39 | 10 | 31 | 1.79 | 44 | 252 |
| B2 | 0.50 | 0.15 | 35 | 20 | 3 | 5.50 | 129 | 0.87 | 91063 | 110 | -38 | 20 | 37 | 2.44 | 76 | 296 |
| | 0.45 | 0.20 | 15 | 20 | 3 | 5.03 | 163 | 0.50 | | 111 | -28 | -10 | 22 | 1.13 | 16 | 179 |
| B3 | 0.50 | 0.15 | 35 | 15 | 3 | 5.46 | 132 | 0.78 | 108212 | 109 | -35 | 14 | 34 | 2.17 | 66 | 271 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 6.01 | 154 | 0.92 | | 132 | -38 | 7 | 29 | 1.62 | 38 | 238 |
| | | | | | Results using Wolfram Mathematica 9.0 | | | | | | | | | | | |
| | | | | | 3 | 5.46 | 133 | 0.78 | 108198 | 109 | -35 | 13 | 34 | 2.16 | 65 | 270 |
| | | | | | 3 | 6.02 | 154 | 0.92 | | 132 | -38 | 7 | 29 | 1.62 | 38 | 238 |
| B4 | 0.50 | 0.50 | 20 | 20 | 1 | 5.48 | 72 | 0.50 | 157470 | 110 | 11 | 0 | 66 | 23.97 | 588 | 1587 |
| | 0.45 | 0.20 | 15 | 25 | 2 | 6.00 | 133 | 0.50 | | 132 | -37 | -5 | 18 | 1.66 | 34 | 211 |
| B5 | 0.50 | 0.50 | 20 | 15 | 1 | 5.49 | 70 | 0.50 | 178342 | 110 | 12 | -3 | 64 | 23.99 | 568 | 1533 |
| | 0.45 | 0.20 | 25 | 25 | 3 | 6.04 | 141 | 0.84 | | 133 | -39 | 9 | 30 | 1.84 | 44 | 246 |
| B6 | 0.50 | 0.50 | 35 | 20 | 1 | 5.48 | 84 | 0.58 | 178136 | 110 | 12 | 10 | 77 | 23.99 | 682 | 1841 |
| | 0.45 | 0.20 | 15 | 20 | 2 | 6.03 | 135 | 0.50 | | 133 | -38 | -4 | 18 | 1.71 | 37 | 219 |
| B7 | 0.50 | 0.50 | 35 | 15 | 1 | 5.48 | 83 | 0.62 | 198014 | 110 | 11 | 10 | 76 | 23.99 | 677 | 1828 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 6.03 | 146 | 0.80 | | 133 | -38 | 8 | 30 | 1.72 | 40 | 238 |
| B8 | 0.50 | 0.15 | 20 | 20 | 3 | 5.50 | 141 | 0.50 | 160170 | 110 | -35 | 0.0 | 20 | 1.16 | 20 | 159 |
| | 0.45 | 0.55 | 15 | 25 | 1 | 6.01 | 62 | 0.50 | | 132 | 1 | -5 | 56 | 23.70 | 426 | 1343 |
| B9 | 0.50 | 0.15 | 20 | 15 | 3 | 5.48 | 141 | 0.51 | 177995 | 110 | -35 | 0 | 20 | 1.17 | 121 | 160 |
| | 0.45 | 0.55 | 25 | 25 | 1 | 6.05 | 71 | 0.70 | | 133 | 0 | 5 | 65 | 23.98 | 496 | 1560 |
| B10 | 0.50 | 0.15 | 35 | 20 | 3 | 5.48 | 134 | 0.51 | 180587 | 110 | -35 | 8 | 28 | 1.75 | 47 | 223 |
| | 0.45 | 0.55 | 15 | 20 | 1 | 6.06 | 63 | 0.51 | | 133 | 0 | -3 | 57 | 23.95 | 440 | 1385 |
| B11 | 0.50 | 0.15 | 35 | 15 | 3 | 5.44 | 134 | 0.55 | 196960 | 109 | -34 | 9 | 30 | 1.87 | 53 | 237 |
| | 0.45 | 0.55 | 25 | 20 | 1 | 6.02 | 72 | 0.66 | | 132 | 0 | 6 | 66 | 23.95 | 502 | 1580 |
| B12 | 0.50 | 0.50 | 20 | 20 | 1 | 4.79 | 63 | 0.96 | 133937 | 96 | 0 | 0 | 55 | 23.13 | 491 | 1328 |
| | 0.45 | 0.55 | 15 | 25 | 1 | 4.42 | 63 | 0.97 | | 97 | 9.0 | -9 | 60 | 23.78 | 455 | 1432 |
| B13 | 0.50 | 0.50 | 20 | 15 | No Feasible Solution | | | | | | | | | | | |
| | 0.45 | 0.55 | 25 | 25 | | | | | | | | | | | | |
| | 0.50 | 0.50 | 20 | 15 | 1 | 5.33 | 63 | 0.98 | 152543 | 107 | -10 | 12 | 57 | 23.74 | 510 | 1376 |
| | 0.45 | 0.55 | 25 | 35 | 1 | 3.96 | 63 | 0.50 | | 87 | 19 | -22 | 57 | 23.95 | 439 | 1383 |
| B14 | 0.50 | 0.50 | 35 | 20 | No Feasible Solution | | | | | | | | | | | |
| | 0.45 | 0.55 | 15 | 20 | | | | | | | | | | | | |
| | 0.50 | 0.50 | 35 | 35 | 1 | 4.93 | 59 | 0.50 | 193061 | 99 | 12 | -15 | 52 | 23.35 | 463 | 1250 |
| | 0.45 | 0.55 | 15 | 20 | 1 | 5.58 | 69 | 0.93 | | 123 | -1 | 4 | 63 | 23.93 | 479 | 1510 |

In case -B3, contribution of qualities to the demand functions for two products are positive i.e. $Dq_1 = 14$ and $Dq_2 = 7$. This is because of the contributions of prices i.e.

$Dp_1 = -35$ and $Dp_2 = -38$ (these buck-aways customers due to prices again go back to the items due to qualities and for that $Dq_1 = 14$, $Dq_2 = 7$ ) and as a result, resultant demands are $D_1 = 34$, $D_2 = 29$ which are less than the prime demands $d_{10} = 55$, $d_{20} = 60$ units. It is interesting to note that in case -B12, the contribution of prices and qualities are reversed (+9 and -9 due to prices and qualities respectively) and sum total of both contribution is zero. As a result, the RDs are equal to the prime / base demands (55 and 60 units). In this case, qualities are almost equal to 1. In the cases -B4, -B5, -B6, -B7, -B9 and -B11, one of the RDs is more than the corresponding base demand but the sum total of RDs is less than that of base demands of two products. This condition holds good due to assumption -(xi). Due to this assumption, for some values of $d_{ij}, i = 1, 2; j = 1, 2, 3, 4$, there is no feasible solution for same cases (-B13 and -B14).

#### Comparison of optimum results by two methods

Optimum results of the system for different cases with different parametric values have been evaluated by the proposed Genetic Algorithm (cf. Table 4.17). To verify the results, the problem given by case-B3 of the Model 4.2C have been solved by Wolfram Mathematica 9.0 (Random Search Method) and the results are presented in Table 4.17. It is seen that the proposed GA gives better result than the Mathematica.

#### Effect of learning parameter on Model 4.2C (case-B3)

To evaluate the effect of learning parameter introduced in the set-up and maintenance costs, we took the most general Model 4.2C. The Model 4.2C was evaluated with and without learning effects in the above costs and the optimum results are presented in Table 4.18. It is observed that as expected, profits are less in all cases without learning effects.

**Table** 4.18: Results without learning effect for Model 4.2C (case-B3)

| Case | Responsiveness | | | | Optimum results | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_{11}$ / $d_{21}$ | $d_{12}$ / $d_{22}$ | $d_{13}$ / $d_{23}$ | $d_{14}$ / $d_{24}$ | $m_1^*$ / $m_2^*$ | $M_1^*$ / $M_2^*$ | $P_1^*$ / $P_2^*$ | $q_1^*$ / $q_2^*$ | $Z_3^*$ | $s_1^*$ / $s_2^*$ | $Dp_1^*$ / $Dp_2^*$ | $Dq_1^*$ / $Dq_2^*$ | $D_1^*$ / $D_2^*$ | $t_1^*$ / $t_2^*$ | $N_1^*$ / $N_2^*$ | $Q_1^*$ / $Q_2^*$ |
| | | | | | Results without learning effect on set-up cost | | | | | | | | | | | |
| B3 | 0.50 | 0.15 | 35 | 15 | 3 | 5.48 | 1323 | 0.78 | 107448 | 110 | -35 | 14 | 34 | 2.17 | 66 | 270 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 6.01 | 156 | 0.91 | | 132 | -38 | 7 | 29 | 1.59 | 37 | 237 |
| | | | | | Results without learning effect on maintenance cost | | | | | | | | | | | |
| | | | | | 3 | 5.49 | 128 | 0.80 | 107635 | 110 | -35 | 15 | 35 | 2.28 | 68 | 278 |
| | | | | | 3 | 5.94 | 154 | 0.91 | | 131 | -37 | 7 | 30 | 1.62 | 38 | 238 |
| | | | | | Results without learning effect on both set-up and maintenance cost | | | | | | | | | | | |
| | | | | | 3 | 5.49 | 131 | 0.79 | 106762 | 110 | -35 | 14 | 34 | 2.20 | 67 | 272 |
| | | | | | 3 | 6.00 | 155 | 0.91 | | 132 | -38 | 7 | 29 | 1.60 | 37 | 236 |

Table 4.19: Results of Model 4.2C (case-B3) without $p_m$ reduction

| | Responsiveness | | | | Optimum results | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_m$ | $d_{11}$ / $d_{21}$ | $d_{12}$ / $d_{22}$ | $d_{13}$ / $d_{23}$ | $d_{14}$ / $d_{24}$ | $m_1^*$ / $m_2^*$ | $M_1^*$ / $M_2^*$ | $P_1^*$ / $P_2^*$ | $q_1^*$ / $q_2^*$ | $Z_3^*$ | $s_1^*$ / $s_2^*$ | $Dp_1^*$ / $Dp_2^*$ | $Dq_1^*$ / $Dq_2^*$ | $D_1^*$ / $D_2^*$ | $t_1^*$ / $t_2^*$ | $N_1^*$ / $N_2^*$ | $Q_1^*$ / $Q_2^*$ |
| 0.90 | 0.50 | 0.15 | 35 | 15 | 3 | 5.44 | 125 | 0.78 | 107884 | 109 | -35 | 14 | 34 | 2.31 | 68 | 273 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 5.92 | 168 | 0.89 | | 130 | -37 | 7 | 30 | 1.48 | 34 | 239 |
| 0.70 | 0.50 | 0.15 | 35 | 15 | 3 | 5.48 | 116 | 0.78 | 108149 | 110 | -35 | 14 | 34 | 2.46 | 70 | 269 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 5.92 | 188 | 0.89 | | 130 | -37 | 7 | 30 | 1.32 | 3 | 240 |
| 0.50 | 0.50 | 0.15 | 35 | 15 | 3 | 5.48 | 136 | 0.78 | 108186 | 110 | -35 | 14 | 34 | 2.10 | 64 | 269 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 5.92 | 169 | 0.92 | | 130 | -37 | 7 | 30 | 1.48 | 34 | 240 |
| 0.30 | 0.50 | 0.15 | 35 | 15 | 3 | 5.48 | 119 | 0.80 | 108164 | 110 | -35 | 15 | 35 | 2.44 | 71 | 274 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 5.90 | 156 | 0.89 | | 130 | -37 | 6 | 29 | 1.60 | 37 | 238 |
| 0.10 | 0.50 | 0.15 | 35 | 15 | 3 | 5.48 | 121 | 0.80 | 108152 | 110 | -35 | 15 | 35 | 2.42 | 70 | 274 |
| | 0.45 | 0.20 | 25 | 20 | 3 | 5.91 | 149 | 0.89 | | 130 | -37 | 6 | 29 | 1.67 | 39 | 238 |

### Effect of $p_m$ reduction on optimum profit for Model 4.2C in case B3

It is difficult to choose the system parameters of a GA. Normally, probability of mutation for a problem is assumed to be low ($\leq 0.50$). We performed the optimization of the case-B3 of Model 4.2C with different values $p_m$ from 0.90 to 0.10 (cf. Table 4.19). It is seen that as $p_m$ reduces from 0.90 to 0.10 by 0.20, the optimum value of $Z_3$ (objective) increases initially and becomes maximum at $p_m = 0.50$ and then decreases. Thus the optimum value of $p_m$ for the present model is 0.50. It may be noted that the optimum results are obtained with a particular value of $p_m$ throughout the optimization of the system. But, in our proposed GA, the value of $p_m$ has been reduced at each iteration of the execution of the optimization process from 0.90 to 0.01 and it yields better result (cf. Table 4.17) than the result obtained with fixed $p_m$ (cf. Table 4.19).

### Pictorial representations of optimum results for Model 4.2C

(i) Considering the case-B3 from Table 4.17 of the Model 4.2C, optimum profit $Z_3^* = 108212$ units is obtained for $m_1^* = 3$, $m_2^* = 3$, $M_1^* = 5.46$, $M_2^* = 6.01$, $P_1^* = 132$, $P_2^* = 154$, $q_1^* = 0.78$ and $q_2^* = 0.92$. Taking number of cycles as variable and others by their optimum values, the total profit for the Model 4.2C is plotted in Fig.4.7 against the different values of $m_1$ and $m_2$. In the similar fashion Fig. 4.8 is plotted against the mark-ups ($M_1$, $M_2$) of two products. In this figure, it is noted that global optimum values ($Z_1^*$=112341, $M_1^*$=6.44, $M_2^*$=6.46 ) lie on Feasible Unconstrained Solution Space(FUSS) but within Feasible Constrained Solution Space(FCSS) region $Z_1^*$=108212 units is the local optimum for $M_1^*$=5.46, $M_2^*$=6.01. Figs. 4.9 and 4.10 are plotted for the total profit against production rates ($P_1$, $P_2$) and quality levels ($q_1$, $q_2$) as variables and others as constant by their optimum values respectively. These figures show that the objective function is concave.

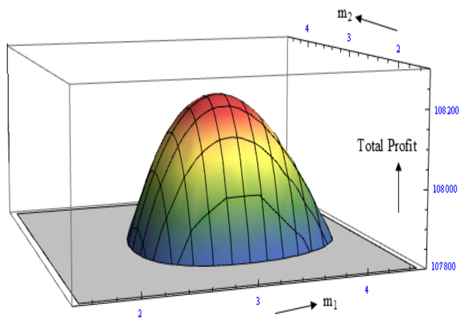



**Figure** 4.7: Total profit against the number of cycles.

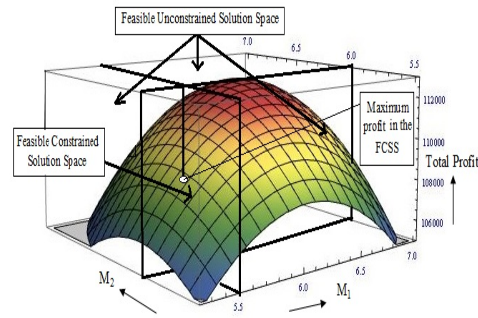



**Figure** 4.8: Total profit against the Mark-ups.

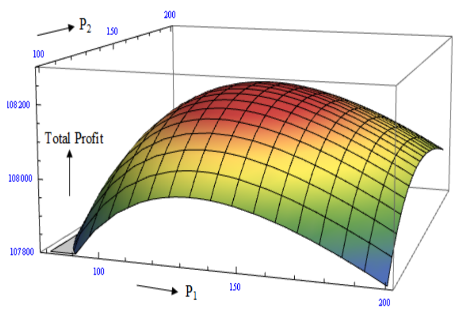



**Figure** 4.9: Total profit against the Production Rates.

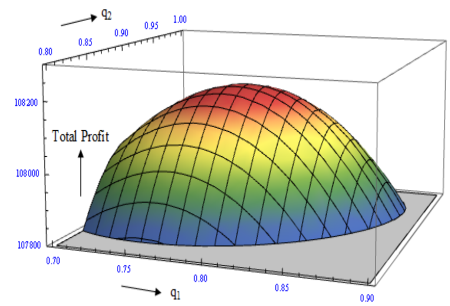



**Figure** 4.10: Total profit against the Quality levels.

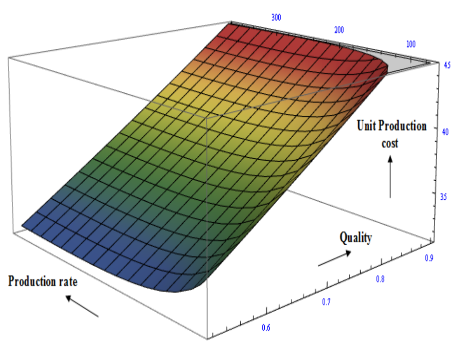



**Figure** 4.11: UPC against the production rate and quality level of a product.

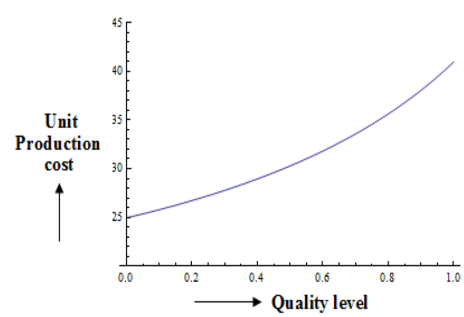



**Figure** 4.12: UPC against the quality level of a product.

(ii) Fig. 4.11 is obtained by plotting the UPC $C_1(P_1, q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20P_1^{\frac{1}{2}}$ against the different values of production rate and quality of product-1. This UPC is a convex function against production rate only (cf. Figs. 4.13 and 4.14).

(iii) Fig. 4.12 represents UPC $C_1(q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20\sqrt{P_1}$ against the quality of product-1 when production rate $P_1 = 272$. This figure suggests that UPC is increasing function with respect to quality.

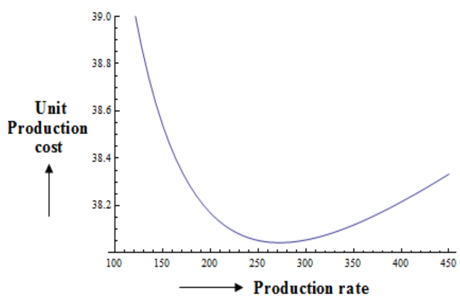



**Figure** 4.13: UPC including quality improvement cost against production rate.

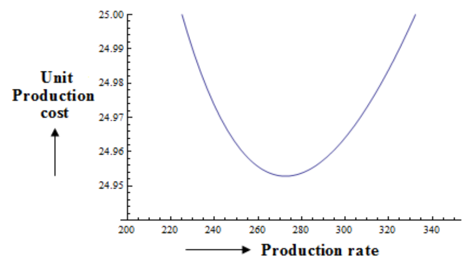



**Figure** 4.14: UPC without quality improvement cost against production rate.

(iv) Fig. 4.13 and Fig. 4.14 represent the UPC against the production rate with and without quality improvement cost in the UPC respectively. Here $C_1(P_1) = 20 + \frac{450}{P_1} + \frac{8.00*q_1}{1-0.50*q_1} + 0.20P_1^{\frac{1}{2}}$ (taking $q_1 = 0.90$) for Fig. 4.13 and $C_1(P_1) = 20 + \frac{450}{P_1} + 0.20P_1^{\frac{1}{2}}$ for Fig. 4.14 are considered. In these figures, UPC is a convex function with respect to production rate. $C_1(P_1^*)$ have the minimum values 38.04 and 24.95 at $P_1^* = 272$ for the above two cases respectively. Though normally it is assumed that minimum value of UPC $(C_1(P_1))$ leads to maximum profit, in this case, the above value of $P_1^*$ is not equal to the corresponding optimal values obtained by optimizing total profits. (example-$P_1^* = 129$ for the case -B2 in Table 4.17).

### 4.3.11 Practical Implication

In a sugar mill where two types of sugar- good quality sugar and low quality sugar are produced or in the rice mills where two types of rice- fine quality and raw quality rices are produced, the products are substitutable and the customers (i.e. retailers) very often change the brand on the basis of prices and qualities. This analysis will be helpful for the production managers of the said mills to fix the optimum prices, qualities, production rates, etc for maximum profit. The responsiveness parameters $(d_{11}, d_{12}, d_{13}, d_{14}, d_{21}, d_{22}, d_{23}, d_{24})$ to prices and qualities can be obtained from the experts or may be calculated from past data. The present problem can also be applied for the managers of big departmental stores like Big Bazar, Pentaloons, etc, where several substitutable products are sold. In these stores also, customers of one brand very often change over to other brand. Here, the replenishment may be considered as procurements/ productions with infinite rate.

## 4.4 Model-4.3 : Optimum ordering for two substitute items in a news-vendor management with promotional effort on demand using Rough Age based Genetic Algorithm[3]

### 4.4.1 Assumptions and Notations

The models are formulated with the help of following assumptions.

(i) Demands of the items are assumed to be uniform and random.

(ii) Shortages are allowed

(iii) Promotional effort does not affect the basic demand but affects the effort-induced demand.

(iv) Items are substitutable. Substitution are made only when a item is exhausted and other excessed.

(v) Selling price of substitutable items is lower than the normal selling price.

The models are formulated with the help of following notations.

| | |
|---|---|
| $i$ | A subscript identifying item's index (i=1, 2). |
| $Q_i$ | Order quantity (units/cycle) (DV). |
| $P_i$ | Purchasing cost per unit item (\$/unit). |
| $\xi_i D_i$ | Random continuous demand which follows uniform distribution. Here, $D_i$ is the base demand (units/cycle) and $\xi_i \geq 1$ is the promotional effort (unit/effort) (DV). The expected effort induced demand is E[$\xi_i D_i$]. |
| $S_i$ | Selling price per unit item (\$/unit). |
| $Ss_i$ | Selling price per unit item for substitution (\$/unit). |
| $Sc_i$ | Shortage cost per unit item (\$/unit). |
| $H_i$ | Holding cost per unit item per unit time (\$/unit/time unit). |
| $T_i$ | Time when inventory reaches to zero (time unit). |
| $T$ | Inventory cycle time for both items (time unit). |
| $P_c$ | Purchasing cost of the items (\$). |
| $P_m$ | Promotional cost (\$). |
| $B_1$ | Budget amount on purchasing cost (\$). |
| $B_2$ | Budget amount on promotional cost (\$). |
| $B$ | Total budget amount on both purchasing and promotional cost (\$). |
| $PF_j^{(R_k)}$ | Profit function of $j^{th}$ scenario in the $R_k$ region (\$). |
| $EPF$ | Expected total profit (\$). |

[3]This model has been communicated in **International Journal of Industrial Engineering-Theory, Application and Practice**, Univ. Cincinnati Industrial Engineering

### 4.4.2 Mathematical Model Development

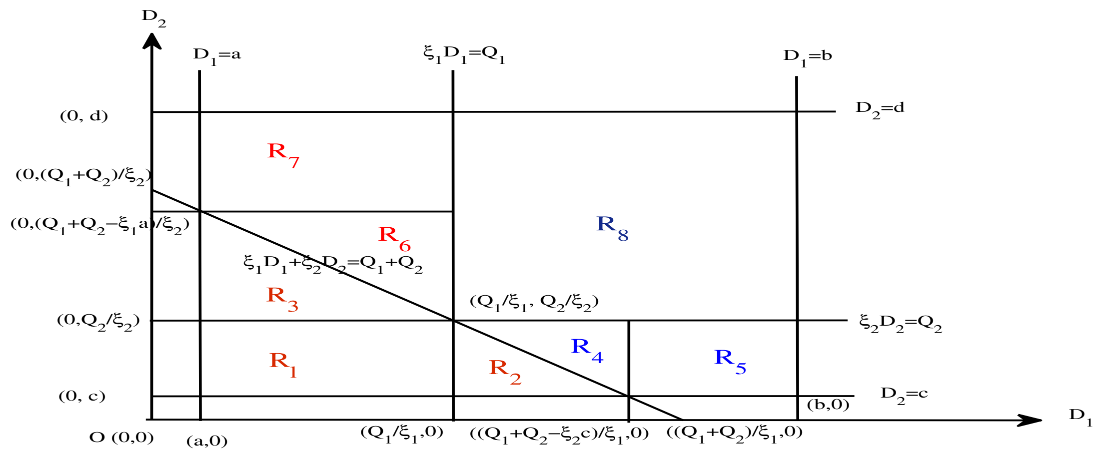



**Figure** 4.15: Eight regions giving rise to the total expected profit function.

Let the ordered quantity for $i^{th}$ item be $Q_i$. Then total purchasing cost is $P_c = P_1Q_1+P_2Q_2$ and promotional cost is $P_m = M_1(\xi_1 - 1)^2 D_1^{\alpha_1} + M_2(\xi_2 - 1)^2 D_2^{\alpha_2}$ where, $M_i > 0$ and $\alpha_i$'s are constants. A higher value of $M_i$ or $\alpha_i$ means a greater difficulty in attracting customers. This cost is convex in promotional effort $\xi_i$, by which the basic demand $D_i$ is enhanced.

The derivation of the profit function is based on considering the six possible events in eight regions $R_1$-$R_8$ depicted in Fig. 4.15. In the first of three regions $R_1$-$R_3$, the system is with excess quantity and in the last five regions $R_4$-$R_8$, the system is under shortage. i.e. $Q_1 + Q_2 - (\xi_1 D_1 + \xi_2 D_2) > 0\, and\, < 0$ respectively. In these regions, the possible mutually exclusive six events are described as follows:

$R_1$ : Demands for both items are less than their stock levels, $(\xi_i D_i < Q_i,\ for\, i = 1, 2)$ .

$R_2$ : Demand for item 1 exceeds its inventory level and the excess demand can be fully satisfied by item 2, $(\xi_1 D_1 > Q_1, \xi_2 D_2 < Q_2, Q_1 + Q_2 - (\xi_1 D_1 + \xi_2 D_2) > 0)$.

$R_3$ : Demand for item 2 exceeds its inventory level and the excess demand can be fully satisfied by item 1, $(\xi_1 D_1 < Q_1, \xi_2 D_2 > Q_2, Q_1 + Q_2 - (\xi_1 D_1 + \xi_2 D_2) > 0)$.

$R_4$ & $R_5$ : Demand for item 1 exceeds its inventory level but the excess demand can only be partially satisfied by item 2, $(\xi_1 D_1 > Q_1, \xi_2 D_2 < Q_2, Q_1 + Q_2 - (\xi_1 D_1 + \xi_2 D_2) < 0)$.

$R_6$ & $R_7$ : Demand for item 2 exceeds its inventory level but the excess demand can only be partially satisfied by item 1, $(\xi_1 D_1 < Q_1, \xi_2 D_2 > Q_2, Q_1 + Q_2 - (\xi_1 D_1 + \xi_2 D_2) < 0)$.

$R_8$ : Demands for both items are greater than their inventory levels $(\xi_i D_i > Q_i,\ for\, i = 1, 2)$.

There will be six scenarios for six different events depicted in Figs. 4.16 - 4.19. Profit function for all scenarios are obtained subtracting cost price from selling price, where sales process includes the normal and substitution salvage sale and cost price appear due to the purchasing and holding of the items and promotional effort. The purchasing cost of retailers is the sale's proceeds of the wholesaler and promotional cost is spent by retailer due to more sale of the items. According to our assumption, the six different events/scenarios (depicted in Figs. 4.16, 4.17, 4.18 and 4.19) are observed in the above described eight regions. These six events are mutually exclusive. The profit function obtained for these different scenarios are as follows:

**Scenario 1:** In the region $R_1$=$\{\xi_1 D_1 < Q_1, \xi_2 D_2 < Q_2\}$ both the items are in excess and the excess units are lost, i.e. retailer does not get any sales revenue for these excess units (cf. Fig. 4.16). In this case, the profit function is

$$\begin{aligned} PF_1^{(R_1)} &= S_1\xi_1 D_1 + S_2\xi_2 D_2 - \frac{H_1}{2}(2Q_1 - \xi_1 D_1)T - \frac{H_2}{2}(2Q_2 - \xi_2 D_2)T - P_c - P_m \\ &= (S_1 + \frac{H_1}{2}T)\xi_1 D_1 + (S_2 + \frac{H_2}{2}T)\xi_2 D_2 - (H_1 Q_1 T + H_2 Q_2 T + P_c + P_m) \end{aligned} \quad (4.35)$$

[See the Appendix A.1 for details calculation.]

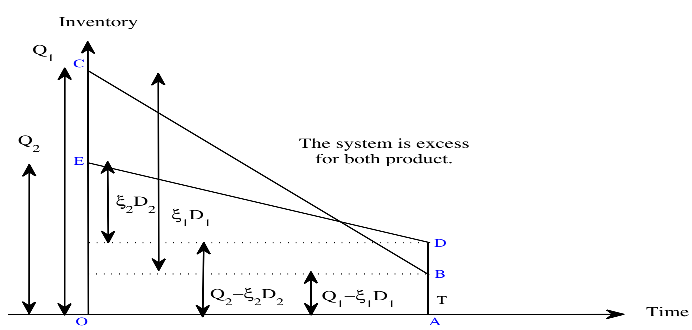



**Figure** 4.16: Scenario-1 : The system is excess for both item.

**Scenario 2:** In the region $R_2$=$\{\xi_1 D_1 > Q_1, \xi_2 D_2 < Q_2$ and $Q_2 - \xi_2 D_2 > \xi_1 D_1 - Q_1\}$, the excess units of item 2 is sufficient to fill up shortages of the item 1 and after fulfilling the shortages at substitution price, the rest excess units of item 2 are considered as lost sale (cf. Fig. 4.17). Therefore, profit function in this case is

$$PF_2^{(R_2)} = S_1 Q_1 + S_2\xi_2 D_2 + Ss_2(\xi_1 D_1 - Q_1) - \frac{H_1 Q_1^2}{2\xi_1 D_1}T - \frac{H_2}{2}(2Q_2 - \xi_2 D_2)T$$

$$+\frac{H_2(\xi_1 D_1 - Q_1)^2}{2\xi_1 D_1}T - P_c - P_m \text{ [See the Appendix A.2 for details calculation.]}$$
$$= (S_2 + \frac{H_2}{2}T)\xi_2 D_2 + (Ss_2 + \frac{H_2}{2}T)\xi_1 D_1 + (S_1 Q_1 - Ss_2 Q_1$$
$$- H_2 Q_2 T - H_2 Q_1 T - P_c - P_m) + \frac{1}{2\xi_1 D_1}(H_2 - H_1)Q_1^2 T \quad (4.36)$$

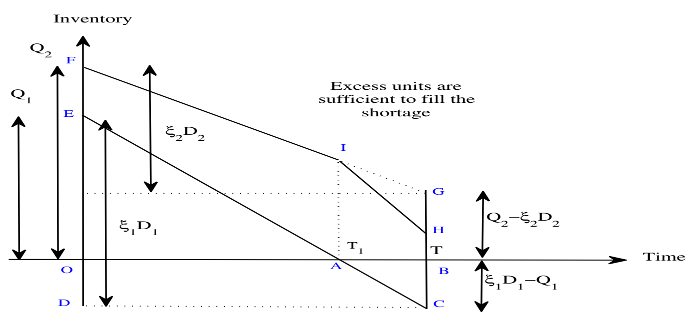



Figure 4.17: Scenario-2 and 3 : Excess units are sufficient to fill the shortage.

**Scenario 3:** In the region $R_3$=$\{\xi_1 D_1 < Q_1, \xi_2 D_2 > Q_2$ and $Q_1 - \xi_1 D_1 > \xi_2 D_2 - Q_2\}$, the excess units of item 1 is sufficient to fill the shortages of item 2 (it is like Scenario 2). Thus profit is

$$PF_3^{(R_3)} = S_1\xi_1 D_1 + S_2 Q_2 + Ss_1(\xi_2 D_2 - Q_2) - \frac{H_1}{2}(2Q_1 - \xi_1 D_1)T - \frac{H_2 Q_2^2}{2\xi_2 D_2}T$$
$$+\frac{H_1(\xi_2 D_2 - Q_2)^2}{2\xi_2 D_2}T - P_c - P_m$$
$$= (S_1 + \frac{H_1}{2}T)\xi_1 D_1 + (Ss_1 + \frac{H_1}{2}T)\xi_2 D_2 + (S_2 Q_2 - Ss_1 Q_2$$
$$-H_1 Q_1 T - H_1 Q_2 T - P_c - P_m) + \frac{1}{2\xi_2 D_2}(H_1 - H_2)Q_2^2 T \quad (4.37)$$

**Scenario 4:** In the regions $R_4$ and $R_5$=$\{\xi_1 D_1 > Q_1, \xi_2 D_2 < Q_2$ and $Q_2 - \xi_2 D_2 < \xi_1 D_1 - Q_1\}$, the excess units of item 2 is not sufficient to fill the shortages of item 1. i.e., the system is under shortage for item 1 and after fulfilling the customer's demand of item 2, it is substituted for shortages of item 1 and rate of shortages will remain same as before in spite of this partial substitution. [cf. Fig. 4.18]. Against these assumptions the profit function is,

$$PF_4^{(R_4\&R_5)} = S_2\xi_2D_2 + S_1Q_1 + Ss_2(Q_2-\xi_2D_2) - \frac{H_2}{2}(2Q_2-\xi_2D_2)T - \frac{H_1Q_1^2}{2\xi_1D_1}T$$
$$+\frac{H_2}{2}\left[\frac{(\xi_1D_1-Q_1)^2}{\xi_1D_1} - \frac{\{(\xi_1D_1-Q_1)-(Q_2-\xi_2D_2)\}^2}{\xi_1D_1}\right]T$$
$$-Sc_1\{(\xi_1D_1-Q_1)-(Q_2-\xi_2D_2)\} - P_c - P_m \qquad \text{[See Appendix A.3 for details]}$$
$$=(S_2-\frac{H_2}{2}T - Sc_1 - Ss_2)\xi_2D_2 - Sc_1\xi_1D_1 + (Ss_2Q_2 + S_1Q_1 + Sc_1(Q_2+Q_1) - P_c$$
$$-P_m) - \frac{1}{\xi_1D_1}(\frac{H_2}{2}Q_2^2T + H_2Q_2Q_1T + \frac{H_1}{2}Q_1^2T) + \frac{\xi_2D_2}{\xi_1D_1}H_2(Q_2+Q_1)T - \frac{\xi_2^2D_2^2}{2\xi_1D_1}H_2T \tag{4.38}$$

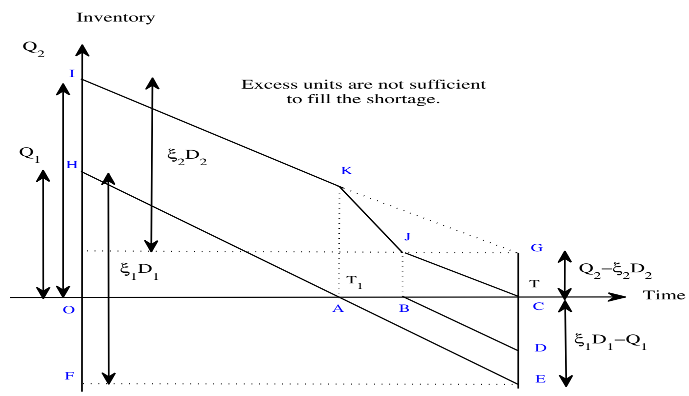



**Figure** 4.18: Scenario-4 and 5 : Excess units are not sufficient to fill the shortage.

**Scenario 5:** In the regions $R_6$ and $R_7$=$\{\xi_1D_1 < Q_1, \xi_2D_2 > Q_2$ and $Q_1-\xi_1D_1 < \xi_2D_2 - Q_2\}$, the excess units of item 1 is not sufficient to fill the shortages of item 2. The profit is like scenario 4. i.e.,

$$PF_5^{(R_6\&R_7)} = S_1\xi_1D_1 + S_2Q_2 + Ss_1(Q_1-\xi_1D_1) - \frac{H_1}{2}(2Q_1-\xi_1D_1)T - \frac{H_2Q_2^2}{2\xi_2D_2}T$$
$$+\frac{H_1}{2}\left[\frac{(\xi_2D_2-Q_2)^2}{\xi_2D_2} - \frac{\{(\xi_2D_2-Q_2)-(Q_1-\xi_1D_1)\}^2}{\xi_2D_2}\right]T$$
$$-Sc_2\{(\xi_2D_2-Q_2)-(Q_1-\xi_1D_1)\} - P_c - P_m$$
$$=(S_1-\frac{H_1}{2}T - Sc_2 - Ss_1)\xi_1D_1 - Sc_2\xi_2D_2 + (Ss_1Q_1 + S_2Q_2 + Sc_2(Q_1+Q_2) - P_c$$
$$-P_m) - \frac{1}{\xi_2D_2}(\frac{H_1}{2}Q_1^2T + H_1Q_1Q_2T + \frac{H_2}{2}Q_2^2T) + \frac{\xi_1D_1}{\xi_2D_2}H_1(Q_1+Q_2)T - \frac{\xi_1^2D_1^2}{2\xi_2D_2}H_1T \tag{4.39}$$

**Scenario 6:** For the $R_8$=$\{\xi_1 D_1 > Q_1,\, \xi_2 D_2 > Q_2\}$, the system is under shortages for both items (cf. Fig. 4.19). In this scenario, profit is obtained as

$$
\begin{aligned}
PF_6^{(R_8)} &= S_1Q_1 + S_2Q_2 - \frac{H_1Q_1^2}{2\xi_1D_1}T - \frac{H_2Q_2^2}{2\xi_2D_2}T \\
&- Sc_1(\xi_1D_1 - Q_1) - Sc_2(\xi_2D_2 - Q_2) - P_c - P_m \\
&= (S_1Q_1 + S_2Q_2 + Sc_1Q_1 + Sc_2Q_2 - P_c - P_m) - Sc_1\xi_1D_1 - Sc_2\xi_2D_2 \\
&- \frac{H_1}{2\xi_1D_1}Q_1^2T - \frac{H_2}{2\xi_2D_2}Q_2^2T \qquad \text{[cf. Appendix A.4 for details.]} \qquad (4.40)
\end{aligned}
$$

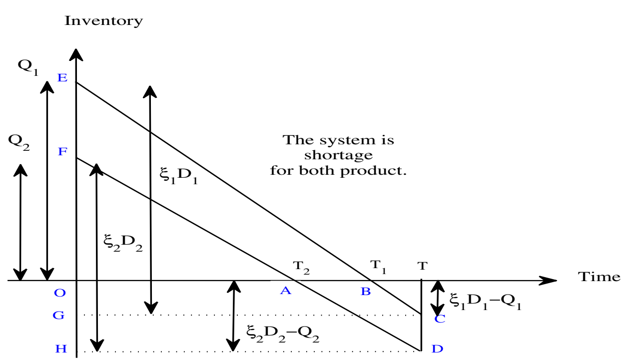



**Figure** 4.19: Scenario-6 : The system is under shortage for both item.

**Expected profit**

If the joint probability density function of demand is of the form

$$
f(D_1, D_2) = \begin{cases} \frac{1}{(b-a)(d-c)}, & \text{for } a \le D_1 \le b,\ c \le D_2 \le d \\ 0, & \text{elsewhere.} \end{cases}
$$

then the expression for the total expected profit is obtained by integrating the corresponding profit expressions over their respective regions. The expected profit $EPF(Q_1, Q_2, \xi_1, \xi_2)$ for the retailer is

$$
\begin{aligned}
EPF(Q_1, Q_2, \xi_1, \xi_2) = \Bigg[ & \int_a^{Q_1/\xi_1}\int_c^{Q_2/\xi_2} PF_1^{(R_1)} + \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2c)/\xi_1}\int_c^{(Q_1+Q_2-\xi_1D_1)/\xi_2} PF_2^{(R_2)} \\
& + \int_a^{Q_1/\xi_1}\int_{Q_2/\xi_2}^{(Q_1+Q_2-\xi_1D_1)/\xi_2} PF_3^{(R_3)} + \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2c)/\xi_1}\int_{(Q_1+Q_2-\xi_1D_1)/\xi_2}^{Q_2/\xi_2} PF_4^{(R_4)} \\
& + \int_{(Q_1+Q_2-\xi_2c)/\xi_1}^{b}\int_c^{Q_2/\xi_2} PF_4^{(R_5)} + \int_{(Q_1+Q_2-\xi_2D_2)/\xi_1}^{Q_1/\xi_1}\int_{Q_2/\xi_2}^{(Q_1+Q_2-\xi_1a)/\xi_2} PF_5^{(R_6)} \\
& + \int_a^{Q_1/\xi_1}\int_{(Q_1+Q_2-\xi_1a)/\xi_2}^{d} PF_5^{(R_7)} + \int_{Q_1/\xi_1}^{b}\int_{Q_2/\xi_2}^{d} PF_6^{(R_8)} \Bigg] f(D_1, D_2)d(D_1)d(D_2)
\end{aligned}
$$

$$= \frac{1}{(b-a)(d-c)}\Bigg[(S_1+\frac{H_1}{2}T)\xi_1 I_1^{(R_1)} + (S_2+\frac{H_2}{2}T)\xi_2 I_2^{(R_1)} - (H_1Q_1T+H_2Q_2T)I_3^{(R_1)}$$

$$+(S_2+\frac{H_2}{2}T)\xi_2 I_2^{(R_2)} + (Ss_2+\frac{H_2}{2}T)\xi_1 I_1^{(R_2)} + (S_1Q_1 - Ss_2Q_1 - H_2Q_2T - H_2Q_1T)I_3^{(R_2)}$$
$$+\frac{1}{2\xi_1}(H_2-H_1)Q_1^2 T I_4^{(R_2)}$$

$$+(S_1+\frac{H_1}{2}T)\xi_1 I_1^{(R_3)} + (Ss_1+\frac{H_1}{2}T)\xi_2 I_2^{(R_3)} + (S_2Q_2 - Ss_1Q_2 - H_1Q_1T - H_1Q_2T)I_3^{(R_3)}$$
$$+\frac{1}{2\xi_2}(H_1-H_2)Q_2^2 T I_4^{(R_3)}$$

$$+(S_2-\frac{H_2}{2}T - Sc_1 - Ss_2)\xi_2 I_2^{(R_4)} - Sc_1\xi_1 I_1^{(R_4)} + (Ss_2Q_2 + S_1Q_1 + Sc_1(Q_2+Q_1))I_3^{(R_4)}$$
$$-\frac{1}{\xi_1}(\frac{H_2}{2}Q_2^2T + H_2Q_2Q_1T + \frac{H_1}{2}Q_1^2T)I_4^{(R_4)} + \frac{\xi_2}{\xi_1}H_2(Q_2+Q_1)TI_5^{(R_4)} - \frac{\xi_2^2}{2\xi_1}H_2TI_6^{(R_4)}$$

$$+(S_2-\frac{H_2}{2}T - Sc_1 - Ss_2)\xi_2 I_2^{(R_5)} - Sc_1\xi_1 I_1^{(R_5)} + (Ss_2Q_2 + S_1Q_1 + Sc_1(Q_2+Q_1))I_3^{(R_5)}$$
$$-\frac{1}{\xi_1}(\frac{H_2}{2}Q_2^2T + H_2Q_2Q_1T + \frac{H_1}{2}Q_1^2T)I_4^{(R_5)} + \frac{\xi_2}{\xi_1}H_2(Q_2+Q_1)TI_5^{(R_5)} - \frac{\xi_2^2}{2\xi_1}H_2TI_6^{(R_5)}$$

$$+(S_1-\frac{H_1}{2}T - Sc_2 - Ss_1)\xi_1 I_1^{(R_6)} - Sc_2\xi_2 I_2^{(R_6)} + (Ss_1Q_1 + S_2Q_2 + Sc_2(Q_1+Q_2))I_3^{(R_6)}$$
$$-\frac{1}{\xi_2}(\frac{H_1}{2}Q_1^2T + H_1Q_1Q_2T + \frac{H_2}{2}Q_2^2T)I_4^{(R_6)} + \frac{\xi_1}{\xi_2}H_1(Q_1+Q_2)TI_5^{(R_6)} - \frac{\xi_1^2}{2\xi_2}H_1TI_6^{(R_6)}$$

$$+(S_1-\frac{H_1}{2}T - Sc_2 - Ss_1)\xi_1 I_1^{(R_7)} - Sc_2\xi_2 I_2^{(R_7)} + (Ss_1Q_1 + S_2Q_2 + Sc_2(Q_1+Q_2))I_3^{(R_7)}$$
$$-\frac{1}{\xi_2}(\frac{H_1}{2}Q_1^2T + H_1Q_1Q_2T + \frac{H_2}{2}Q_2^2T)I_4^{(R_7)} + \frac{\xi_1}{\xi_2}H_1(Q_1+Q_2)TI_5^{(R_7)} - \frac{\xi_1^2}{2\xi_2}H_1TI_6^{(R_7)}$$

$$\begin{aligned}&+(S_1Q_1 + S_2Q_2 + Sc_1Q_1 + Sc_2Q_2)I_3^{(R_8)} - Sc_1\xi_1 I_1^{(R_8)} - Sc_2\xi_2 I_2^{(R_8)}\\ &-\frac{H_1}{2\xi_1}Q_1^2TI_4^{(R_8)} - \frac{H_2}{2\xi_2}Q_2^2TI_5^{(R_8)}\Bigg]\\ &-P_1Q_1 - P_2Q_2 - [\frac{M_1(\xi_1-1)^2(b^{\alpha_1+1}-a^{\alpha_1+1})}{(\alpha_1+1)(b-a)} + \frac{M_2(\xi_2-1)^2(d^{\alpha_2+1}-c^{\alpha_2+1})}{(\alpha_2+1)(d-c)}]\end{aligned} \tag{4.41}$$

where $I_1$, $I_2$, $I_3$, $I_4$, $I_5$ and $I_6$ with their respective regions are calculated in Appendix A.5. Mathematical expressions of the models under different conditions and constraints are as follows:

**Model 4.3A1 : With promotional effort and with budget constraint on purchasing cost**

$$\begin{aligned}&\text{Maximize } EPF(Q_1,Q_2,\xi_1,\xi_2),\\ &\text{Subject to} \quad P_c \text{ i.e. } P_1Q_1 + P_2Q_2 \le B_1,\\ &\qquad\qquad Q_1 \ge a,\ Q_2 \ge c \quad \text{and} \quad \xi_1,\ \xi_2 > 1.\end{aligned} \tag{4.42}$$

**Model 4.3A2 : With promotional effort and with budget constraint on promotional cost**

$$\begin{aligned}&\text{Maximize } EPF(Q_1,Q_2,\xi_1,\xi_2),\\ &\text{Subject to} \quad P_m \text{ i.e. } \frac{M_1(\xi_1-1)^2(b^{\alpha_1+1}-a^{\alpha_1+1})}{(\alpha_1+1)(b-a)} + \frac{M_2(\xi_2-1)^2(d^{\alpha_2+1}-c^{\alpha_2+1})}{(\alpha_2+1)(d-c)} \le B_2,\\ &\qquad\qquad Q_1 \ge a,\ Q_2 \ge c \quad \text{and} \quad \xi_1,\ \xi_2 > 1.\end{aligned} \tag{4.43}$$

**Model 4.3A3 : With promotional effort and with budget constraint on both purchasing and promotional cost**

$$\begin{aligned}&\text{Maximize } EPF(Q_1, Q_2, \xi_1, \xi_2),\\ &\text{Subject to} \quad P_c + P_m \leq B,\\ &\qquad\qquad Q_1 \geq a,\ Q_2 \geq c \quad \text{and} \quad \xi_1,\ \xi_2 > 1.\end{aligned} \tag{4.44}$$

**Model 4.3A4 : With promotional effort and without budget constraint**

$$\begin{aligned}&\text{Maximize } EPF(Q_1, Q_2, \xi_1, \xi_2),\\ &\text{Subject to} \quad Q_1 \geq a,\ Q_2 \geq c \quad \text{and} \quad \xi_1,\ \xi_2 > 1.\end{aligned} \tag{4.45}$$

**Model 4.3B1 : Without promotional effort and with budget constraint on purchasing cost**

$$\begin{aligned}&\text{Maximize } EPF(Q_1, Q_2),\\ &\text{Subject to} \quad P_1Q_1 + P_2Q_2 \leq B_1,\\ &\qquad Q_1 \geq a,\ Q_2 \geq c \text{ and } \xi_1 = \xi_2 = 1.\end{aligned} \tag{4.46}$$

**Model 4.3B2 : Without promotional effort and without budget constraint**

$$\begin{aligned}&\text{Maximize } EPF(Q_1, Q_2),\\ &\text{Subject to} \quad Q_1 \geq a,\ Q_2 \geq c \text{ and } \xi_1 = \xi_2 = 1.\end{aligned} \tag{4.47}$$

### 4.4.3 Solutions methodology

To solve above maximization problems, we have applied the Rough Age based Genetic Algorithm (RAGA) as described in § 2.3.2.3.

### 4.4.4 Numerical Experiments and Results

In this section, we perform two real life experiments to illustrate the proposed models.

Experiment 1: A fish merchant at the railway city market, Kharagpur, India sells two types of fishes- "Rui" and "Katla" for the customers every morning from 6:00 A.M. to 12:00 Noon. The random demand of rui and katla are same and follows an uniform distribution within 25 and 900 kilogram (kg). He purchases rui at the rate \$5.20/kg and katla at \$5.25/kg from a wholesaler and sells the said fishes to the customers at \$8.65/kg and \$8.70/kg respectively. If one type of fish is exhausted, shortage fishes are substituted by the available one at lower prices at \$7.65/kg and \$7.70/kg respectively. He offers lower prices for substitution to the customer for maintaining the goodwill. The shortages cost for the said fishes are at \$0.55/kg and \$0.65/kg respectively. To preserve the fishes with a temporary cooling system, \$0.50 and \$0.60 are spent per kg per hour. Again, he spends some money to boost the demand. From previous experience, the parameters of promotional cost are as

$M_1 = 25.0$, $M_2 = 30.0$, $\alpha_1 = 0.75$ and $\alpha_2 = 0.60$. Now the decisions to be taken by the merchant are (i) How much does he order for these fishes to get maximum profit? (ii) How much to spend on promotional effort to boost the demand in the market?

Fish merchant also wants to make the above decisions under the budget constraints on purchase or promotional cost or jointly on both. These decisions are derived for the above input data using RAGA and the optimal results for the models 4.3A1 - 4.3A4, 4.3B1 and 4.3B2 are presented in Table 4.20.

**Table** 4.20: Optimum results of Experiment 1 for different models

| Budget constraint | $Model$ | Budget | $Q_1^*$ | $Q_2^*$ | $\xi_1^*$ | $\xi_2^*$ | $EPF^*$ | $P_c^*$ | $P_m^*$ |
|---|---|---|---|---|---|---|---|---|---|
| Without | 4.3A4 | | 498.62 | 426.38 | 1.3148 | 1.6392 | 951.69 | 4831.32 | 704.20 |
| | 4.3B2 | | 347.64 | 241.80 | | | 732.54 | 3077.20 | |
| On purchasing cost $(B_1)$ | 4.3A1 | 4500 | 474.43 | 387.22 | 1.2545 | 1.5061 | 873.48 | | 447.94 |
| | 4.3A1 | 3000 | 330.25 | 244.32 | 1.0682 | 1.1083 | 759.49 | | 24.59 |
| | 4.3A1 | 2276 | | | ....Infeasible Solution.... | | | | |
| | 4.3B1 | 2800 | 313.42 | 222.89 | | | 720.15 | | |
| | 4.3B1 | 2000 | 217.60 | 165.41 | | | 554.07 | | |
| | 4.3B1 | 812 | | | ....Infeasible Solution.... | | | | |
| On Promotional Cost $(B_2)$ | 4.3A2 | 600 | 502.55 | 422.44 | 1.2910 | 1.5896 | 947.24 | 4831.12 | |
| | 4.3A2 | 500 | 507.14 | 417.85 | 1.2661 | 1.5378 | 933.05 | 4830.89 | |
| | 4.3A2 | 400 | 512.81 | 412.18 | 1.2385 | 1.4804 | 905.96 | 4830.61 | |
| On both Purchasing and Promotional Cost $(B)$ | 4.3A3 | 5100 | 492.60 | 399.67 | 1.2512 | 1.5029 | 897.24 | 4659.85 | 440.15 |
| | 4.3A3 | 3500 | 375.70 | 281.85 | 1.1055 | 1.1869 | 794.02 | 3433.38 | 66.62 |
| | 4.3A3 | 2276 | | | ....Infeasible Solution.... | | | | |

**E**xperiment 2: In the above mentioned problem (Experiment 1), let the fish merchant sales the substitute items with current price instead of reduced price. i.e., $Ss_1 = S_1 = 8.65$, $Ss_2 = S_2 = 8.70$. Other input data remain same. The optimal results for the models 4.3A1 - 4.3A4, 4.3B1 and 4.3B2 are presented in Table 4.21.

### 4.4.5 Discussion

- Tables 4.20 and 4.21 furnish the following optimal policies for the fish merchant.

(i) Promotional effort gives more expected profit due to the increase of demands, henceforth order should be placed for more quantities of the items (By comparing the Models 4.3A1, 4.3A2, 4.3A3, 4.3A4 with Models 4.3B1, 4.3B2).

(ii) As expected, it is also seen that, more investment on purchasing cost yields more profit.

(iii) As expected, the expected profit is maximum when the system has no constraint and expected profit decreases with the increasing of the number of constraints, which suggests that, more number of constraints reduce the feasibility of the region.

**Table** 4.21: Optimum results of Experiment 2 for different models

| Budget constraint | $Model$ | Budget | $Q_1^*$ | $Q_2^*$ | $\xi_1^*$ | $\xi_2^*$ | $EPF^*$ | $P_c^*$ | $P_m^*$ |
|---|---|---|---|---|---|---|---|---|---|
| Without | 4.3A4 | | 562.53 | 362.46 | 1.3146 | 1.6586 | 1073.69 | 4828.12 | 32.52 |
| | 4.3B2 | | 395.09 | 211.75 | - | - | 788.05 | 3166.20 | - |
| On purchasing cost $(B_1)$ | 4.3A1 | 4500 | 534.95 | 327.27 | 1.2523 | 1.5192 | 974.25 | | 460.50 |
| | 4.3A1 | 3000 | 357.20 | 217.62 | 1.0645 | 1.1038 | 808.69 | | 22.30 |
| | 4.3A1 | 2319 | | | ....Infeasible Solution.... | | | | |
| | 4.3B1 | 2800 | 338.53 | 198.01 | - | - | 766.34 | | - |
| | 4.3B1 | 2000 | 227.62 | 155.50 | - | - | 578.80 | | - |
| | 4.3B1 | 807 | | | ....Infeasible Solution.... | | | | |
| On Promotional Cost $(B_2)$ | 4.3A2 | 600 | 570.38 | 354.61 | 1.2849 | 1.5958 | 1066.73 | 4827.73 | |
| | 4.3A2 | 500 | 578.12 | 346.87 | 1.2603 | 1.5437 | 1050.35 | 4827.31 | |
| | 4.3A2 | 400 | 588.20 | 386.79 | 1.2330 | 1.4861 | 1021.01 | 4826.84 | |
| On both Purchasing and Promotional Cost $(B)$ | 4.3A3 | 5100 | 562.73 | 331.87 | 1.2430 | 1.5038 | 1005.53 | 4668.56 | 431.44 |
| | 4.3A3 | 3500 | 413.47 | 245.33 | 1.1001 | 1.1821 | 854.72 | 3438.08 | 61.92 |
| | 4.3A3 | 2319 | | | ....Infeasible Solution.... | | | | |

(iv) Here, for the given set of input data, the promotional budget is less important than the budget of purchasing.

(v) The infeasible solution also reveals the need of minimum budget on purchasing cost or both purchasing and promotional cost. Therefore, a minimum budget is required to introduce a new business.

(vi) Comparing the results between Tables 4.20 and 4.21, the profits of Experiment 2 are higher than those of Experiment 1 for all models because of higher selling prices of the substitute items.

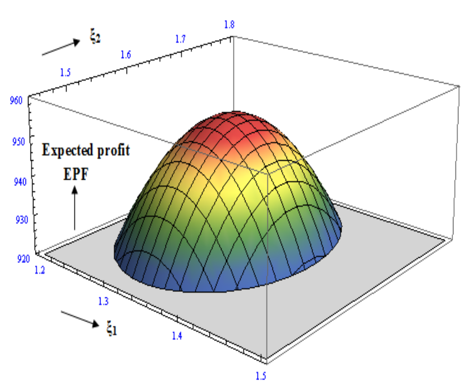



**Figure** 4.20: Expected profit against promotional efforts for Model 4.3A4

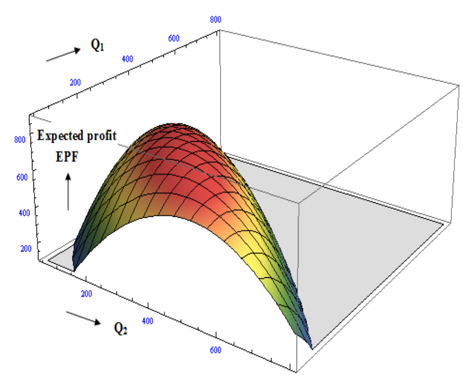



**Figure** 4.21: Expected profit against order quantities for Model 4.3B2

• Considering the order quantities as constant at optimum values ($Q_1^* = 498.62$, $Q_2^* = 426.38$) for Model 4.3A4, the expected profit function is drawn as in Fig. 4.20 with respect

to promotional efforts $\xi_1$ and $\xi_2$. The Fig. 4.20 depicts that the expected profit function is concave against promotional efforts.
• Fig. 4.21 represents the concavity property of expected profit with respect to order quantities $(Q_1, Q_2)$ for Model 4.3B2.

**Table** 4.22: Optimum results of Model 4.3A4 for different values of $M_1$ and $M_2$

| $M_1$ | $M_2$ | $Q_1^*$ | $Q_2^*$ | $\xi_1^*$ | $\xi_2^*$ | $EPF^*$ | $P_m^*$ |
|---|---|---|---|---|---|---|---|
| 10 | 10 | 473.58 | 451.41 | 1.7399 | 2.7846 | 2141.94 | 1736.64 |
| 10 | 20 | 496.29 | 428.70 | 1.7489 | 1.9262 | 1514.66 | 1191.87 |
| 10 | 30 | 521.48 | 403.51 | 1.7553 | 1.6244 | 1295.09 | 993.88 |
| 20 | 10 | 465.79 | 459.20 | 1.3779 | 2.7857 | 1872.77 | 1485.72 |
| 20 | 20 | 477.76 | 447.23 | 1.3840 | 1.9387 | 1237.63 | 953.24 |
| 20 | 30 | 505.33 | 419.66 | 1.3905 | 1.6357 | 1010.95 | 754.30 |
| 30 | 10 | 462.12 | 462.85 | 1.2545 | 2.7853 | 1780.06 | 1397.15 |
| 30 | 20 | 467.29 | 457.70 | 1.2585 | 1.9444 | 1141.95 | 870.34 |
| 30 | 30 | 493.34 | 431.65 | 1.2637 | 1.6419 | 911.67 | 670.32 |
| 40 | 10 | 460.12 | 464.12 | 1.1920 | 2.7848 | 1732.94 | 1351.37 |
| 40 | 20 | 461.19 | 463.80 | 1.1949 | 1.9475 | 1093.38 | 828.00 |
| 40 | 30 | 485.78 | 439.31 | 1.1991 | 1.6455 | 861.06 | 627.38 |

### 4.4.6 Sensitivity Analysis

Considering the different values of the promotional cost parameters $M_1, M_2$ and $\alpha_1, \alpha_2$ for the Model 4.3A4, the optimal results are presented in Table 4.22 and Table 4.23 respectively.

(i) From Table 4.22, we conclude that as $M_i$'s increase, $\xi_i$'s decrease i.e. total demand $\xi_i D_i$ decreases, therefore order quantities decrease and finally expected profit is decreased.

(ii) The increasing values of $\alpha_i$'s decrease the values of $\xi_i$'s, hence the quantities $Q_i$'s also decrease. Therefore, expected profit decreases (cf. Table 4.23).

## 4.5 Conclusion

This investigation is proposed for the substitute products in random environment using various inventory models including :

• The first model proposed here derives production policies of multi-item imperfect production inventory systems with complementary, substitute or independent items under

Table 4.23: Optimum results of Model 4.3A4 for different values of $\alpha_1$ and $\alpha_2$

| $\alpha_1$ | $\alpha_2$ | $Q_1^*$ | $Q_2^*$ | $\xi_1^*$ | $\xi_2^*$ | $EPF^*$ | $P_m^*$ |
|---|---|---|---|---|---|---|---|
| 0.30 | 0.30 | 443.13 | 481.86 | 5.4693 | 4.6527 | 6049.76 | 5426.24 |
| 0.30 | 0.50 | 410.59 | 514.02 | 5.4048 | 2.1507 | 4256.07 | 3739.15 |
| 0.30 | 0.70 | 398.81 | 526.18 | 5.3662 | 1.3598 | 3641.59 | 3148.93 |
| 0.50 | 0.30 | 495.77 | 429.22 | 2.3724 | 4.5881 | 3860.09 | 3293.91 |
| 0.50 | 0.50 | 489.27 | 435.72 | 2.3625 | 2.1264 | 2125.11 | 1729.92 |
| 0.50 | 0.70 | 530.97 | 394.02 | 2.3597 | 1.3394 | 1553.05 | 1190.42 |
| 0.70 | 0.30 | 511.28 | 413.71 | 1.4196 | 4.5515 | 3145.63 | 2593.22 |
| 0.70 | 0.50 | 470.22 | 454.77 | 1.4152 | 2.1442 | 1419.13 | 1108.61 |
| 0.70 | 0.70 | 435.56 | 291.70 | 1.2091 | 1.1548 | 789.35 | 128.03 |
| 0.90 | 0.30 | 516.97 | 408.02 | 1.2449 | 4.5322 | 2915.85 | 2353.83 |
| 0.90 | 0.50 | 448.70 | 476.26 | 1.1217 | 2.1545 | 1197.16 | 910.09 |
| 0.90 | 0.70 | 365.34 | 266.77 | 1.0403 | 1.0952 | 755.73 | 29.25 |

out-of-control state production and budget constraints. Here, both UPC and quality of an item are production dependent. For the first time, these types of production-inventory systems for complementary and substitute items with resource constraint has been formulated and solved via GRG technique. The optimum production policies for the imperfect production system and decision on the prices, qualities and greenness of the complementary and substitute items while assuming that the customers' behaviour arbitrarily can be outlined.

• The second model proposed here shows the production cum sale of imperfect products substitute on the basis of the prices and qualities considered over a random time horizon and the optimum prices, qualities, production rates and cycles are determined so that total profit is maximum. Joint and separate effects of price and quality on the substitution are taken into account in this production-marketing system. Here it is assumed that price and quality of a product are independent. The virgin ideas presented in this paper are (i) imperfect production cum sale of two substitute products with the provision of repair of imperfect products, (ii) substitutability of the products on the basis of selling prices and qualities separately and jointly, (iii) allotment of some expenditure against improvement of quality and environment protection and (iv) uncertain planning horizon with normal distribution. This model upholds the following factors: (i) reliability for the production process, (ii) more substitute products and (iii) supply-chain system incorporating retailers and customers. New investigation also can be performed introducing price discounts (AUD/IQD) on the substitute products, taking imprecise time horizon, etc.

• Finally in the third model, it is assumed that the news-vendor management system for two substitute items with uniform stochastic demand in open market and a cost was spent for both items to promote the items for more sale. We further have found the maximum expected profit for two models- with and without promotional effort under with and without budget constraint using a Rough Age based Genetic Algorithm developed for single objective optimization. Moreover, the discussion of the models and results give a way-out for the introduction of new business system for such type of substitute items having random demand like daily newspapers, X-mass trees, fishes, etc. The models can be illustrated with other types of probability distribution functions. Moreover, the present solution procedure also can be applied to other types of EOQ, EPQ and supply-chain models.

# Chapter 5

# Inventory Problems with Carbon Emission in Fuzzy Environment

## 5.1 Introduction

In recent years, besides economic criteria (cost minimization or profit maximization), the social impact of production system for a long period has become a major topic in research and industrial application. The environmental concerns are becoming increasingly relevant for firms due to more stringent various rules and regulations on carbon particles imposed by government and growing customer's awareness to the social welfare. Recently, in 2015 United Nations Climate Change Conference held in Paris, France, 196 parties attending it made a global agreement on the reduction of green house gas emission. To improve the environmental performance as well as economic criteria, it has become important and challenging for firms worldwide to incorporate CE management (minimization of emission) into their production system and business decisions. Many countries have enacted legislations to mitigate global warming by reducing CE. A number of countries have imposed carbon taxes for every unit of CE whatever be the amount- low or high. Some countries have introduced carbon cap and trade scheme for their industries. Emission trading i.e. cap and trade is a market based approach used to control pollution by providing economic incentives for achieving reductions in the emission of pollutants. "Cap" means a legal limit on the quantity of carbon which an industry can emit each year. Now two cases may arise- (i) if the emission is exceeded by cap, additional amount over the cap may be bought from trade market or paid penalty to the government; (ii) firms that keep their emission levels below their allotted level (i.e. cap), may sell their surplus permit to other firms or use them to offset excess emission in other parts of their facilities. Again, some manufacturing firms don't like to be involved with carbon cap and trade system. They prefer to produce the exact amount of the product so that emission is almost equal to the allowed cap. In this case, the cap is used as a constraint. These four cases are considered in the present investigation.

Moreover, in a production system, all produced units are not perfect and the production of imperfect units commences after the passage of some time from the start of the production. This instant is uncertain- may be taken as fuzzy. This leads the mathematical formulation of the system to FDE and the objective function (maximization / minimization) also becomes fuzzy. The solution of FDE and optimization of objective function require some special techniques. Till now, the carbon management in fuzzy production system has not been reported in the literature. The above realistic phenomena promoted us to take up the present investigation. Here production policies for maximum profit for a firm in a developed or developing country producing imperfect units with time-dependent fuzzy defective rate under the four possible CE regulations are outlined and illustrated.

## 5.2 Literature Review

There are some fuzzy inventory/fuzzy production inventory models taking demand [71], deterioration [66], etc. as fuzzy. Normally, these problems have been formulated with crisp differential equation and solved using extension principle and/ or defuzzification techniques. But this is not correct. If a parameter involving in the differential equation is fuzzy, the said differential equation becomes a FDE which has been ignored by previous several researchers. A few researchers solved the FDEs using different techniques during last two decades [4, 26, 27, 40, 176]. Buckley and Feuring [27] presented a new solution technique to solve the governing FDE. Using this technique, the $\alpha$-cut of the fuzzy average profit i.e. the equivalent crisp multi-objective problem (MOP) is obtained. Recently some investigators [92, 178] have tackled this type of inventory models formulating as FDE models and solved using $\alpha$-cuts technique. The earlier formulation and analysis furnish approximate solutions where as FDE formulation gives better results. In the present investigation, FDE technique following Buckley and Feuring [27] is implemented. Recently some works on stability, consistency and convergence of FDE [83] are also available in the literature.

The concept of an intuitionistic fuzzy set (IFS) can be seen as an alternative approach to define a fuzzy set in cases where available information is not sufficient for the definition of an imprecise concept by means of conventional fuzzy sets. Therefore it is expected that IFS could be used to simulate the human decision-making process and any activity requiring human expertise and knowledge which are inevitably imprecise or not totally reliable. Here the degrees of rejection and satisfaction are such that the sum of both values is always less than one. Angelov [6] implemented the optimization in an intuitionistic fuzzy environment. Wei [266] used the maximizing deviation methods to solve the intuitionistic fuzzy multiple attribute decision making problems with incomplete weight information. Pramanik and Roy [204] solved a vector optimization problem using an intuitionistic fuzzy goal programming. A transportation model was solved by Jana and Roy [116] using multi-objective intuitionistic fuzzy linear programming. The advantage of the IFOT is twofold: It gives the richest apparatus for formulation of optimization problems and the solutions of intuitionistic fuzzy optimization problems can satisfy the objective(s) in greater degree compared to the analogous fuzzy optimization problem and the crisp one. Recently, Dutta and Guha [81] presented an approach to derive the weights of the decision criteria or alternatives in multi-attribute decision making with goals described by IFNs. In the proposed method, aggregation of fuzzy goals and application of the max-min principle lead to a non-linear optimization problem whose solution gives the desired crisp priorities. Till now, very few might [38] have used IFOT for the optimization of fuzzy decision making problem with constraint, specially in the area of fuzzy production system with CE regulations.

Due to complex nature of the objective functions, it is difficult to find the optimal strategy of the reduced problems using traditional optimization techniques. GAs are extensively used to face these types of situations during the last decades by researchers (cf. Mondal and Maiti [179], Roy *et al.* [222], Maiti [161], Sawyerr *et al.* [235], Thakur *et al.* [255]). Here, a GA with rough age based criteria is used to reproduce a new chromosome in crossover level.

In the literature of models with imperfect production process (cf. § 1.3.2), several EPL models are available for imperfect units. Taleizadeh et al. [249] presented an algorithm to determine the optimum values of manufacturing lot size and unit price to have maximum profit for an EPQ inventory model with reworkable defective items under a given multi-shipment policy. Recently, Manna et al. [172] developed an EPQ model with promotional demand in random planning horizon with reworking of the imperfect items including waste disposal and vending the units. Wang and Tang [262] investigated the EPQ model characterizing the set-up cost, holding cost and elapsed time as fuzzy variables. Till now, none has considered the production-inventory system with time-dependent imprecise i.e. fuzzy defective rate.

In the literature, some inventory models have been formulated and analysed under distinct CE regulation policies. Unlike the previous research (cf. § 1.3.4), Absi et al. [1] did not consider a global emission limit but an average limit per item. Zhang and Xu [290] investigated the multi-item production planning problem with carbon cap and trade mechanism, where the firm can buy or sell the right to emit carbon in a trading market of CE. They presented a profit-maximization model to characterize the optimization problem and analysed the optimal policy of production and carbon trading decisions. Benjaafar et al. [17] considered three different carbon footprint policies and showed how CE concerns can be integrated into operational decision-making. He et al. [101] investigated the impact of production and regulation (i.e. cap-and-trade and carbon tax) parameters on the optimal lot-size and CEs and compared the firm's optimal CEs under the above two regulations. But

Table 5.1: Literature Review for Model-5.1

| Authors with year | Environmental effect | UPC | Single/ Multi objective | Method of solution |
|---|---|---|---|---|
| Khouja and Mehrez [127], 1994 | No | Production rate dependent | Single | Analytic |
| Hayek and Salameh [100], 2001 | No | Constant | Single | Analytic |
| Ben-Daya [16], 2002 | No | No | Single | Pattern Search Technique |
| Chiu [55], 2003 | No | Constant | Single | Analytic |
| Sana [227], 2010 | No | Production rate dependent | Single | Analytic |
| Hua et al. [107], 2011 | Carbon footprints | No | Single | Analytic |
| Guchhait et al. [92], 2012 | No | Constant | Multi | Interval compared Genetic Algorithm(ICGA) |
| Bouchery et al. [25], 2012 | CE | No | Multi | Interactive procedure |
| Mondal et al., [178], 2013 | No | Production rate dependent | Multi | Genetic Algorithm |
| Chen et al. [50], 2013 | CE | No | Single | Analytic |
| Zhang and Xu [290], 2013 | CE | Constant | Single | Analytic |
| Du et al., [75], 2013 | CE | Constant | Single | Stackelberg game |
| Benjaafar et al. [17], 2013 | CE | No | Single | ILOG CPLEX version 11.1 |
| Present Model 5.1 | CE | Production rate dependent | Single & Multi | RMOGA |

none considered/ determined the carbon policies under fuzzy production system.

Summarizing the above mentioned literature, the systematic chronological developments of the Model 5.1 in the related areas are presented in Table 5.1. These vacuums promoted us to consider the Model 5.1 and to present a guideline for a firm.

Identifying the gaps in the above developments as presented in Table 5.2, some new concepts have been introduced in the Model 5.2., after formulating the model as a FDE.

Table 5.2: Literature Review for Model-5.2

| Authors with year | Out of control State | Environmental effect | UPC | Formulation using FDE | Single/ Multi objective | Method of solution |
|---|---|---|---|---|---|---|
| Khouja and Mehrez [127], 1994 | Random | No | Production rate dependent | No | Single | Analytic |
| Hayek and Salameh [100], 2001 | Random | No | Constant | No | Single | Analytic |
| Ben-Daya [16], 2002 | Random | No | No | No | Single | Pattern Search Technique |
| Chiu [55], 2003 | Random | No | Constant | No | Single | Analytic |
| Wang and Tang [262], 2009 | Fuzzy | No | Constant | No | Single | Numerical search Procedure |
| Zhang et al. [289], 2009 | Random fuzzy | No | No | No | Single | Random fuzzy simulation(SPSA) |
| Sana [227], 2010 | Random | No | Production rate dependent | No | Single | Analytic |
| Hu et al. [106], 2011 | Fuzzy random | No | No | No | Single | Analytic |
| Hua et al. [107], 2011 | No | Carbon footprints | No | No | Single | Analytic |
| Guchhait et al. [92], 2012 | Fuzzy | No | Constant | Yes | Multi | Interval compared Genetic Algorithm(ICGA) |
| Bouchery et al. [25], 2012 | No | CE | No | No | Multi | Interactive procedure |
| Mondal et al. [178], 2013 | No | No | Production rate dependent | Yes | Multi | Genetic Algorithm |
| Chen et al. [50], 2013 | No | CE | No | No | Single | Analytic |
| Zhang and Xu [290], 2013 | No | CE | Constant | No | Single | Analytic |
| Du et al. [75], 2013 | No | CE | Constant | No | Single | Stackelberg game |
| Benjaafar et al. [17], 2013 | No | CE | No | No | Single | ILOG CPLEX version 11.1 |
| Present Model 5.2 | Fuzzy | CE | Production rate dependent | Yes | Multi | IFOT |

There may be some firms producing the imperfect products after the passage of some time from the commencement of production and at the same time, facing the problem of CE under government regulations. At beginning, a firm does not know what should be its CE for optimum production and which government regulation will be most beneficial for its maximum profit. Till now, there is no systematic procedures/ algorithms which a firm should follow for its maximum benefit following the various rules and regulations of a government. This vacuum promoted us to develop two models to present a guideline for a firm.

In the first model, an imperfect EPL model is considered with time dependent defective rate. Produced defective units are partially reworked and are sold as fresh units. Under the environmental regulation, a cost (say carbon tax) which is charged by the government to mitigate global warming by reducing CEs is taken into account. The models are formulated as a profit maximization problem. The models are illustrated with some numerical examples.

In the second model, objective is to present the appropriate production schemes for firms to achieve maximum profit when the production system is imperfect with time dependent fuzzy defective production rate and forced to follow the country's CE rules. Here produced defective units are partially reworked instantly and treated as fresh units. Rest defective units are sold at a reduced price. Under the environmental regulation, a cost (say carbon tax) which is charged by the government to mitigate global warming by reducing CE is taken into account. The models are formulated as profit maximization using FDE and the corresponding inventory and environmental costs are derived using fuzzy Riemann-integration. $\alpha$-cuts of average profits are obtained and the reduced multi-objective crisp problems are solved using IFOT. The models are illustrated with some numerical experiments. For different values of $\alpha$ ($\alpha$-cut), pictorial representations of average profit and CE are presented. A general algorithm to be followed by a production firm for maximum profit is outlined under the cloud of country's CE rules. For illustration, two different firms- one from developed and other from developing countries are considered and optimum carbon policies for maximum profit are outlined. Thus the present investigation helps a firm to determine its carbon policy taking the appropriate rules and regulations in vogue in that country for the firm's maximum profit.

## 5.3 Model-5.1 : Green logistics under imperfect production system: A Rough age based Multi-Objective Genetic Algorithm approach [1]

### 5.3.1 Assumptions and Notations

*The following assumptions are used to develop the proposed models:*

(i) Single item production with infinite time horizon.

(ii) Production run time is taken as DV.

[1]This model has been communicated in **Computers & Industrial Engineering** , ELSEVIER

(iii) Production rate is finite and taken as a DV.

(iv) Lead time is zero and no shortages are allowed.

(v) Demand is constant.

(vi) The production process shifts from "In-control" state to "Out-of-control" state after a certain period of time. Imperfect units are produced during the "Out-of-control" state only.

(vii) There is immediate partial reworking for the defective units at a cost and the defective units which are not reworked, are sold at a lower price and reworked units are treated as fresh ones.

(viii) UPC is raw material, labour cost dependent and one part of it is spent against wear and tear of the equipments also.

*The following notations are used to develop the proposed models:*

- $P$ Production rate $(tons/time\,unit)$ (DV).
- $t_1$ Production run-time in one cycle $(years)$ (DV).
- $D$ Market demand $(tons/time\,unit)$.
- $T$ Cycle time $(years)$.
- $\tau$ elapsed time $(years)$, measured from the commencement of production, at which defective production begins. i.e., the beginning time of the "out-of-control" state.
- $\lambda$ The machine produces imperfect units at this rate $(tons/time\,unit)$ when the machinery system is in "out-of-control" state. Here $\lambda$ is defined as $\lambda(t,\tau) = \gamma(t-\tau), \gamma > 0$.
- $\theta$ Percentage of reworked defective units.
- $S$ Selling price per unit perfect product $(\$/unit)$.
- $Sa$ Salvage price per unit imperfect product which are not reworked $(\$/unit)$.
- $N$ Defective units in a production cycle $(tons)$.
- $Q$ Total perfect units $(tons)$.
- $I(t)$ Inventory level at time t $(tons)$.
- $C(P)$ UPC $(\$)$ , $C(P) = r + \frac{g}{P^{\delta_1}} + \eta P^{\delta_2}$ where, $\delta_1, \delta_2 > 0$ and $r$ is the raw material cost $(\$)$ per unit, $g$ is the total labour/energy costs $(\$)$ per unit time in a production system which is equally distributed over the produced units. So,$(\frac{g}{P^{\delta_1}})$ decreases with increases of $P$. The third term $\eta P^{\delta_2}$ is the wear and tear cost $(\$)$, proportional to the positive power of production rate $P$.
- $Ch$ Holding cost $(\$/unit/time\,unit)$.
- $Cr$ Cost to rework an imperfect unit $(\$/unit)$.
- $Cs$ Set up cost $(\$/cycle)$.

| | |
|---|---|
| $\widetilde{\hat{C}h}$ | Fuzzy amount of CE associated per unit held inventory per unit time $(tons/unit/time\,unit)$. |
| $\widetilde{\hat{C}r}$ | Fuzzy amount of CE associated to rework an imperfect product $(tons/unit)$. |
| $\widetilde{\hat{C}s}$ | Fuzzy amount of CE associated per set-up $(tons/cycle)$. |
| $\widetilde{\hat{r}}, \widetilde{\hat{g}}, \widetilde{\hat{\eta}}$ | Fuzzy amounts of CE $(tons)$ associated with raw material, energy/ labour and wear and tear per unit product per unit time respectively. |
| $HC, PC, RC$ | Holding, Production, Rework cost $(\$)$ respectively. |
| $TC, TR$ | Total cost and Total sale revenue $(\$)$ respectively. |
| $\widetilde{CO_2}$ | Fuzzy CE amount $(tons)$. |
| $\widetilde{CEC}, \widetilde{CER}$ | Fuzzy CE cost and CE reward $(\$)$ respectively. |
| $\widetilde{ATP}$ | Fuzzy Average Total Profit (ATP) |
| $\widetilde{ACEC}, \widetilde{ACER}$ | Fuzzy Average Carbon Emission Cost (ACEC) and Revenue (ACER) $(\$)$ respectively. |

$\hat{}, \tilde{}$ These are used on the top of some parameters to represent CE and fuzzy nature respectively.

### 5.3.2 Mathematical Model Development

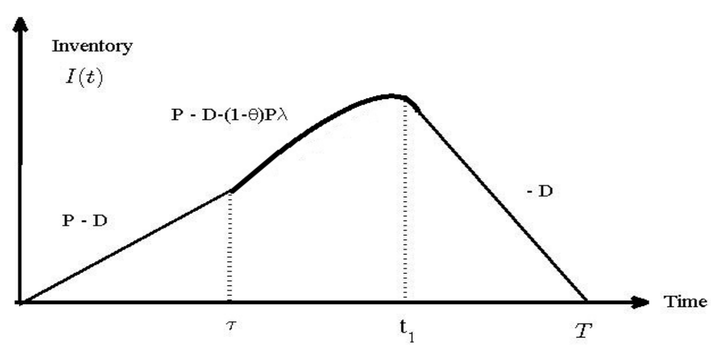



Figure 5.1: Inventory versus time.

In this investigation, an imperfect EPL model is assumed over an infinite planning horizon. In this production process, the production starts at a rate P at time $t = 0$ and runs up to time $t = t_1$. The system produces perfect quality units up to a certain time $\tau$ (i.e., "in-control" state), after that, the production system shifts to an "out-of-control" state $[\tau, t_1]$. In this "out-of-control" state, some of the produced units are of non-conforming quality (i.e., defective units) and some of these defective units are reworked immediately. The inventory piles up during the time interval $[0, t_1]$ adjusting market demand $D$ against the production and reworking processes, upto the perfect product $Q$ units, i.e., when the system stops the production. The stock at $t = t_1$ is depleted satisfying the demand $D$ in the market and it reaches zero level at time $T$ (cf. Fig. 5.1). For the single item imperfect production process, the governing differential equations are:

$$\frac{dI(t)}{dt} = \begin{cases} P-D, \quad 0 \leq t \leq \tau \\ P-D-(1-\theta)\gamma P(t-\tau), \quad \tau \leq t \leq t_1 \\ -D, \quad t_1 \leq t \leq T \end{cases} \tag{5.1}$$

with the boundary conditions $I(t) = 0, \quad at \ t = 0$ and $T$. The solution $I(t)$ of the above differential equations are given by

$$I(t) = \begin{cases} (P-D)t, & 0 \leq t \leq \tau \\ (P-D)t - \frac{(1-\theta)\gamma P}{2}(t-\tau)^2, & \tau \leq t \leq t_1 \\ D(T-t), & t_1 \leq t \leq T \end{cases} \tag{5.2}$$

Applying continuity condition at $t_1$ we have,

$$T = \frac{1}{D}\{Pt_1 - \frac{(1-\theta)\gamma P}{2}(t_1-\tau)^2\} \tag{5.3}$$

**Holding cost**

Holding cost during $[0, T]$ is

$$\begin{aligned} HC =& Ch\int_0^T I(t)dt = Ch[\int_0^\tau I(t)dt + \int_\tau^{t_1} I(t)dt + \int_{t_1}^T I(t)dt] \\ &= Ch\Big[\frac{P-D}{2}t_1^2 - \frac{(1-\theta)\gamma P}{6}(t_1-\tau)^3 + \frac{D}{2}(T-t_1)^2\Big] \end{aligned} \tag{5.4}$$

**Production cost**

The production cost during $[0, t_1]$ is

$$PC = C(P)\int_0^{t_1} Pdt = C(P)P[\int_0^\tau dt + \int_\tau^{t_1} dt] = C(P)Pt_1 \tag{5.5}$$

**Rework cost**

The rework cost during the time span $[\tau, t_1]$ is $RC = Cr\theta N$, where $N = \int_\tau^{t_1} \lambda(t,\tau)Pdt = \frac{1}{2}\gamma P(t_1-\tau)^2$ then,

$$RC = \frac{1}{2}Cr\theta\gamma P(t_1-\tau)^2 \tag{5.6}$$

Therefore, the total fresh units including reworked products are

$$Q = Pt_1 - \frac{1}{2}(1-\theta)\gamma P(t_1-\tau)^2 \tag{5.7}$$

**Total cost (without CE part)**

As a result, due inventory the total cost = Set-up cost + Holding cost + Production cost + Rework cost. i.e.,

$$\begin{aligned} TC &= Cs + HC + PC + RC \\ &= Cs + Ch\Big[\frac{P-D}{2}t_1^2 - \frac{(1-\theta)\gamma P}{6}(t_1-\tau)^3 + \frac{D}{2}(T-t_1)^2\Big] \\ &\quad +(r + \frac{g}{P^{\delta_1}} + \eta P^{\delta_2})Pt_1 + \frac{1}{2}Cr\theta\gamma P(t_1-\tau)^2 \end{aligned} \tag{5.8}$$

**Cost due to CE**

The CE associated with set-upping, inventory holding, producing, reworking is as $\widetilde{CO_2} = \widetilde{\hat{C}s} + \widetilde{\hat{HC}} + \widetilde{\hat{PC}} + \widetilde{\hat{RC}}$. Then,

$$\begin{aligned} \widetilde{CO_2} &= \widetilde{\hat{C}s} + \widetilde{\hat{C}h}\Big[\frac{P-D}{2}t_1^2 - \frac{(1-\theta)\gamma P}{6}(t_1-\tau)^3 + \frac{D}{2}(T-t_1)^2\Big] \\ &\quad +(\widetilde{\hat{r}} + \frac{\widetilde{\hat{g}}}{P^{\delta_1}} + \widetilde{\hat{\eta}}P^{\delta_2})Pt_1 + \frac{1}{2}\widetilde{\hat{C}r}\theta\gamma P(t_1-\tau)^2 \end{aligned} \tag{5.9}$$

In practice, there are several ways to deal with the CE. These are as follows:

**Penalty due to CE**

In this case, regulatory authority simply imposes some penalty for CE, irrespective of the emission amount. The objective is to bring the CE to zero level. The CEC is

$$\widetilde{CEC} = tax * \widetilde{CO_2} \tag{5.10}$$

where 'tax' is the penalty ($\$/ton$) per unit of carbon emitted.

**Under carbon trading**

In this system, the production firm is permitted to emit a fixed amount C (i.e. cap) of carbon per unit time by the regulatory authority. Normally a heavy penalty is charged by the authority if the emission by the firm is more than permitted cap. Here, again two cases may arise:

**Case 1 - Cap and penalty / purchase:** In this case, a penalty (fixed by the regulatory authority) or purchasing price (purchasing from other firms emitting less carbon) for per unit excess emitted carbon is paid by the firm. This amount of cost is evaluated as

$$\widetilde{CEC} = tax * (\widetilde{CO_2} - CT) \tag{5.11}$$

where 'tax' is the penalty / purchasing price ($\$/ton$) per unit excess of carbon.

**Case 2 - Cap and reward / sale:** To encourage the less CE, regulatory authority gives cash rewards to the firm which emits less carbon than the cap. Some times, the firm may sale its excess carbon to the co-firms, not claiming the rewards from regulatory authority. In this case,

$$\widetilde{CER} = rew * (CT - \widetilde{CO_2}) \tag{5.12}$$

is the CE rewards or revenue, where 'rew' is per unit carbon reward or selling price ($\$/ton$).

### Strictly under permitted cap

In this case, regulatory authority strictly impose the restriction of emission upto cap. i.e. if emission exceeds cap the firm will be asked to close. Hence the firm is compelled to maximize the profit subject to the constraint,

$$\widetilde{CO_2} \leq CT \tag{5.13}$$

### Total Sale Revenue

**Revenue for perfect products:** Total sales revenue of perfect products is

$$SR_P = S\int_0^T Ddt = SDT$$

**Sales Revenue for imperfect products:** The defective products which are not reworked is disposed by a lower price and total sales revenue is

$$SR_D = Sa(1-\theta)N$$

. Therefore, total sales revenue $TR = SR_P + SR_D$ for the system is as

$$TR = SDT + \frac{1}{2}Sa(1-\theta)\gamma P(t_1-\tau)^2 \tag{5.14}$$

### Constraint for "Out-of-control" state

In this production system, it is expected to have total production time greater than the time of occurrence of out-of-control state. This requirement acts as a constraint and is expressed as

$$t_1 \geq \tau \tag{5.15}$$

### 5.3.3 Optimization Problems

**Model 5.1A (EPL with carbon tax)**

Model 5.1A is formulated considering CEC in § 5.3.2 (Penalty due to CE). Here, average total profit $\widetilde{ATP}(P,t_1)$ for the system is

$$\widetilde{ATP}(P,t_1) = \frac{TR - TC - \widetilde{CEC}}{T} \tag{5.16}$$

where $TR$, $TC$, $\widetilde{CEC}$ and $T$ are given by Eqs. (5.14), (5.8), (5.10) and (5.3) respectively. Now the problem is to find ($P$, $t_1$) for which objective $\widetilde{ATP}(P,t_1)$ given by Eq. (5.16), is maximized subject to the constraint (5.15).

**Model 5.1B(EPL with cap and penalty)**

Model 5.1B is formulated considering the CEC as describe in § 5.3.2 (Cap and penalty / purchase) for case 1. This model is same as Model 5.1A except the expressions of $\widetilde{CEC}$. This expression is changed by Eq. (5.11).

**Model 5.1C (EPL with cap and reward)**

As describe about CEC in § 5.3.2 for case 2 (Cap and reward / sale), Model 5.1C is formulated. In this model, average total profit $\widetilde{ATP}(P,t_1)$ for the system is as

$$\widetilde{ATP}(P,t_1) = \frac{TR - TC + \widetilde{CER}}{T} \tag{5.17}$$

where $TR$, $TC$, $\widetilde{CER}$ and $T$ are given by Eqs. (5.14), (5.8), (5.12) and (5.3) respectively. The problem is to find ($P$, $t_1$) for which objective $\widetilde{ATP}(P,t_1)$ given by Eq. (5.17), is maximized subject to the constraint (5.15).

**Model 5.1D (EPL with carbon constraint)**

Model 5.1D is formulated when CE is strictly under permitted cap as described in § 5.3.2. Average total profit $ATP(P,t_1)$ for this system is

$$ATP(P,t_1) = \frac{TR - TC}{T} \tag{5.18}$$

where $TR$, $TC$ and $T$ are given by Eqs. (5.14), (5.8) and (5.3) respectively.

Thus the problem is to find ($P$, $t_1$) for which objective $ATP(P,t_1)$ given by Eq. (5.18), is maximized subject to the constraint (5.15) and (5.13).

**Model 5.1E (EPL with unlimited emission)**

Model 5.1E is developed when the firm is permitted to emit unlimited carbon. Thus the profit function is maximized without CE i.e., there is no penalty (tax=\$0.0) for CE i.e. cap (C) tends to infinity. Hence, Model 5.1E is formulated by either putting tax=\$0.0 in Model 5.1A or without carbon constraint (5.13) of Model 5.1D.

### 5.3.4 Solution Methodology

Now, the objective functions for the Models 5.1A-5.2E are defuzzified by taking fuzzy expectation (cf. § 2.1.2 using Lemma 2.8) and optimized using the RMOGA (cf. § 2.3.2.3).

### 5.3.5 Numerical Experiments and Results

In this section, we develop numerical experiments and results which illustrate the application of the proposed models.

**Table** 5.3: Optimal results for Models 5.1A, 5.1B, 5.1C, 5.1D and 5.1E

| $Models$ | $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $N^*$ | $Q^*$ | $T^*$ | $E[\widetilde{ACO_2}^*]$ | $E[\widetilde{ACEC}^*]/E[\widetilde{ACER}^*]$ |
|---|---|---|---|---|---|---|---|---|
| 5.1A | 618.43 | 3.40 | 3679.96 | 357.86 | 2014.18 | 13.42 | 6791.36 | 5093.52 |
| 5.1B | 618.43 | 3.40 | 7429.96 | 357.86 | 2014.18 | 13.42 | 6791.36 | 1343.52 |
| 5.1C | 640.38 | 3.58 | 9588.95 | 436.69 | 2187.04 | 14.58 | 6863.47 | 784.13 |
| 5.1D | 622.66 | 3.42 | 8779.43 | 366.89 | 2038.49 | 13.58 | 6800.00 | |
| 5.1E | 659.20 | 3.99 | 8820.99 | 617.58 | 2474.30 | 16.49 | 7034.84 | |

**Experiment 1 (Model 5.1A)**

For the Model 5.1A, we consider the following input data in appropriate units as mentioned in § 5.3.1.
$\theta = 0.75$, $\gamma = 0.25$, $d = 150$, $\tau = 1.25$, $S = 100$, $Sa = 20$, $Cs = 15000$, $Ch = 0.60$, $Cr = 2.0$, $\widetilde{\hat{C}s} = (28000, 30000, 32000)$, $\widetilde{\hat{C}h} = (2.5, 3.0, 3.5)$, $\widetilde{\hat{C}r} = (5.0, 6.0, 7.0)$, $tax = \$0.75$ and UPC as: $C(P) = 5.0 + \frac{8000}{P} + 0.02P$, CE per unit production: $\hat{C}(P) = (3.0, 4.0, 5.0) + \frac{(2500,3000,3500)}{P} + (0.007, 0.008, 0.009)P$. With these input data, we find the optimum production rate, optimum production run time and optimum profit which are presented in Table 5.3.

**Experiment 2 (Model 5.1B), 3 (Model 5.1C), 4 (Model 5.1D) and 5 (Model 5.1E)**

Taking all the input data same as Model 5.1A, we solve Models 5.1B, 5.1C, 5.1D and 5.1E with C=5000.00,(C=10000.00, rew=\$0.25), C=6800.00 and tax=\$0.0 respectively. The optimal results of these models are presented in Table 5.3.

### Experiment 6

We now construct the objective functions Average Carbon Emission Cost (ACEC) and Average Carbon Emission Revenue (ACER) as $\widetilde{ACEC}(P,t_1) = \frac{\widetilde{CEC}}{T}$ and $\widetilde{ACER}(P,t_1) = \frac{\widetilde{CER}}{T}$ where $\widetilde{CEC}$, $\widetilde{CER}$ and T are given by Eqs. (5.10), (5.12) and (5.3) respectively. Considering the same input data we minimize $E[\widetilde{ACEC}(P,t_1)]$ for Model 5.1A and 5.1B and maximize $E[\widetilde{ACER}(P,t_1)]$ for Model 5.1C, as single-objective optimization problems with constraint (5.15). The optimum results of this experiment are presented in Table 5.4 and compared with optimum results of the earlier models.

**Table** 5.4: Optimum of results of Experiment 6

| *Objective* | *Model* | $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $N^*$ | $Q^*$ | $T^*$ | $E[\widetilde{ACO_2}^*]$ | $E[\widetilde{ACEC}^*]/E[\widetilde{ACER}^*]$ |
|---|---|---|---|---|---|---|---|---|---|
| Min $E[\widetilde{ACEC}]$ | 5.1A | 521..92 | 3.66 | 3574.44 | 379.81 | 1816.76 | 12.11 | 6740.87 | 5055.65 |
| Min $E[\widetilde{ACEC}]$ | 5.1B | 521..92 | 3.66 | 7324.44 | 379.81 | 1816.76 | 12.11 | 6740.87 | 1305.65 |
| Max $E[\widetilde{ACER}]$ | 5.1C | 521..92 | 3.66 | 9444.88 | 379.81 | 1816.76 | 12.11 | 6740.87 | 814.78 |

### Experiment 7

Considering the same input data as in Experiment 1, we construct the new problems for the Models 5.1A, 5.1B and 5.1C in the following forms respectively.

$$\begin{cases} \text{Max} \quad E[\widetilde{ATP}(P,t_1)] \\ \text{Min} \quad E[\widetilde{ACEC}(P,t_1)] \\ \text{Sub to constraint (5.15) for Model 5.1A and 5.1B} \end{cases} \tag{5.19}$$

$$\begin{cases} \text{Max} \quad E[\widetilde{ATP}(P,t_1)] \\ \text{Max} \quad E[\widetilde{ACER}(P,t_1)] \\ \text{Sub to constraint (5.15) for Model 5.1C} \end{cases} \tag{5.20}$$

The optimum results for the above constructed problems (5.19) and (5.20) are presented in Table 5.5.

**Table** 5.5: Optimal results for Model 5.1A, 5.1B and 5.1C for Experiment 7

| Model 5.1A | | | | Model 5.1B | | | | Model 5.1C | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $E[\widetilde{ACEC}^*]$ | $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $E[\widetilde{ACEC}^*]$ | $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $E[\widetilde{ACER}^*]$ |
| 616.94 | 3.41 | 3679.94 | 5092.47 | 607.55 | 3.40 | 7428.35 | 1333.93 | 626.86 | 3.49 | 9584.85 | 795.26 |
| 535.02 | 3.58 | 3598.06 | 5056.09 | 561.05 | 3.40 | 7379.12 | 1309.02 | 567.33 | 3.42 | 9515.54 | 813.07 |
| 581.70 | 3.35 | 3654.82 | 5064.50 | 593.30 | 3.36 | 7417.46 | 1320.41 | 614.11 | 3.38 | 9571.01 | 804.19 |

### 5.3.6 Discussion

Table 5.3 presents the optimal result of Models 5.1A, 5.1B, 5.1C, 5.1D and 5.1E. Here, the profit \$3679.96 is obtained for optimum values of $P* = 618.43\ tons/year$ and $t_1^* = 3.40\ years$. Average 6791.36 $tons$ carbon is emitted in the system and the CE tax is charged @ \$0.75 per ton CE per year. Total 357.86 $tons$ defective products are produced and $75\%$ of these defectives are reworked immediately. When CEC is not taken into account (i.e. Model 5.1E or Model 5.1A with tax=\$0.0), the profit increases to \$8820.99. This result is obtained for $P^* = 659.20\ tons/year$ and $t_1^* = 3.99\ year$ and corresponding defective units are 617.58 $tons$ which is more than Model 5.1A. These results are as per expectation. Once tax on CE is removed, the system is free from any restriction on production and has worked for unlimited CE. As a result, the rate of production has gone up, profit is higher and production time is marginally more.

Profits of Models 5.1B and 5.1C are greater than that of Model 5.1A because of zero cap (C=0.0 $ton$) has been considered in Model 5.1A. Model 5.1C gives the highest profit among all models as the firm gets some rewards from regulatory authority i.e. earns by selling left out CE to other firm for less CE than cap. On the other hand, for other models, the firm pays penalty for CE. Model 5.1D gives the satisfactory profit \$8779.43 under the strict carbon cap (C=6800.00 $tons$) constraint. It is to be noted that these results of Model 5.1D are in between the results of Models 5.1B and 5.1C. It is as per expectation. In Model 5.1D, the inequality $\widetilde{CO}_2 \leq CT$ is taken in equality sense of CT where Models 5.1B and 5.1C assume $\widetilde{CO}_2$, more and less than CT respectively.

The results furnished in the Table 5.4 are due to the carbon minimization/maximization only of Models 5.1A, 5.1B and 5.1C and the corresponding profits $E[\widetilde{ATP}^*]$ are given. It is to be noted that for all models, system optimization furnish more profits that the $E[\widetilde{ACEC}]$ optimization. This means that in the system, production-inventory plays an important role along with the CE part.

Now from the Table 5.4, it is seen that the production rate P and production run time $t_1$ which minimize the $E[\widetilde{ACEC}]$ are quite different from P and $t_1$ which are obtained maximizing the profit $E[\widetilde{ATP}]$ for Models 5.1A, 5.1B and 5.1C. It is interesting to note that for the optimum production rate $P^* = 521.92\ tons/year$ and production run time $t_1^* = 3.66\ years$ which minimize $E[\widetilde{ACEC}]$, the corresponding profit \$3574.44 is lower than the profit $E[ATP]$ =\$3679.96 obtained maximizing $E[\widetilde{ATP}]$ directly, although for the optimum values, $P^* = 618.43\ tons/year$ and $t_1^* = 3.40\ years$ which maximize $E[\widetilde{ATP}]$, corresponding $E[\widetilde{ACEC}]$ \$5093.52 is higher than the minimum value \$5055.65 of $E[\widetilde{ACEC}]$. Therefore with respect to Model 5.1A, it is a better choice for a firm's manager to take the profit function $E[\widetilde{ATP}]$ as the objective to get more profit for the system.

Similar behaviours are observed for Models 5.1B and 5.1C.

Normally, no production-inventory firm management prefers any restriction on CE produced by the firm. From Table 5.4, it is interesting to note that the Model 5.1E in which CE is not taken into consideration, fetches a profit from the production-inventory point of view only, which is not the maximum profit for all possible systems. Rather, under the carbon restriction (i.e. present cap), if the firm management decides for less CE than cap and sells the saved carbon to others or gets the reward for this, then the profit is maximum. In the case of Model 5.1E, rate of production is highest and the duration of production is longest. This means that adjusting the production rate and production time, a firm can fulfil its social responsibility by emitting less carbon and at the same time, achieve its target by making maximum profit. However, all these depend on the permitted cap.

Following pictorial representations about the system are presented:

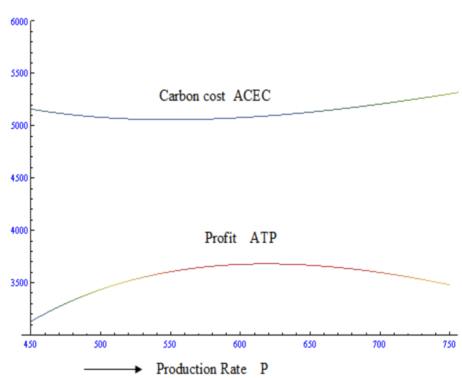



**Figure** 5.2: Concavity of $E[\widetilde{ATP}]$ and convexity of $E[\widetilde{ACEC}]$ against P for Model 5.1A.

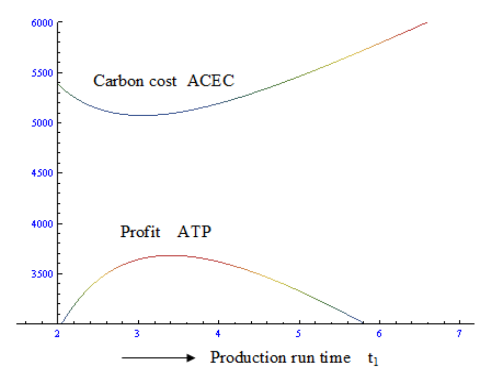



**Figure** 5.3: Concavity of $E[\widetilde{ATP}]$ and convexity of $E[\widetilde{ACEC}]$ against $t_1$ for Model 5.1A.

- For the given set of parameters in the above Experiment 1 for Model 5.1A, Fig. 5.2 represents the concavity property of the objective function $E[\widetilde{ATP}]$ and convexity property of $E[\widetilde{ACEC}]$ against the production rate (P).

- For the given set of parameters in the above Experiment 1 for Model 5.1A, Fig. 5.3 represents the concavity property of the objective function $E[\widetilde{ATP}]$ and convexity property of $E[\widetilde{ACEC}]$ against the production run time ($t_1$).

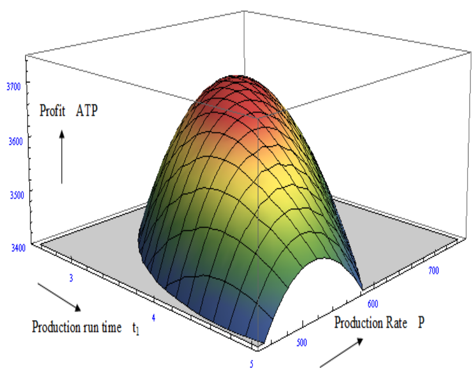



**Figure** 5.4: Concavity $\widetilde{ATP}$ against P and $t_1$ for Model 5.1A.

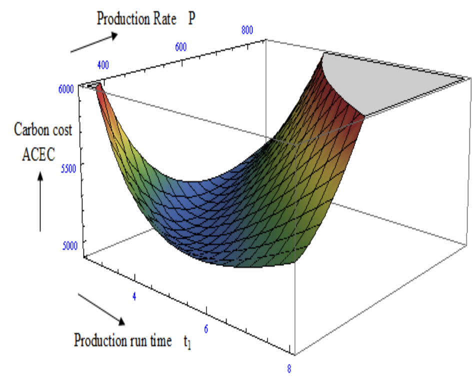



**Figure** 5.5: Convexity of $\widetilde{ACEC}$ against P and $t_1$ for Model 5.1A.

• For the given set of parameters in the above Experiment 1 for Model 5.1A, Fig. 5.4 represents the concavity property of the objective function $E[\widetilde{ATP}]$ against the production rate (P) and production run time ($t_1$) jointly.

• Fig. 5.5 represents the convexity property of $E[\widetilde{ACEC}]$ against the production rate (P) and production run time ($t_1$) for input data of Model 5.1A.

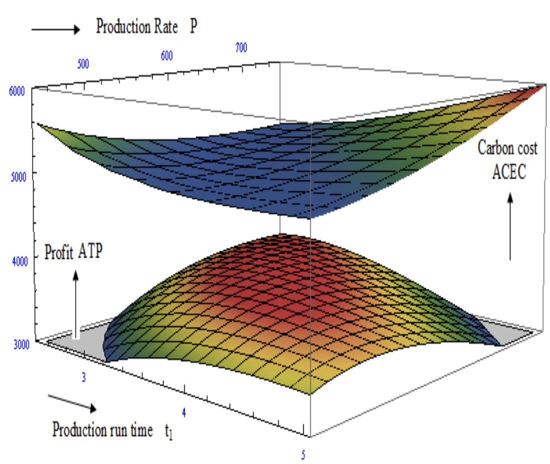



**Figure** 5.6: Concavity of $\widetilde{ATP}$ and convexity of $\widetilde{ACEC}$ against P and $t_1$ for Model 5.1A.

• For the given set of parameters in the Experiment 1 for Model 5.1A, Fig. 5.6 represents the concavity property of the objective function $E[\widetilde{ATP}]$ and convexity property of $E[\widetilde{ACEC}]$ against the production rate (P) and production run time ($t_1$).

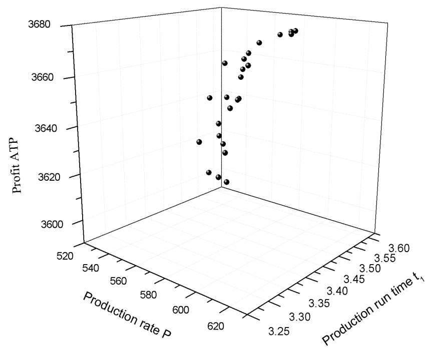



**Figure** 5.7: Optimum profits for Model 5.1A due to Experiment 7

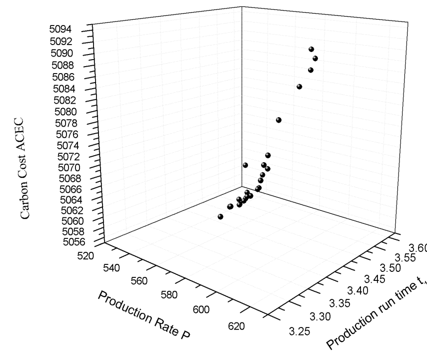



**Figure** 5.8: Optimum carbon cost for Model 5.1A due to Experiment 7

The results furnished in the Table 5.5 are obtained due to Experiment 7 for Models 5.1A, 5.1B and 5.1C. As the Experiment 7 is a multi-objective optimization problem, it gives pareto solution and is difficult to choose most optimum solutions. Here, three optimum solutions are given depending on the importance (priority) given to the objectives. Two results are for highest priority to one objective and lowest to other and third result with in between priorities. It is seen that the results of an objective taking as a single objective is also available in this pareto solution set when its takes the priorities for the objectives as (1, 0). Otherwise, it gives the compromise solutions. Here also, Model 5.1C furnishes maximum profit than the Models 5.1A and 5.1B, as revenue from carbon sale is taken into account in Model 5.1C. The fronts of pareto solutions of the multi-objectives for Models 5.1A, 5.1B and 5.1C are pictorially presented in Figs. 5.7 to 5.12.

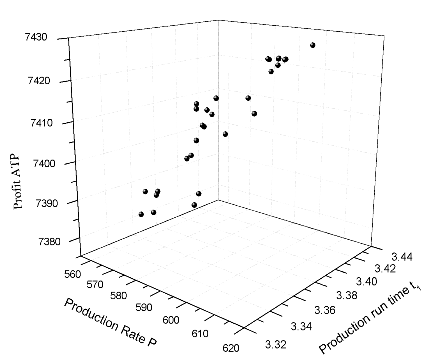



**Figure** 5.9: Optimum profits for Model 5.1B due to Experiment 7

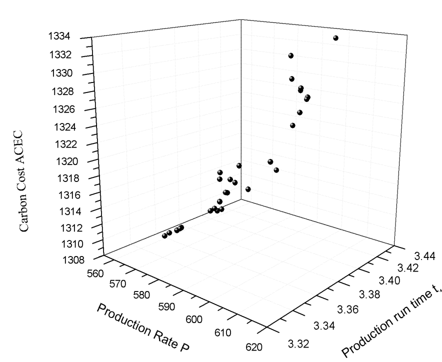



**Figure** 5.10: Optimum carbon cost for Model 5.1B due to Experiment 7

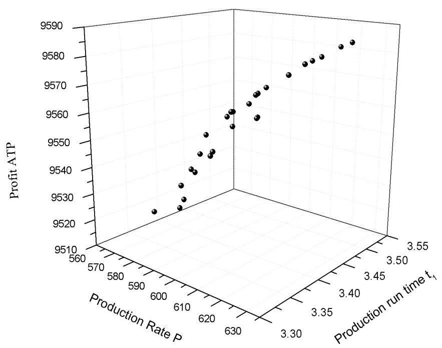



**Figure** 5.11: Optimum profits for Model 5.1C due to Experiment 7

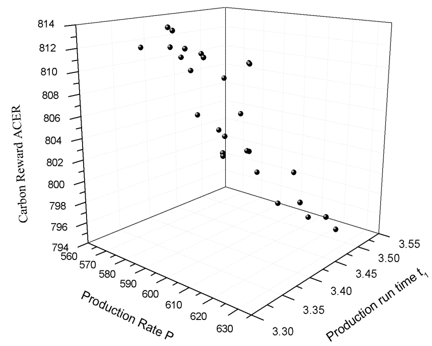



**Figure** 5.12: Optimum carbon reward for Model 5.1C due to Experiment 7

### 5.3.7 Practical Implication

In different countries, different types of regulations for CE are in vogue. There may be four types of regulations- (i) average tax on CE, whatever be the amount, (ii) $\&$ (iii) carbon trading- purchase (penalty) or sale (revenue) and (iv) no restriction on CE (i.e. unlimited CE is permitted). Thus, a production firm management is in a fix i.e. does not know which regulation should be followed so that firm's profit is maximum. Considering this situation, an algorithm (with example) for the management is given for his/her maximum profit in Algorithm 3.

**Example:** Let a company A from Annex I countries (Australia, Austria, Belgium, Canada, Germany, Italy, etc.) produces iron bars using coal based production system. Initially, perfect bars were produced upto $1.25$ years from the commencement of production. After that, some imperfect bars were produced $(25\%)$, some of which were reworked $(75\%)$ and sold as a new one ($\$100$ per bar). The rest were sold at a reduced price $\$20$ per bar. There are $250$ units demand per year in the open market.

Let P units be the production rate per year and production process continues up to $t_1$. To produce one unit of iron bar, the required raw material costs $\$2.50$; total energy $\$4000$ and minor repair cost $\$0.02$. The set-up cost for one business cycle is $\$15000$, storage cost per unit bar is $\$0.50$ per year, rework cost is $\$2.50$ per defective unit. Let CEs due to set-up be $(18000, 20000, 22000)$ tons per cycle, holding $(1.0, 2.0, 3.0)$ tons per unit bar per year, rework $(0.5, 1.0, 1.5)$ tons per unit, raw material $(4.0, 5.0, 6.0)$ tons per amount of raw materials required to produce one unit, $(4000, 5000, 6000)$ tonnes per year for total energy, $(0.07, 0.08, 0.09)$ ton per one unit for wear and tear.

Let the country has the following clean energy regulation. The country allows a firm to emit carbon of $20000$ tonnes per year and permits $1$ tonne of $CO_2$ at the cost $\$1.0$ and rewarded for less CE at $0.25.

**Algorithm 3:** MANAGERIAL DECISIONS TO CHOOSE A MODEL FOR HIGHEST PROFIT

```
Step 1: Check the regulatory system
if No restriction on CE i.e., Unlimited cap then
    Use Model 5.1E : EPL with unlimited CE
else if Tax is imposed due to CE (whatever be the amount) then
    Use Model 5.1A : EPL with carbon tax
else
    Go to step 2
Step 2: Check CE with limited carbon cap C
Calculate the total CE using Model 5.1E
if CE < C then
    Use Model 5.1C : EPL with cap and reward
else
    Go to step 3
Step 3: Check Optimum profit due to paying penalty for excess CE or use cap as a
constraint
Find optimum profit for Model 5.1B
Find optimum profit for Model 5.1D
if Profit of Model 5.1B > Profit of Model 5.1D then
    Use Model 5.1B : EPL with cap and penalty
else
    Use Model 5.1D : EPL with carbon constraint
End
```

Table 5.6: Optimal results of practical implication for Model 5.1E and 5.1C

| $Model$ | $P^*$ | $t_1^*$ | $E[\widetilde{ATP}^*]$ | $N^*$ | $Q^*$ | $T^*$ | $E[\widetilde{ACO_2}]*$ | $E[\widetilde{ACER}]$ |
|---|---|---|---|---|---|---|---|---|
| 5.1E | 483.86 | 7.76 | 17938.55 | 2565.54 | 3114.79 | 12.46 | 19402.57 | 149.50 |
| 5.1C | 406.55 | 6.66 | 18389.10 | 1490.41 | 2337.28 | 09.34 | 17492.79 | 626.80 |

The objective of this firm is to determine P and $t_1$ following the existing emission rule of the country so that the profit of the firm is maximum. The detailed optimum results are presented in Table 5.6 and following the prescribed Algorithm 3, the steps are:

**Step 1:** Since the regulatory system imposes limited carbon cap 20000.00 tons, go to Step 2.

**Step 2:** Average CE 19402.57 tons is obtained from optimizing Model 5.1E, which is less than limited carbon cap 20000.00 tons. Therefore, the firm use the Model 5.1C.

From Table 5.6, it is seen that the average CE per year is 19402.57 $tons$, (obtaining from optimizing the Model 5.1E) which is less than the permitted cap 20000.00 $tons$. It is also noted that if the firm management use the Model 5.1E, he/she gets reward or can sell the save-carbon to its co-firm in amount $\$149.50$ which is less than the amount $\$626.80$ obtained by using Model 5.1C. In that case, the average profit for Model 5.1C is $\$18389.10$ whereas $\$18088.05(= 17938.55 + 149.50)$ is the profit for Model 5.1E. Therefore, the firm management uses the Model 5.1C for highest profit. This means that adjusting the production rate and production time, a firm can fulfil its social responsibility by emitting less carbon and at the same time, can achieve its target by making maximum profit.

## 5.4 Model-5.2 : EPL models with fuzzy imperfect production system including carbon emission : A fuzzy differential equation approach [2]

In the present model, an imperfect EPL model is developed over an infinite planning horizon. In this production process, the production starts at time t=0 with a particular fixed production rate, P, say and runs upto a time, say $t_1$. During this production period, the system initially produces perfect units upto an uncertain (fuzzy) time, say $t = \tilde{\tau}$ (i.e. "in-control" state) and after that, the system goes to "out-of-control" state producing some imperfect/defective units. Some of these units are reworked instantly and treated as fresh units. Rest defective units are sold at a reduced price. In this way, the built-up stock are depleted at a demand rate D and exhausted after some uncertain (fuzzy) time, $\tilde{T}$ say. This whole process is repeated again. Here, UPC is raw material, total energy and wear and tear costs dependent following Khouja and Mehrez [127].

The production firm emits $CO_2$ in the process of production and pays tax / accepts carbon trading following the country's energy regulation. The purpose of this investigation is to find the optimum production rate so that the total profit out of production and carbon trading is maximum. To have a mathematical presentation of the above production process, the following notations and assumptions are used.

### 5.4.1 Assumptions and Notations

**Assumptions:**

(i) Single item production with infinite time horizon.

(ii) Production run time is taken as DV.

(iii) Production rate is finite and taken as a DV.

(iv) Lead time is zero and no shortages are allowed.

[2]This model has been communicated in **Journal of Intelligent Manufacturing**, SPRINGER

(v) Demand is constant.

(vi) The production process shifts from "In-control" state to "Out-of-control" state after an uncertain period of time. Imperfect units are produced during the "Out-of-control" state only.

(vii) There is immediate partial reworking for the defective units at a cost and the defective units which are not reworked, are sold at a lower price and reworked units are treated as fresh ones.

(viii) UPC is raw material, total energy and wear and tear cost dependent.

**Notations:**

Decision variables:

| | |
|---|---|
| $P$ | Production rate in units per unit time. |
| $t_1$ | Production run-time in one cycle. |

Parameters:

| | |
|---|---|
| $D$ | Market demand. |
| $\tilde{T}$ | Cycle time. |
| $\tilde{\tau}$ | Fuzzy time (measured from the commencement of production), at which defective production begins. i.e., beginning time of the "out-of-control" state. |
| $\tilde{\lambda}$ | The machine produces imperfect units at this rate when the machinery system is in "out-of-control" state. Here $\tilde{\lambda}$ is defined as $\tilde{\lambda}(t, \tilde{\tau}) = \gamma(t - \tilde{\tau})$, $\gamma > 0$. |
| $\theta$ | Percentage of reworked defective units. |
| $S$ | Selling price per unit perfect product. |
| $Sa$ | Salvage price per unit imperfect products which are not reworked. |
| $\tilde{N}$ | Defective units in a production cycle. |
| $\tilde{Q}$ | Total good quantities. |
| $\tilde{I}(t)$ | Inventory level at time t. |
| $C(P)$ | UPC, $C(P) = r + \frac{g}{P^{\delta_1}} + \eta P^{\delta_2}$ where, $\delta_1$, $\delta_2(> 0)$ are the elasticity parameters and so chosen to provide the feasible solution of the models. and $r$ is the raw material cost per unit item, $g$ is the total labour/energy costs per unit time in a production system which is equally distributed over the produced units. So,$(\frac{g}{P^{\delta_1}})$ decreases with increases of $P$. The third term $\eta P^{\delta_2}$ is the wear and tear cost, proportional to the positive power of production rate $P$. |
| $Ch$ | Holding cost per unit per unit time. |
| $Cr$ | Cost to rework an imperfect unit. |
| $Cs$ | Set up cost. |
| $\hat{C}h$ | Amount of CE associated per unit held inventory per unit time. |
| $\hat{C}r$ | Amount of CE associated to rework an imperfect product. |
| $\hat{C}s$ | Amount of CE associated per set-up. |
| $\hat{r}, \hat{g}, \hat{\eta}$ | Amounts of CE associated with raw material, energy/ labour and wear and tear per unit product per unit time respectively. |

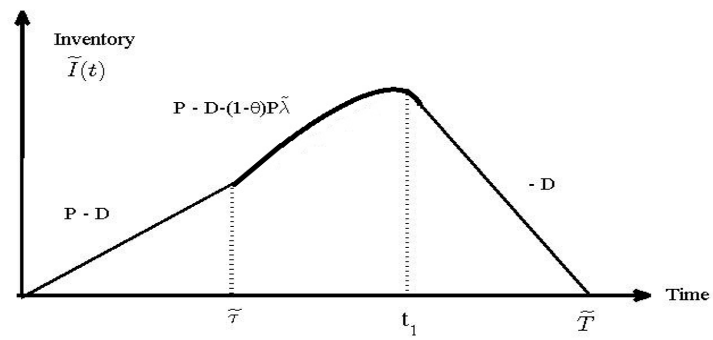



**Figure** 5.13: Inventory versus time.

$\widetilde{HC}, \widetilde{PC}, \widetilde{RC}, \widetilde{TC}, \widetilde{TR}$ Fuzzy Holding, Production, Rework, Total cost and Total sale revenue respectively.

$\widetilde{CO_2}, \widetilde{CEC}, \widetilde{CER}$ Fuzzy CE amount, CE Cost (CEC) and CE reward respectively.

$\widetilde{ATP}, \widetilde{ACEC}$ Fuzzy Average Total Profit (ATP) and Average Carbon Emission Cost (ACEC) for the system respectively.

$\tilde{}, \hat{}$ These symbols are used on the top of some parameters to represent fuzzy parameters and CE respectively.

Here parameters and variables are in appropriate units.

### 5.4.2 Mathematical Model Formulation

Using the above notations and assumptions, the production-inventory process with respect to time is given in Fig. 5.13 where $\tilde{Q}$ is the maximum perfect stock at $t = t_1$. This is built up at $P - D$ rate upto $t = \tilde{\tau}$, then at rate $P - D - (1-\theta)\tilde{\lambda}P$ upto $t = t_1$ and then depleted to zero at D rate by $t = \tilde{T}$. Therefore, for the single item imperfect production process, the governing differential equations are:

$$\frac{d\widetilde{I}(t)}{dt} = \begin{cases} P - D, \quad 0 \le t \le \widetilde{\tau} \\ P - D - (1-\theta)\gamma P(t - \widetilde{\tau}), \quad \widetilde{\tau} \le t \le t_1 \\ -D, \quad t_1 \le t \le \widetilde{T} \end{cases} \tag{5.21}$$

with the boundary conditions $\widetilde{I}(t) = 0, \quad at\ t = 0$ and $\widetilde{T}$. According to Buckley and Feuring [27], $\alpha$-cut of the solution $\widetilde{I}(t)$ of the above differential equations are given by $\widetilde{I}(t)[\alpha] = [I_L(\alpha, t), I_R(\alpha, t)]$, where,

$$I_L(\alpha, t) = \begin{cases} (P - D)t, & 0 \le t \le \widetilde{\tau} \\ (P - D)t - \frac{(1-\theta)\gamma P}{2}(t - \tau_L)^2, & \widetilde{\tau} \le t \le t_1 \\ D(T_L - t), & t_1 \le t \le \widetilde{T} \end{cases} \tag{5.22}$$

$$I_R(\alpha,t)=\begin{cases}(P-D)t, & 0\le t\le\widetilde{\tau}\\ (P-D)t-\frac{(1-\theta)\gamma P}{2}(t-\tau_R)^2, & \widetilde{\tau}\le t\le t_1\\ D(T_R-t), & t_1\le t\le\widetilde{T}\end{cases}\tag{5.23}$$

Applying continuity condition at $t_1$ we have,

$$\begin{cases}T_L=\frac{1}{D}\{Pt_1-\frac{(1-\theta)\gamma P}{2}(t_1-\tau_L)^2\}\\ T_R=\frac{1}{D}\{Pt_1-\frac{(1-\theta)\gamma P}{2}(t_1-\tau_R)^2\}\end{cases}\tag{5.24}$$

Here $\widetilde{I}(t)[\alpha]=[I_L(\alpha,t),I_R(\alpha,t)]$ satisfies all the conditions given by (2.20) and (2.15) [cf. Appendix B.1] according to Buckley and Feuring [27] in the interval $[0,\widetilde{T}]$, so $\widetilde{I}(t)[\alpha]$ is a valid solution of equation (5.21). [For details see Appendix B.2]

**Holding cost**

Holding cost during $[0,\widetilde{T}]$ is $\widetilde{HC}=Ch\int_0^{\widetilde{T}}\widetilde{I}(t)dt=Ch[\int_0^{\widetilde{\tau}}\widetilde{I}(t)dt+\int_{\widetilde{\tau}}^{t_1}\widetilde{I}(t)dt+\int_{t_1}^{\widetilde{T}}\widetilde{I}(t)dt]$. Let $\alpha$-cut set of total holding cost $\widetilde{HC}$ is $\widetilde{HC}[\alpha]=[HC_L(\alpha),HC_R(\alpha)]$. Then

$$\begin{cases}HC_L(\alpha)=Ch[\int_0^{\tau_L}I_L(\alpha,t)dt+\int_{\tau_R}^{t_1}I_L(\alpha,t)dt+\int_{t_1}^{T_L}I_L(\alpha,t)dt]\\ =Ch\left[\frac{P-D}{2}\tau_L^2+\frac{P-D}{2}(t_1^2-\tau_R^2)-\frac{(1-\theta)\gamma P}{6}\{(t_1-\tau_L)^3-(\tau_R-\tau_L)^3\}+\frac{D}{2}(T_L-t_1)^2\right]\\ HC_R(\alpha)=Ch[\int_0^{\tau_R}I_R(\alpha,t)dt+\int_{\tau_L}^{t_1}I_R(\alpha,t)dt+\int_{t_1}^{T_R}I_R(\alpha,t)dt]\\ =Ch\left[\frac{P-D}{2}\tau_R^2+\frac{P-D}{2}(t_1^2-\tau_L^2)-\frac{(1-\theta)\gamma P}{6}\{(t_1-\tau_R)^3-(\tau_L-\tau_R)^3\}+\frac{D}{2}(T_R-t_1)^2\right]\end{cases}\tag{5.25}$$

**Production cost**

The production cost during $[0,t_1]$ is $\widetilde{PC}=C(P)\int_0^{t_1}Pdt=C(P)P[\int_0^{\widetilde{\tau}}dt+\int_{\widetilde{\tau}}^{t_1}dt]$. Let $\alpha$-cut set of the above said cost is $\widetilde{PC_i}[\alpha]=[PC_L(\alpha),PC_R(\alpha)]$.Then,

$$\begin{cases}PC_L(\alpha)=C(P)P[\int_0^{\tau_L}dt+\int_{\tau_R}^{t_1}dt=C(P)P[t_1-(\tau_R-\tau_L)]\\ PC_R(\alpha)=C(P)P[\int_0^{\tau_R}dt+\int_{\tau_L}^{t_1}dt=C(P)P[t_1-(\tau_L-\tau_R)]\end{cases}\tag{5.26}$$

**Rework cost**

The rework cost during the time span $[\widetilde{\tau},\,t_1]$ is $\widetilde{RC}=Cr\theta\widetilde{N}$ where $\widetilde{N}=\int_{\widetilde{\tau}}^{t_1}\widetilde{\lambda}(t,\widetilde{\tau})Pdt$. Let $\alpha$-cut set of the above said cost and the integral are $\widetilde{RC}[\alpha]=[RC_L(\alpha),RC_R(\alpha)]$ and $\widetilde{N}[\alpha]=[N_L(\alpha),N_R(\alpha)]$. Then,

$$RC_L(\alpha)=Cr\theta N_L(\alpha),\qquad RC_R(\alpha)=Cr\theta N_R(\alpha)\tag{5.27}$$

and according to Wu [270] the integrals $N_L(\alpha)$ and $N_R(\alpha)$ are evaluated as

$$\begin{cases} N_L(\alpha) = \gamma P \int_{\tau_R}^{t_1}(t-\tau_R)dt = \frac{\gamma P}{2}(t_1-\tau_R)^2 \\ N_R(\alpha) = \gamma P \int_{\tau_L}^{t_1}(t-\tau_L)dt = \frac{\gamma P}{2}(t_1-\tau_L)^2 \end{cases} \tag{5.28}$$

Therefore, the total $\alpha$-cut set of good inventory including reworked products are

$$\begin{cases} Q_L(\alpha) = Pt_1 - (1-\theta)N_R(\alpha) \\ Q_R(\alpha) = Pt_1 - (1-\theta)N_L(\alpha) \end{cases} \tag{5.29}$$

**Total cost (without CEC)**

As a result, due inventory the total cost = Set-up cost + Holding cost + Production cost + Rework cost. i.e. $\widetilde{TC} = Cs + \widetilde{HC} + \widetilde{PC} + \widetilde{RC}$. Let $\alpha$-cut set of the above said cost is $\widetilde{TC}[\alpha] = [TC_L(\alpha), TC_R(\alpha)]$. Then

$$\begin{cases} TC_L(\alpha) = Cs + HC_L(\alpha) + PC_L(\alpha) + RC_L(\alpha) \\ TC_R(\alpha) = Cs + HC_R(\alpha) + PC_R(\alpha) + RC_R(\alpha) \end{cases} \tag{5.30}$$

**Cost due to CE**

The CE associated with set-upping, inventory holding, producing, reworking is as $\widetilde{CO_2} = \hat{C}s + \widetilde{\hat{H}C} + \widetilde{\hat{P}C} + \widetilde{\hat{R}C}$. Let $\alpha$-cut sets of the above said emission is $\widetilde{CO_2}[\alpha] = [CO_{2L}(\alpha), CO_{2R}(\alpha)]$. Then,

$$\begin{cases} CO_{2L}(\alpha) = \hat{C}s + \hat{C}h\Big[\frac{P-D}{2}\tau_L^2 + \frac{P-D}{2}(t_1^2-\tau_R^2) - \frac{(1-\theta)\gamma P}{6}\{(t_1-\tau_L)^3 \\ -(\tau_R-\tau_L)^3\} + \frac{D}{2}(T_L-t_1)^2\Big] + P(\hat{r} + \frac{\hat{g}}{P^{\delta_1}} + \hat{\eta}P^{\delta_2})\{t_1-(\tau_R-\tau_L)\} + \frac{\hat{C}r\theta\gamma P}{2}(t_1-\tau_R)^2 \\ CO_{2R}(\alpha) = \hat{C}s + \hat{C}h\Big[\frac{P-D}{2}\tau_R^2 + \frac{P-D}{2}(t_1^2-\tau_L^2) - \frac{(1-\theta)\gamma P}{6}\{(t_1-\tau_R)^3 \\ -(\tau_L-\tau_R)^3\} + \frac{D}{2}(T_R-t_1)^2\Big] + P(\hat{r} + \frac{\hat{g}}{P^{\delta_1}} + \hat{\eta}P^{\delta_2})\{t_1-(\tau_L-\tau_R)\} + \frac{\hat{C}r\theta\gamma P}{2}(t_1-\tau_L)^2 \end{cases} \tag{5.31}$$

In practice, there are several ways to deal with the CE. These are as follows:

**Penalty due to CE**

In this case, regulatory authority simply imposes some penalty for CE, irrespective of the emission amount. The objective is to bring the CE to zero level. The CE cost $\widetilde{CEC} = tax * \widetilde{CO_2}$, where 'tax' is the penalty per unit of carbon emitted. Let $\alpha$-cut sets of the above said emission cost is $\widetilde{CEC}[\alpha] = [CEC_L(\alpha), CEC_R(\alpha)]$. Then

$$\begin{cases} CEC_L(\alpha) = tax * CO_{2L}(\alpha) \\ CEC_R(\alpha) = tax * CO_{2R}(\alpha) \end{cases} \tag{5.32}$$

**Under carbon trading**

In this system, the production firm is permitted to emit a fixed amount C (i.e. cap) of carbon per unit time by the regulatory authority. Normally a heavy penalty is charged by the authority if the emission by the firm is more than permitted cap. Here, again two cases may arise:

**Case 1 - Cap and penalty / purchase:** In this case, a penalty (fixed by the regulatory authority) or purchasing price (purchasing from other firm emitting less carbon) for per unit excess emitted carbon is paid by the firm. This amount of cost is evaluated as $\widetilde{CEC} = tax * (\widetilde{CO_2} - C\tilde{T})$, where 'tax' is the penalty / purchasing price per unit excess of carbon. Thus

$$\begin{cases} CEC_L(\alpha) = tax * (CO_{2L}(\alpha) - CT_R) \\ CEC_R(\alpha) = tax * (CO_{2R}(\alpha) - CT_L) \end{cases} \tag{5.33}$$

**Case 2 - Cap and reward / sale:** To encourage the less CE, regulatory authority gives cash rewards to the firm which emits less carbon than the cap. Some times, the form may sale its excess carbon to the co-firms, not claiming the rewards from regulatory authority. In this case, $\widetilde{CER} = rew * (C\tilde{T} - \widetilde{CO_2})$ is the CE rewards or revenue, where 'rew' is per unit carbon reward or selling price. Therefore, we have equivalent crisp form as

$$\begin{cases} CER_L(\alpha) = rew * (CT_L - CO_{2R}(\alpha)) \\ CER_R(\alpha) = rew * (CT_R - CO_{2L}(\alpha)) \end{cases} \tag{5.34}$$

**Strictly under permitted cap**

In this case, regulatory authority strictly impose the restriction of emission upto cap. If emission exceeds cap the firm will asked to close. Hence the firm is compelled to maximize the profit subject to the constraint,

$$\widetilde{CO_2} \leq C\tilde{T}. \qquad \text{i.e.,} \begin{cases} CO_{2L}(\alpha) \leq CT_R \\ CO_{2R}(\alpha) \leq CT_L \end{cases} \tag{5.35}$$

**Total Sale Revenue**

**Revenue for perfect products:** Total sales revenue of perfect products is $\widetilde{PSR} = S\int_0^{\widetilde{T}} Ddt = SD[\int_0^{\tilde{\tau}} dt + \int_{\tilde{\tau}}^{t_1} dt + \int_{t_1}^{\widetilde{T}} dt] = [SD\{T_L - (\tau_R - \tau_L)\},\, SD\{T_R - (\tau_L - \tau_R)\}]$.
**Sales Revenue for imperfect products:** The defective products which are not reworked is disposed by a lower price and total sales revenue is $\widetilde{DSR} = [Sa(1-\theta)N_L(\alpha),\, Sa(1-\theta)N_R(\alpha)]$. Therefore, total sales revenue $\widetilde{TR} = [TR_L(\alpha), TR_R(\alpha)]$ for the system is

$$\begin{cases} TR_L(\alpha) = SD\{T_L - (\tau_R - \tau_L)\} + Sa(1-\theta)N_L(\alpha) \\ TR_R(\alpha) = SD\{T_R - (\tau_L - \tau_R)\} + Sa(1-\theta)N_R(\alpha) \end{cases} \tag{5.36}$$

**Constraint for "Out of control" state**

In this production system, it is expected to have total production time greater than the time of occurrence of out-of-control state. This requirement acts as a constraint and is expressed as $t_1 - \tilde{\tau} \geq \beta$, $\beta > 0$ which is interpreted in the setting of possibility and necessity theory [79]. The above constraint reduces to

$$\text{Pos}(t_1 - \tilde{\tau} \geq \beta) \geq \rho_1, \qquad \text{and} \qquad \text{Nes}(t_1 - \tilde{\tau} \geq \beta) \geq \rho_2$$

where $\rho_1$ and $\rho_2$ represent the degree of impreciseness. For $\tilde{\tau} = (\tau_1, \tau_2, \tau_3)$ being a TFN, using Lemma 2.1 and 2.2, we get

$$t_1 \geq \begin{cases} \beta + \tau_1 + \rho_1(\tau_2 - \tau_1), & \text{in possibility sense} \\ \beta + \tau_3 - (1 - \rho_2)(\tau_3 - \tau_2), & \text{in necessity sense.} \end{cases} \tag{5.37}$$

### 5.4.3 Optimization Problems

**Model 5.2A (EPL with carbon tax)**

Model 5.2A is formulated considering CEC in § 5.4.2 (Penalty due to CE). Here, average total profit $\widetilde{ATP}$ for the system is

$$\widetilde{ATP} = \frac{\widetilde{TR} - \widetilde{TC} - \widetilde{CEC}}{\widetilde{T}}$$

or, $[ATP_L(\alpha), ATP_R(\alpha)]$

$$= \frac{[TR_L(\alpha), TR_R(\alpha)] - [TC_L(\alpha), TC_R(\alpha)] - [CEC_L(\alpha), CEC_R(\alpha)]}{[T_L(\alpha), T_R(\alpha)]}$$

$$\text{or, } ATP_L(\alpha) = \frac{TR_L(\alpha) - TC_R(\alpha) - CEC_R(\alpha)}{T_R(\alpha)} \tag{5.38}$$

$$\text{and } ATP_R(\alpha) = \frac{TR_R(\alpha) - TC_L(\alpha) - CEC_L(\alpha)}{T_L(\alpha)} \tag{5.39}$$

Now we construct an objective function for better approximate solutions as

$$ATP_C(\alpha) = \frac{ATP_L(\alpha) + ATP_R(\alpha)}{2} \tag{5.40}$$

where $[TR_L(\alpha), TR_R(\alpha)]$, $[TC_L(\alpha), TC_R(\alpha)]$, $[CEC_L(\alpha), CEC_R(\alpha)]$ and $[T_L(\alpha), T_R(\alpha)]$ are given by Eqs. (5.36), (5.30), (5.32) and (5.24) respectively.

Now the problem is to find $(P,\ t_1)$ for which multi objectives $ATP_k(\alpha)$ ( for all k=L, C, R) given by Eqs. (5.38), (5.40) and (5.39) respectively, are maximized subject to the constraints (5.37).

**Model 5.2B(EPL with cap and penalty)**

Model 5.2B is formulated considering the CEC as describe in § 5.4.2 (Cap and penalty / purchase) for case 1. This model is same as Model 5.2A except the expressions of $[CEC_L(\alpha), CEC_R(\alpha)]$. These expressions are changed by Eq. (5.33).

**Model 5.2C (EPL with cap and reward)**

As describe about CEC in § 5.4.2 for case 2 (Cap and reward / sale), Model 5.2C is formulated. In this model, average total profit $\widetilde{ATP}$ for the system is as

$$\widetilde{ATP} = \frac{\widetilde{TR} + \widetilde{CER} - \widetilde{TC}}{\widetilde{T}}$$

$$\text{or, } ATP_L(\alpha) = \frac{TR_L(\alpha) + CER_L(\alpha) - TC_R(\alpha)}{T_R(\alpha)} \tag{5.41}$$

$$\text{and } ATP_R(\alpha) = \frac{TR_R(\alpha) + CER_R(\alpha) - TC_L(\alpha)}{T_L(\alpha)} \tag{5.42}$$

Now we construct a new objective function for better approximate solutions as

$$ATP_C(\alpha) = \frac{ATP_L(\alpha) + ATP_R(\alpha)}{2} \tag{5.43}$$

where $[TR_L(\alpha), TR_R(\alpha)]$, $[TC_L(\alpha), TC_R(\alpha)]$, $[CER_L(\alpha), CER_R(\alpha)]$ and $[T_L(\alpha), T_R(\alpha)]$ are given by Eqs. (5.36), (5.30), (5.34) and (5.24) respectively.

The problem is to find $(P, t_1)$ for which multi objectives $ATP_k(\alpha)$ ( for all k=L, C, R) given by Eqs. (5.41), (5.43) and (5.42) respectively, are maximized subject to the constraints (5.37).

**Model 5.2D (EPL with carbon constraint)**

Model 5.2D is formulated when CE is strictly under permitted cap as described in § 5.4.2. Average total profit $\widetilde{ATP}$ for this system is

$$\widetilde{ATP} = \frac{\widetilde{TR} - \widetilde{TC}}{\widetilde{T}}$$

$$\text{or, } ATP_L(\alpha) = \frac{TR_L(\alpha) - TC_R(\alpha)}{T_R(\alpha)} \tag{5.44}$$

$$\text{and } ATP_R(\alpha) = \frac{TR_R(\alpha) - TC_L(\alpha)}{T_L(\alpha)} \tag{5.45}$$

Now we construct a new objective function for better approximate solutions as

$$ATP_C(\alpha) = \frac{ATP_L(\alpha) + ATP_R(\alpha)}{2} \tag{5.46}$$

where $[TR_L(\alpha), TR_R(\alpha)]$, $[TC_L(\alpha), TC_R(\alpha)]$, $[CEC_L(\alpha), CEC_R(\alpha)]$ and $[T_L(\alpha), T_R(\alpha)]$ are given by Eqs. (5.36), (5.30), (5.35) and (5.24) respectively.

Thus the problem is to find $(P,\ t_1)$ for which multi objectives $ATP_k(\alpha)$ ( for all k=L, C, R) given by Eqs. (5.44), (5.46) and (5.45) respectively, are maximized subject to the constraints (5.37) and (5.35).

**Model 5.2E (EPL with unlimited emission)**

Model 5.2E is developed when the firm is permitted to emit unlimited carbon. Thus the profit function is maximized without CE i.e., there is no penalty (tax=0.0) for CE i.e. cap (C) tends to infinity. Hence, Model 5.2E is formulated by either putting tax=0.0 in Model 5.2A or without carbon constraint (5.35) of Model 5.2D.

### 5.4.4 Solution Methodology

To solve multi-objective maximization problems for Models 5.2A-5.2E, we have used the IFOT (cf. § 2.4.2.1).

### 5.4.5 Numerical Experiments and Results

In this section, we develop numerical experiments and results which illustrate the application of the proposed models.

**Experiment 1 (Model 5.2A)**

For the Model 5.2A, we consider the following input data:
$\alpha = 0.50$, $\theta = 0.750$, $\gamma = 0.25$, $d = 150$, $\tau_1 = 1.00 year$, $\tau_2 = 1.25 years$, $\tau_3 = 1.50 years$, $S$ = \$100, $Sa$ = \$20, $Cs$ = \$15000, $Ch$ = \$0.60, $Cr$ = \$2.0, $\hat{Cs} = 30000.00 tonnes$, $\hat{Ch} = 3.0 tonnes$, $\hat{Cr} = 6.0 tonnes$, $\beta = 0.25$, $\rho_1 = 0.90$, $\rho_2 = 0.10$, $tax$ = \$0.75, $w = 0.10$ and unit production cost as: $C(P) = \$(5.0 + \frac{8000}{P} + 0.02P)$, CE per unit production: $\hat{C}(P) = (4.0 + \frac{3000}{P} + 0.008P) tonnes$. With these input data, we find the optimum production rate, optimum production run time and optimum profit.

**Table** 5.7: Individual minimum and maximum of objective functions

| Objective functions | optimize $ATP_L$ | optimize $ATP_C$ | optimize $ATP_R$ |
|---|---|---|---|
| $ATP_L$ | $ATP_L^* = 2791.73$ | $ATP_L = 2754.08$ | $ATP_L = 2534.56$ |
| $ATP_C$ | $ATP_C = 3652.27$ | $ATP_C^* = 3683.78$ | $ATP_C = 3603.69$ |
| $ATP_R$ | $ATP_R = 4512.81$ | $ATP_R = 4613.48$ | $ATP_R^* = 4672.83$ |
| Variables $(P^*, t_1^*)$ | (588.47,3.96) | (618.18,3.40) | (681.80,2.66) |

Individual maximum of the objective functions $ATP_k$ for $k = L, C, R$ are obtained and given in Table 5.7. Now we calculate $L_L = 2534.56$, $L_C = 3603.69$, $L_R = 4512.81$, $U_L = 2791.73$, $U_C = 3683.78$, $U_R = 4672.83$. Using the equation (2.70), we formulate the following problem as :

$$\left.\begin{array}{ll}
\max\ (\mu-\nu) & \\
\text{sub to}\quad \mu \le \dfrac{e^{-w\left(\frac{2791.73-ATP_L}{2791.73-2534.56}\right)}-e^{-w}}{1-e^{-w}}; & \nu \ge \left(\dfrac{2791.73-ATP_L}{2791.73-2534.56}\right)^2 \\
\mu \le \dfrac{e^{-w\left(\frac{3683.78-ATP_C}{3683.78-3603.69}\right)}-e^{-w}}{1-e^{-w}}; & \nu \ge \left(\dfrac{3683.78-ATP_C}{3683.78-3603.69}\right)^2 \\
\mu \le \dfrac{e^{-w\left(\frac{4672.83-ATP_R}{4672.83-4512.81}\right)}-e^{-w}}{1-e^{-w}}; & \nu \ge \left(\dfrac{4672.83-ATP_R}{4672.83-4512.81}\right)^2 \\
t_1 \ge \begin{cases} \beta+\tau_1+\rho_1(\tau_2-\tau_1), & \text{in possibility sense} \\ \beta+\tau_3-(1-\rho_2)(\tau_3-\tau_2), & \text{in necessity sense.} \end{cases} & \\
\mu \ge \nu \text{ and } \mu+\nu \le 1; \qquad \mu,\ \nu \ge 0. &
\end{array}\right\} \tag{5.47}$$

The solutions obtained for Eq. (5.47) are given in Table 5.8.

**Table** 5.8: Optimum results of Eq. (5.47) for w=0.10

| $\mu^*$ | $\nu^*$ | $P^*$ | $t_1^*$ | $[ATP_L^*, ATP_C^*, ATP_R^*]$ | $[N_L^*, N_R^*]$ | $[Q_L^*, Q_R^*]$ | $[CO_{2L}^*, CO_{2R}^*]$ |
|---|---|---|---|---|---|---|---|
| 0.7451 | 0.0603 | 628.29 | 3.25 | [2728.59, 3681.07, 4633.54] | [277.17, 355.85] | [1953.28, 1972.91] | [85777.91, 92004.06] |

Now we perform the Pareto-Optimal Solution test for strong or weak solutions. The Pareto-Optimal results are presented in Table 5.9. In Table 5.9, the value of $V^*$ is quite small and hence, the optimal results in Table 5.9 are strong Pareto-optimal solution and can be accepted.

**Table** 5.9: Pareto-Optimal results

| $V^*$ | $P^*$ | $t_1^*$ | $[ATP_L^*, ATP_C^*, ATP_R^*]$ | $[N_L^*, N_R^*]$ | $[Q_L^*, Q_R^*]$ | $[CO_{2L}^*, CO_{2R}^*]$ |
|---|---|---|---|---|---|---|
| 0.0000 | 627.48 | 3.25 | [2728.59, 3681.06, 4633.53] | [277.33, 355.98] | [1950.76, 1970.37] | [85700.90, 91918.67] |

### Experiment 2 (Model 5.2B), 3 (Model 5.2C), 4 (Model 5.2D) and 5 (Model 5.2E)

Taking all the input data same as Model 5.2A, we solve Models 5.2B, 5.2C, 5.2D and 5.2E with C=7500.00 tonnes, ( C=20000.00 tonnes, rew=$0.25), C=7000.00 tonnes and tax=$0.0 respectively. The optimal results of these models are presented in Table 5.10.

**Table** 5.10: Optimal results for Model 5.2B, 5.2C, 5.2D and 5.2E

| $Model$ | $P^*$ | $t_1^*$ | $[ATP_L^*, ATP_C^*, ATP_R^*]$ | $[N_L^*, N_R^*]$ | $[Q_L^*, Q_R^*]$ | $[CO_{2L}^*, CO_{2R}^*]$ |
|---|---|---|---|---|---|---|
| 5.2B | 558.97 | 1.70 | [5902.40, 7536.62, 9170.84] | [7.26, 22.89] | [943.11, 947.01] | [47350.92, 52111.93] |
| 5.2C | 647.81 | 3.40 | [11217.54, 12090.03, 12962.52] | [331.02, 417.05] | [2096.05, 2117.78] | [92732.74, 99295.30] |
| 5.2D | 535.26 | 3.71 | [7888.76, 8674.84, 9460.92] | [365.24, 447.58] | [1874.69, 1895.28] | [82053.46, 87485.37] |
| 5.2E | 680.04 | 3.38 | [8045.07, 8807.12, 9569.18] | [341.72, 432.25] | [2190.47, 2213.10] | [97705.31, 104655.30] |

### Experiment 6

We now construct the objective functions Average Carbon Emission Cost (ACEC) and Average Carbon Emission Revenue as $ACEC_L(\alpha) = \frac{CEC_L(\alpha)}{T_R(\alpha)}$, $ACEC_R(\alpha) = \frac{CEC_R(\alpha)}{T_L(\alpha)}$ and $ACEC_C = \frac{ACEC_L(\alpha)+ACEC_R(\alpha)}{2}$ and $ACER_L(\alpha) = \frac{CER_L(\alpha)}{T_R(\alpha)}$, $ACER_R(\alpha) = \frac{CER_R(\alpha)}{T_L(\alpha)}$ and $ACER_C = \frac{ACER_L(\alpha)+ACER_R(\alpha)}{2}$, where $[CEC_L(\alpha), CEC_R(\alpha)]$, $[CER_L(\alpha), CER_R(\alpha)]$ and $[T_L(\alpha), T_R(\alpha)]$ are given by Eqs. (5.32), (5.34) and (5.24) respectively. Considering the same input data we minimize $\widetilde{ACEC}$ =$[ACEC_L, ACEC_C, ACEC_R]$ for Model 5.2A and 5.2B and maximize $\widetilde{ACER}$ =$[ACER_L, ACER_C, ACER_R]$ for Model 5.2C, as multi-objective optimization problems with constraints (5.37) using IFOT (see Appendix B.3 for solution procedure for minimization problem). The optimum results of this experiment are presented in Table 5.11 and compared with optimum results of the earlier models.

**Table** 5.11: Comparison of results optimising individually $\widetilde{ACEC}$ and $\widetilde{ATP}$ for Model 5.2A

| Objective | $P^*$ | $t_1^*$ | $[T_L^*, T_R^*]$ | $[CO_{2L}^*, CO_{2R}^*]$ | $[TC_L^*, TC_R^*]$ | $[ACEC_L^*, ACEC_R^*]$ | $[ATP_L^*, ATP_R^*]$ |
|---|---|---|---|---|---|---|---|
| Min $\widetilde{ACEC}$ | 529.67 | 3.59 | [12.01, 12.13] | [78711.12, 84012.93] | [74328.26, 82828.56] | [4865.61, 5249.18] | [2646.23, 4532.46] |
| Max $\widetilde{ATP}$ | 627.48 | 3.25 | [13.00, 13.14] | [85700.90, 91918.67] | [78117.89, 88047.81] | [4885.71, 5292.94] | [2728.59, 4633.53] |

## 5.4.6 Discussion

• Table 5.9 presents the optimal results of Model 5.2A. Here, the profit span $[2728.59, 4633.53] is obtained for optimum values of $P* = 627.48$ and $t_1^* = 3.25 years$. Total [85700.90, 91918.67] tonnes carbon is emitted in the system and the CE tax is charged @

$\$0.75$ per unit emission per unit time. Total [277.33, 355.98] units defective products are produced and $75\%$ of these defectives are reworked immediately. When CEC is not taken into account (i.e. Model 5.2E or Model 5.2A with tax=\$0.0), the profit is increased and its span is $\$[8045.00, 9569.00]. This result is obtained for $P* = 680.04$ and $t_1^* = 3.38 years$ and corresponding defective units are [340.70, 431.10] (cf. Table 5.10) which is more than Model 5.2A. These results are as per expectation. Once tax on CE is removed, the system is free from any restriction on production and has worked with unlimited CE. As a result, the rate of production has gone up, profit is higher and production time is marginally more.

• For the given set of parameters in the above Experiment 1 for Model 5.2A, Fig. 5.14 represents the concavity property of the objective function $ATP_C$ against the production rate (P) and production run time ($t_1$).

• Fig. 5.15 represents the convexity property of ACEC, $ACEC_C = \frac{ACEC_L + ACEC_R}{2}$ against the production rate (P) and production run time ($t_1$) for input data of Model 5.2A.

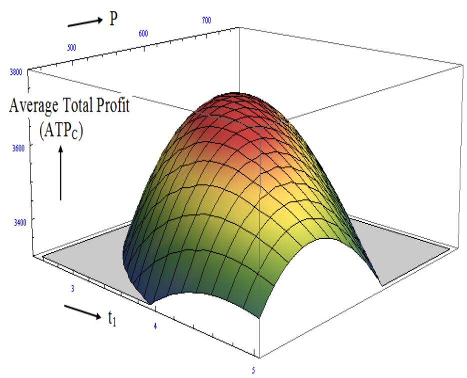



**Figure** 5.14: Concavity $ATP_C$ against P and $t_1$ for Model 5.2A.

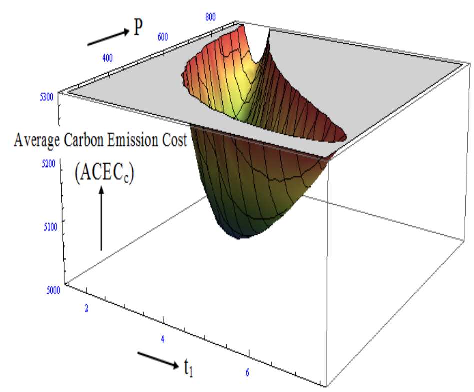



**Figure** 5.15: Convexity of $ACEC_c$ against P and $t_1$ for Model 5.2A.

• $\alpha$-cuts $[ATP_L(\alpha), ATP_R(\alpha)]$ and $[ACEC_L(\alpha), ACEC_R(\alpha)]$ of observed profit ($\widetilde{ATP}$) and ACEC ($\widetilde{ACEC}$) for the Model 5.2A due to Experiment 1 are plotted in Figs. 5.16 and 5.17 respectively . It is interesting to note that the figures represent almost triangular fuzzy numbers for fuzzy profit and carbon cost. Assuming these fuzzy quantities to be perfect triangular numbers, the corresponding membership functions $\mu_{\widetilde{ATP}}(x)$ and $\mu_{\widetilde{ACEC}}(x)$ for profit and carbon cost respectively are formulated as

$$\mu_{\widetilde{ATP}}(x) = \begin{cases} \frac{x-1427.21}{2252.75}, & 1427.21 \le x \le 3679.96 \\ \frac{5767.30-x}{2087.34}, & 3679.96 \le x \le 5767.30 \end{cases} \tag{5.48}$$

$$\mu_{\widetilde{ACEC}}(x) = \begin{cases} \frac{x-4653.45}{440.00}, & 4653.45 \le x \le 5093.45 \\ \frac{5571.04-x}{477.59}, & 5093.45 \le x \le 5571.04 \end{cases} \tag{5.49}$$

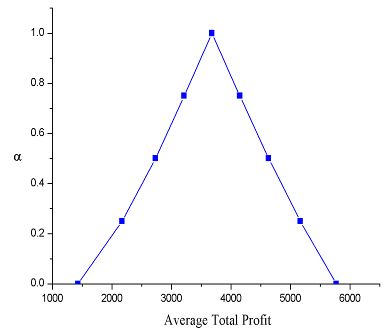



**Figure** 5.16: Membership function of $\widetilde{ATP}$ for Model 5.2A.

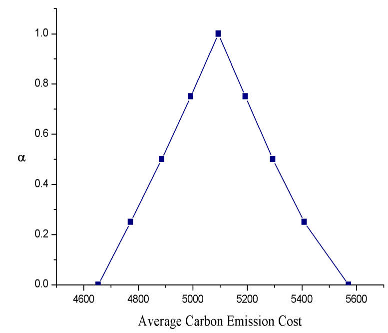



**Figure** 5.17: Membership function of $\widetilde{ACEC}$ for Model 5.2A.

In the formulation of proposed model, we started with TFN membership function of $\tau$, out-of-control time. Thus, it is expected that the optimum results- profit and ACEC will be also fuzzy in nature and represented by a TFN. Hence, the expressions (5.48) and (5.49) are as per expectation and the justification valid formulation and analysis.

• Fig. 5.18 represents the changes in $\%$ of optimum $P^{*}$, $t_1^{*}$, $\alpha$-cut of $\widetilde{ATP}^{*}$ and $\widetilde{ACEC}^{*}$ with respect to change of $\alpha$ in $\%$. With increase of $\alpha$, right spreads of both ATP and ACEC decreases, but the rate of decrease of ATP is more than ACEC. With $\alpha$, the left spreads of these two quantities increase and rate of increase of ATP is higher than that of ACEC. Again production P and production time $t_1$ decreases and increases respectively with $\alpha$. It is expected that if rate of production is reduced, then production period will be higher. Hence the system behaviour is normal.

• Table 5.10 gives the optimum results solving the Models 5.2B, 5.2C, 5.2D and 5.2E. Profits of Models 5.2B and 5.2C are greater than the profit of Model 5.2A because of zero cap (C=0.0tonnes) has been considered in Model 5.2A. Model 5.2C gives the highest profit among all models as the firm gets some rewards from regulatory authority for less CE than cap. On the other hand, for other models, the firm pays penalty for CE. Model 5.2D gives the satisfactory profit $[7888.76, 8674.84, 9460.92] under the strict carbon cap (C=7000.00tonnes) constraint. It is to be noted that these results of Model 5.2D are in between the results of Models 5.2B and 5.2C. It is as per expectation in Model 5.2D, the inequality $CE \leq C$ is taken in equality sense of C where Models 5.2B and 5.2C assume CE, more and less than C respectively.

• Now from the Table 5.11, it is seen that the production rate P and production run time $t_1$ which minimize the $\widetilde{ACEC}$ are quite different from P and $t_1$ which are obtained maximizing the profit $\widetilde{ATP}$ for Models 5.2A, 5.2B and 5.2C. It is interesting to note that for the optimum

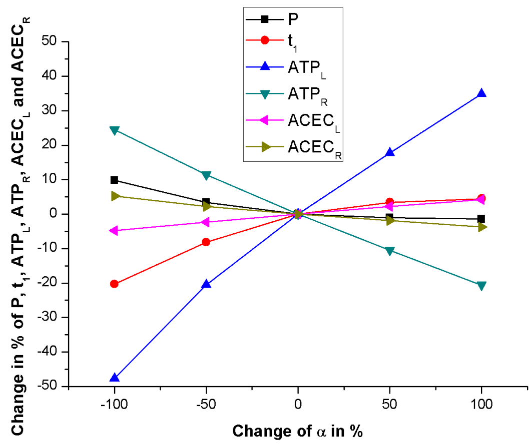



**Figure** 5.18: Sensitivity of $\alpha$ for Model 5.2A

production rate $P^* = 529.67$ and production run time $t_1^* = 3.59 years$ which minimize $\widetilde{ACEC}$, the corresponding profit \$[2646.23, 3589.34, 4532.46] are lower than the profit $\widetilde{ATP}$ =\$$[2728.59, 3681.06, 4633.53]$ obtained maximizing $\widetilde{ATP}$ directly, although for the optimum values, $P^* = 627.48$ and $t_1^* = 3.25 years$ which maximize $\widetilde{ATP}$, corresponding $\widetilde{ACEC}$ \$[4885.71, 5089.32, 5292.94] is higher than the minimum value \$[4865.61, 5057.39, 5249.18] of $\widetilde{ACEC}$. Therefore with respect to Model 5.2A, it is a better choice for a firm's manager to take the profit function $\widetilde{ATP}$ as the objective to get more profit for the system.

### 5.4.7 Practical Implication

In different countries, different types of regulations for CE are in vogue. To the best of our knowledge, there may be four types of regulation- (i) no restriction on CE (i.e. unlimited CE is permitted), (ii) average tax on CE, whatever be the amount, carbon trading- (iii)purchase (penalty) or (iv) sale (revenue). Thus, a production firm management is in a fix i.e. does not know which regulation should be followed so that firm's profit is maximum. Considering this situation, an algorithm (with example) for the management is given for his/her maximum profit in Algorithm 3.

### 5.4.8 Real-life Illustration

**Example 1: For firms from Annex I countries**

Let a company ABC from Annex I countries (Australia, Austria, Belgium, Canada, Germany, Italy, etc.) produces iron bars using coal based production system. Initially, perfect bars were produced upto nearly $[1.00, 1.25, 1.50]$ years from the commencement of production. After that, some imperfect $(25\%)$ bars were produced, some $(75\%)$ of which were reworked and sold as a new one ($100 per bar). The rest were sold at a reduced price $20 per bar at the open market with demand $150$ units per year.

Let P units be production rate per year and production process is continued up to $t_1$. To produce one unit of iron bar, the required raw material costs $5.0; total energy $8000 and minor repair/replacement cost $0.02. The set-up cost for one production/ business cycle is $25000, storage cost per unit bar is $0.30 per year, rework cost is $2.0 per defective unit. Let CEs due to set-up be $10000$ tonnes, holding $0.50$ tonne per unit bar per year, rework $2.0$ tonne per unit, raw material $55.00$ tonnes per amount of raw materials required to produce one unit, total energy $30000$ tonnes per year, wear and tear $0.08$ tonne per one unit.

Let the country has the following clean energy regulation. The country allows a firm to emit carbon of $25000$ tonnes per year and permits $1$ tonne of $CO_2$ at the cost $2.0.

The objective of this firm is to determine P and $t_1$ following the existing emission rule of the country so that the profit of the firm is maximum. Following the prescribed algorithm 3, the detailed optimum results are presented in Table 5.12 and its implementation is given in Table 5.13.

**Table** 5.12: Optimal results of practical implication for Example 1 and 2

| $Example$ | $Model$ | $P^*$ | $t_1^*$ | [$ATP_L^*$, $ATP_C^*$, $ATP_R^*$] | Average Emission | [$N_L, N_R$] | [$Q_L, Q_R$] |
|---|---|---|---|---|---|---|---|
| | 5 | 712.01 | 5.648 | [8093.26, 8672.86, 9252.45] | 27120.39 | [1625, 1821] | [3566, 3615] |
| 1 | 2 | 653.75 | 5.082 | [2357.65, 5801.90, 9246.14] | 26431.34 | [1123, 1280] | [3002, 3042] |
| | 4 | 689.51 | 3.012 | [7456.18, 8261.58, 9066.98] | 25000.00 | [231, 307] | [2000, 2019] |
| 2 | 1 | 300.28 | 8.840 | [2377.98, 4842.41, 7306.84] | 11163.24 | [2106, 2249] | [2110, 2146] |

Therefore, the firm management chooses Model 5.2D to achieve maximum profit as well as it plays an important role for social welfare.

**Example 2: For firms from the developing country like India** In the developing country, India, to improve the environmental performance in future following Kyoto protocol, the government imposes some penalty on the firms for every units of CE, whatever be the amount- high or low.

Let a big iron producing firm from Durgapur, India produces iron bars using coal based production system. Initially, perfect bars were produced upto nearly $[1.00, 1.25, 1.50]$ years from the commencement of production. After that, some imperfect $(25\%)$ bars were produced, some $(75\%)$ of which were reworked and sold as a new one ($200 per bar). The rest were sold at a reduced price $100 per bar at the open market with demand $200$ units per year. To produce one unit of iron bar, the required raw material costs $2.50; total energy $10000 and minor repair cost $0.02. The set-up cost for one production/ business cycle is $15000, storage cost per unit bar is $0.50 per year, rework cost is $2.50 per defective unit. Let CEs due to set-up be $20000$ tonnes, holding $0.50$ tonne per unit bar per year, rework $1.0$

Table 5.13: Algorithm-wise Implementation

| Algorithm | Implementation Example 1 | Implementation Example 2 |
|---|---|---|
| **Step 1: Check the regulatory system** | : Under Carbon Cap and Trading | : Penalty due to CE |
| **if** No restriction on CE i.e., Unlimited cap **then** | : NO | : NO |
| **Use Model 5.2E : EPL with unlimited CE** | : NOT APPLICABLE (NA) | : NA |
| **else if** Tax is imposed due to CE (whatever be the amount) **then** | : NO | : YES |
| **Use Model 5.2A : EPL with carbon tax** | : NA | : APPLICABLE (EXIT) |
| **else** | | |
| **Go to step 2** | : YES | |
| **Step 2: Check CE with limited carbon cap C** | : YES, C=25000.00 tonnes | |
| **Calculate the Average CE using Model 5.2E** | : 27120.39 tonnes of CE | |
| **if** $CE < C$ **then** | : NO | |
| **Use Model 5.2C : EPL with cap and reward** | : NA | |
| **else** | | |
| **Go to step 3** | : YES | |
| **Step 3: Check Optimum profit due to paying penalty for excess CE or use cap as a constraint** | : YES | |
| **Find optimum profit for Model 5.2B** | : $ATP_C = \$5801.90$ | |
| **Find optimum profit for Model 5.2D** | : $ATP_C = \$8261.58$ | |
| **if** Profit of Model 5.2B > Profit of Model 5.2D **then** | : NO | |
| **Use Model 5.2B : EPL with cap and penalty** | : NA | |
| **else** | | |
| **Use Model 5.2D : EPL with carbon constraint** | :APPLICABLE (EXIT) | |
| **End** | | |

tonne per unit, raw material $5.00$ tonnes per amount of raw materials required to produce one unit, total energy $2000$ tonnes per year, wear and tear $0.08$ tonne per one unit.

Let the government of India introduces a nationwide carbon tax of $\$2.0$ per tonne CE. Following the prescribed algorithm 3, the firm management is compelled to use the Model 5.2A. The detailed optimum results are presented in Table 5.12 and the supporting implementation is given in Table 5.13.

## 5.5 Conclusion

This investigation is proposed for the CE in fuzzy environment using various inventory models including :

- The first model proposed here is a production inventory model where defective production rate and CEC are considered in the model formulation. All possible regulations for CE are considered and maximum profits of a firm under each regulation are evaluated. An algorithm for a firm management for maximum profit taking all possible management decisions into consideration against different regulations are given. There is an example with a real life practical case faced by the production firms of different countries.

**Here, for multi-objective optimization technique, RMOGA is also presented.** It is interesting to note that maximum production does not always fetch maximum profit for a firm. Rather, regulated production in which carbon sale (i.e. carbon trade) is permitted / done, gives maximum profit for the system along with the performance of its social responsibilities.

Thus, it is a practical solution to prevent global pollution. Here, all illustrations are with the hypothetical data. Once the real life practical data are available, a firm can implement it following the Algorithm 3 in § 5.3.7.

• The second model proposed here is an imperfect production inventory model where defective production rate is imprecise in nature and CEC is included in it's cost. Using FDE and FRI, an approach is proposed, where $\alpha$-cuts of fuzzy profit are optimized through IFOT to get optimal decision. Optimal profit functions and CEC have been graphically presented as TFNs.

**Thus the outcomes of the present investigation is two fold.**

(i) For defective/ imperfect production system, deterministic model formulation and the corresponding solutions are approximate ones as the time at which imperfect production begins is normally uncertain. Here the consideration of uncertain (imprecise) imperfect production instant, formulation of the models through FDE and its solutions furnish much more realistic and correct solutions of the imperfect production problem.

(ii) For the first time, almost all possible regulations for CE are considered and production rate and production duration for a firm to have maximum profit under each regulation are evaluated. An algorithm for a firm management for the maximum profit taking all possible management decisions into consideration against different regulations is given. This is also illustrated with examples. Now-a-days, these are the real-life emergent problems faced by the production firms throughout the world. In the present formulation, four carbon policies have been outlined which appears to be exhaustive. If in a country, any other type of $CO_2$ policy is in vogue, that can be easily incorporated in the presented algorithm and optimum production rate and duration for maximum profit can be obtained.

Here for a feasible solution of the production system, a fuzzy constraint on the uncertain imperfect production instant is imposed and transformed to a crisp one using possibility and necessity measures. However, in these cases, fuzziness also can be removed using credibility measure. Moreover, for the MOOP, here IFOT has been used. This method is quite young for the solution of MOOPs. The use of IFOT is quite general and can be used for MOOPs in others research areas such as supply-chain, transportation, portfolio management, etc. The present investigation can be extended to include the CE due to transportation of raw materials to production firms, transportation of finish goods to sales counter, etc.

# Chapter 6

# Inventory Problems with Trade Credit Policy in Fuzzy Environment

## 6.1 Introduction

In today's highly competitive commercial market, many suppliers and manufacturers would like to make a co-operative relationship to get tensionless steady sources of supply, production and demand of goods to maximize profits and improve overall quality. Normally, payments are made for goods immediately after receiving the consignment. To avoid stiff price competition, business houses use credit as part of the pricing strategy and provide credit terms to their customers to gain a competitive edge. They often provide credit terms to allow their customers to make purchases today and pay at a later date without any additional charges. Therefore, trade credit plays an important role in a broad range of modern industries and economies. Today, the suppliers offer to manufacturer-cum-retailers a delay period known as trade credit to encourage sales, promote market share, and reduce on-hand stock levels of raw materials within the fixed permitted settlement period. During this period, as the supplier does not charge any interest, manufacturer-cum-retailer can earn interest by depositing the generated sales revenue into an interest bearing account. However, if the payment is not paid in full by the end of the permissible delay period, then suppliers charge the manufacturer-cum-retailers an interest on the outstanding amount. In real practice, manufacturer-cum-retailer also offers a credit period to his/her customers to stimulate own demand. Generally, credit period offered by the supplier is greater than or equal to credit period offered by manufacturer to customers. As in reality, demands of the customers during credit period increases and depends on the credit period. Normally, as mentioned above, a retailer shares a part of supplier's offered credit period with the customers, thus credit offered by the retailer is less than the supplier's credit. But, in some cases, to get the cash quickly or to avoid the deterioration of the item, etc, retailer may give more credit period than that given by the supplier to create high demand for the item so that the item is sold quickly.

Conventionally, if there is any due after the credit period, the manufacturer cum retailer clears the dues at the end of the business period. As it is assumed that the unit selling price is greater than the unit purchasing cost, the manufacturer must have sufficient amounts before the end of business period and he may pay that amount to the supplier some time before the end of the total cycle and in this situation, he will have to pay less interest to the supplier. Moreover,the manufacturer can earn more interest after that time till the end of the business period. This new approach to calculate the interest earned by the retailer is considered in this paper and the result is compared with the above mentioned conventional approach also. Till now, very few have used this new approach in the analysis of inventory model with trade credit.

It has been recognized that one's ability to make precise statement concerning different parameters of an inventory model diminishes with the increasing complexities of world economy throughout the year. As a result it is very difficult to estimate the parameters of an inventory model precisely. Here, we consider that manufacturer cum retailer's demand is trade credit dependent and its parameters are fuzzy in nature. Further more, the rate of producing defective units is fuzzy.

In the formulation of the model, presence of fuzzy demand as well as fuzzy defective production rate leads to fuzzy differential equation of instantaneous state of inventory level. Till now fuzzy differential equation formulation is not much used to solve fuzzy inventory models though the topics on fuzzy differential equations have been rapidly growing in the recent years.

Present investigation is motivated by the fact that many sales organizations offer trade credit periods and production firms produce and sell the produced items in imprecise environment. In developing countries, vendor's providing credit to their buyers is an important form of financing for business and particularly role of trade credit is immense where growth of financial institutions is less compared to developed nations. Moreover, the volume of trade credit in aggregate represents 17.8% of total assets for US firms, 22% for UK firms, and more than 25% for the countries such as Germany, France and Italy [208].

## 6.2 Literature Review

The main objective of inventory management deals with maximization of the total inventory profit for which it is required to determine the optimal inventory policy to meet the future demand. Generally a manufacturer produces an item and it is sold at different markets. But at-times, it has been observed that the markets have different selling seasons. Hence, manufacturer/supplier has to adopt the appropriate management policies/strategies in the business with the different markets. In a production inventory model with deteriorating items, He et al. [102] considered multiple-market demands. Krichen et al. [132] described a single supplier and multiple cooperative retailers inventory model under permissible delay in payments. Then Pal et al. [194] researched on multi-echelon supply chain model in multiple markets with supply disruption. Many researchers (as mentioned in § 1.3.5)incorporated the idea of trade credit in different types of inventory models (EOQ, EPQ, EPL, etc.) in one or two level policy. One of the drawbacks of the models [89, 111, 154] is that the assumed demand during retailer's credit period is constant, which is not realistic. Teng et al. [253] obtained the optimal ordering policy for stock-dependent demand under progressive payment scheme. Further, Teng et al. [254] extended the demand from constant to non-decreasing pattern. Lately, Dye and Yang [74] considered issues of sustainability in the context of joint trade credit and inventory management in which the demand depends on the length of the credit period offered by the retailer to its customers.

As mentioned in § 1.3.2, Salameh and Jaber [225], Maddah and Jaber [153], Sana [227], Ouyang and Chang [188], Sarkar et al. [228] and others are addressed the production inventory model for imperfect item with either lost sale or repairing, but till now, none has considered the production-inventory system with fuzzy defective rate. During last few decades, due to high inflation and consequent sharp decline in the purchasing power of money in the developing countries like India, Bangladesh etc., the financial situation has been changed and it is not possible to ignore the effect of inflation and time value of money any further. Recently, Tiwari et al. [256] developed a two warehouse inventory model for non-instantaneous deteriorating items with permissible delay in payments under inflationary conditions. Mousavi et al. [183] presented a seasonal multi-product multi-period inventory control model with inventory costs obtained under inflation and all-unit discount policy.

**Table** 6.1: Literature Review for Model-6.1

| Authors with year | Model type | Item's character | Credit period | Learning effect | Inflation | Multiple markets |
|---|---|---|---|---|---|---|
| Goyal [90], 1985 | EOQ | Perfect | Crisp | No | No | No |
| Teng [251], 2002 | EOQ | Perfect | Crisp | No | No | No |
| Mahata and Goswami [155], 2010 | EPQ | Deterioration | Crisp | No | No | No |
| He et al. [102], 2010 | Supply chain | Deterioration | No | No | No | Yes |
| Krichen et al. [132], 2011 | EOQ | Perfect | Crisp | No | No | No |
| Pal et al. [194], 2012 | Supply chain | Defective | No | No | No | Yes |
| Ouyang and Chang [188], 2013 | EPQ | Defective | Crisp | No | No | No |
| Sarkar et al. [228], 2014 | Supply chain | Defective | Crisp | No | No | No |
| Das et al. [62], 2015 | Supply chain | Deterioration | Fuzzy | No | No | Yes |
| Tiwari et al. [256], 2016 | EOQ | Deterioration | Crisp | No | Yes | No |
| Mousavi et al. [183], 2016 | EOQ | Perfect | No | No | Yes | No |
| Present Model 6.1 | EPL | Defective | Fuzzy | on Transportation | Yes | Yes |

The first impetus on solving fuzzy differential equation was made by Kandel and Byatt

**Table** 6.2: Literature Review for Model-6.2

| Author(s) with year | Model type | Product's character | Demand | Trade credit | Payment type |
|---|---|---|---|---|---|
| Huang [109], 2006 | EOQ | Perfect | Constant | Two-levels | Conventional |
| Ho [104], 2011 | Supply chain | Perfect | Price and credit linked | Two-levels | Conventional |
| Mahata [154], 2012 | Supply chain | Deterioration | Constant | Two-levels | Conventional |
| Chen and Wang [51], 2012 | Supply chain | Perfect | Random | Two-levels | Conventional |
| Chung and Cárdenas-Barrón [57], 2013 | Supply chain | Deterioration | Stock-dependent | Two-levels | Conventional |
| Ouyang et al. [188], 2013 | EPQ | Imperfect | Constant | Two-levels | Conventional |
| Ouyang et al. [192] | Supply chain | Perfect | Constant | Two-levels | Conventional |
| Liao et al. [142], 2013 | Supply chain | Deterioration | Constant | Single-level | Conventional |
| Chung et al. [58], 2014 | Supply chain | Deterioration | Constant | Two-levels | Conventional |
| Wu et al. [271], 2014 | Supply chain | Deterioration | Trade credit dependent | Two-levels | Conventional |
| Mohini and Pakkala [177], 2015 | EOQ | Deterioration | Random | Single-level | Against delivery, within trade credit, at the trade credit limit and beyond trade credit limit |
| Majumder et al. [166], 2016 | EPQ | Deterioration | Constant | Two-levels | Conventional and beyond trade credit limit |
| Present Model 6.2 | EPL | Imperfect | Trade credit & fuzzy | Two-levels | Conventional and beyond trade credit limit |

[121]. An extended version of their work was published after 2 years. After that different approaches have been used by several authors to solve fuzzy differential equations [27, 40, 119, 260].

Thus, the comparison investigations of the Model 6.1 are presented in Table 6.1.

Again in the context of earlier investigations as presented in Table 6.2, the new considerations in Model 6.2 are as follows:

- Two levels trade credit offered by supplier and manufacturer-cum-retailer is considered.
- A new approach for maximum profit is proposed and compared with the conventional method.
- Assuming fuzzy demand, the proposed inventory model is formulated through FDE and the appropriate solution is obtained.

## 6.3 Model-6.1 : A learning effected imperfect production inventory model for several markets with fuzzy trade credit period and inflation [1]

In this model, it has been considered that manufacturer produces finished goods along with a constant imperfect rate and delivered it to different seasonal markets where the demand rate

[1]This model has been communicated in **International journal of Uncertainty, Fuzziness and Knowledge based system**, World Scientific

and time duration of each market are different. The manufacturer gives an opportunity of initial part payment to those markets who receive the goods during the production run time and the remaining amount paid at the end of the business period. To produce the finished goods, manufacturer received the raw-material instantly rate from the supplier who offers an imprecise credit period to the manufacturer. The proposed model is formulated in terms of integrated total profit under the following assumptions and notations:

### 6.3.1 Assumptions and Notations

***N**otations:*

$I(t)$ Inventory level at time t.

P Manufacturer's production rate per year.

$t_1$ Length of time from the beginning and end of production. It has been taken as DV.

M Manufacturer's trade credit period offered by raw materials supplier in years.

f Unit usage of raw materials per finished product.

$\lambda$ Percentage of defective rate.

$Cs_p$ Manufacturer's set up cost.

$Cs_m$ Market's set up cost.

$C_r$ Unit purchasing price of raw materials.

$C_p$ Unit production cost.

$S_p$ Manufacturer's unit selling price.

$S_m$ Unit selling price of markets.

$H_r$ Unit stock holding cost of raw materials.

$H_p$ Unit stock holding cost per perfect finished product of manufacturer.

$H_m$ Unit stock holding cost per perfect finished product at markets.

$i_e$ Interest earned per quantity per year by the the retailer.

$i_c$ Interest charged per quantity in stocks per year by the raw materials supplier.

$r$ The discount rate.

$i$ The inflation rate, which is varied by the social economical situations (e.g., consumer price index (CPI)and producer price index (PPI)), and the company operation status(e.g.,operation cost index,and productivity index).

$R$ $= r - i$, is the difference between discount rate and inflation rate.

$d_i$ Customers' demand rate per year for $i^{th}$ market.

$T_i$ Starting time of the business of $i^{th}$ market.

$T_{ei}$ Time at which the selling season ends for $i^{th}$ market.

n Number of markets where the products are transported from the manufacturer.

$Q_i$ Quantity received by $i^{th}$ market from the manufacturer.

WTP Total manufacturer's profit.
MTP Total markets' profit.
ITP Total profit for the integrated system.

***A**ssumptions:*

(i) A single manufacturer-cum-retailer and multiple markets have been assumed for the flow of single product.

(ii) Shortages are not allowed and lead time is negligible.

(iii) Time horizon is finite.

(iv) Interest charged per quantity $(i_c)$ is greater or equal to interest earned per quantity $(i_e)$ i.e., $(i_c \geq i_e)$.

(v) The manufacturer offers an opportunity of initial part payment to those markets who receive the item before the end of production run and the remaining part should be paid at the end of their individual business session. But the markets who receive their required items after the production run time $(t_1)$, don't get this opportunity. Every market receives all items at the starting time of their business period to fulfil their fixed customers' demand.

(vi) Market selling price $(S_m)$ is greater or equal to manufacturer's selling price $(S_p)$ which is also greater than the unit purchase price of raw materials $(C_r)$. i.e., $S_m \geq S_p \geq C_r$.

(vii) The credit period $(M)$ is offered by the supplier to the manufacturer is not fixed i.e., it changes due to various factors according to his/her business policy. So in nature, it is vague and imprecise. For this reason here, it has been considered that raw material supplier offers a fuzzy credit period to the manufacturer.

(viii) Learning effect of the manufacturer-cum-retailer reduced transportation cost in each time of markets' order is reduced at a rate $\gamma$.

### 6.3.2 Raw material's inventory for manufacturer

The manufacturer receives all the required quantity of raw materials instantaneously from the raw material supplier to produce the finished good when he/she is going to start his/her production (cf. Fig. 6.1). Then the inventory of raw materials depletes gradually with time due to production and completely depleted at time $t_1$. So, the raw material's inventory $I_r(t)$ at time t satisfies

$$\frac{dI_r(t)}{dt} = -fP \tag{6.1}$$

with the boundary condition $I_r(t_1) = 0$. Hence, the inventory level $I_r(t)$ at time t is

$$I_r(t) = fP(t_1 - t) \tag{6.2}$$

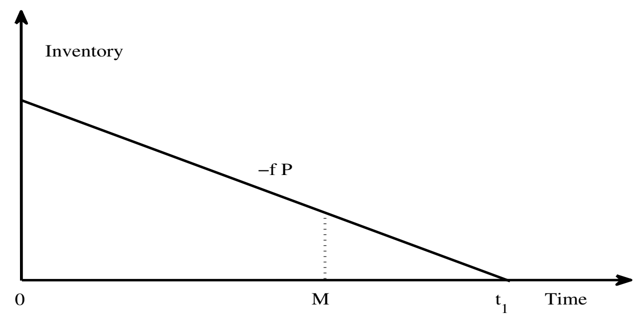



**Figure** 6.1: Raw material's inventory vs. time

The quantity of raw materials received by the manufacturer is

$$Q_r = I_r(0) = fPt_1 \tag{6.3}$$

Present value of raw material's holding cost is

$$\begin{aligned} HC_r &= H_r \int_0^{t_1} e^{-Rt} I_r(t)dt = H_r \int_0^{t_1} e^{-Rt} fP(t_1 - t)dt \\ &= H_r fP \left\{ \frac{t_1}{R} - \frac{1 - e^{-Rt_1}}{R^2} \right\} \end{aligned} \tag{6.4}$$

The manufacturer needs to make the full payments of the raw-material at the end of the credit period M, otherwise manufacturer will have to pay an interest to the supplier. So, present value of interest payable by the manufacturer is

$$\begin{aligned} IP_r &= C_r i_c \int_M^{t_1} e^{-Rt} I_r(t)dt = C_r i_c \int_M^{t_1} e^{-Rt} fP(t_1 - t)dt \\ &= C_r i_c fP \left\{ \frac{(t_1 - M)e^{-RM}}{R} - \frac{e^{-RM} - e^{-Rt_1}}{R^2} \right\} \end{aligned} \tag{6.5}$$

Present value of manufacturer's purchase cost for the raw materials will be

$$PC_r = C_r \int_0^{t_1} fPe^{-Rt}dt = \frac{C_r fP(1 - e^{-Rt_1})}{R} \tag{6.6}$$

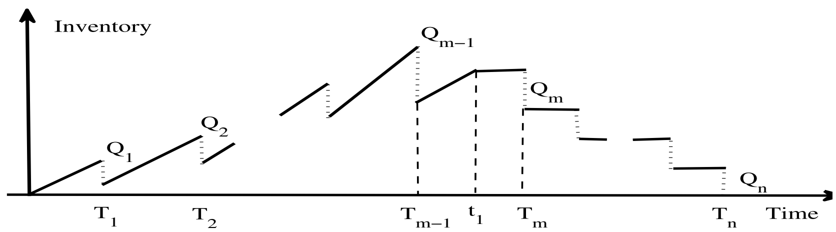



**Figure** 6.2: Manufacturer's finished product inventory vs. time

### 6.3.3 Manufacturer's finished products' inventory

The manufacturer starts production of the finished product along with a constant defective rate $\lambda$ from $t = 0$ and then the inventory level increases over time. After time $T_1$, the first market receives its required quantity $Q_1$ instantaneously, then at time $T_2$, the second market receives its required quantity $Q_2$ instantaneously and so on. Gradually, as the production ceases at time $t_1$, the inventory level decreases due to the remaining markets' instantaneous replenishment and the inventory completely depleted after the receipt of the quantity $Q_n$ by the last market. Let the manufacturer's inventory level in the interval $[T_{k-1}, T_k]$ $(k = 1, 2, \cdots, n)$ be $I_k(t)$ and $I_{m-}(t)$, $I_{m+}(t)$ be the inventory levels in time intervals $[T_{m-1}, t_1]$, $[t_1, T_m]$ respectively. Then the differential equations of the finished products' inventory levels at time t in $[0, T_n]$ are as follows ( cf. Fig. 6.2):

$$
\begin{aligned}
\frac{dI_1(t)}{dt} &= (1-\lambda)P, \qquad 0 \le t \le T_1 \\
\frac{dI_k(t)}{dt} &= (1-\lambda)P, \qquad T_{k-1} \le t \le T_k \qquad [k = 2, 3, \cdots, m-1] \\
\frac{dI_{m-}(t)}{dt} &= (1-\lambda)P, \qquad T_{m-1} \le t \le t_1 \\
\frac{dI_{m+}(t)}{dt} &= 0, \qquad t_1 \le t \le T_m \\
\frac{dI_k(t)}{dt} &= 0, \qquad T_{k-1} \le t \le T_k \qquad [k = m+1, m+2, \cdots, n]
\end{aligned}
$$

with boundary conditions

$$\begin{aligned}
&I_1(0) = 0,\\
&I_k(T_{k-1}) = I_{k-1}(T_{k-1}) - Q_{k-1}, \qquad [k = 2, 3, \cdots, m-1]\\
&I_{m-}(T_{m-1}) = I_{m-1}(T_{m-1}) - Q_{m-1},\\
&I_{m+}(T_m) = I_{m+1}(T_m) + Q_m,\\
&I_k(T_{k-1}) = I_{k-1}(T_{k-1}) - Q_{k-1}, \qquad [k = m+1, m+2, \cdots, n]\\
&\text{and } I_n(T_n) = Q_n
\end{aligned}$$

Solving these equations, we get the level of inventory at different time t as

$$I_k(t) = (1-\lambda)Pt - \sum_{i=1}^{k-1} Q_i \qquad T_{k-1} \le t \le T_k \;\; [k = 1, 2, \cdots, m-1], \tag{6.7}$$

$$I_{m-}(t) = (1-\lambda)Pt - \sum_{i=1}^{m-1} Q_i \qquad T_{m-1} \le t \le t_1 \tag{6.8}$$

$$I_{m+}(t) = \sum_{i=m}^{n} Q_i, \qquad t_1 \le t \le T_m \tag{6.9}$$

$$I_k(t) = \sum_{i=k}^{n} Q_i \qquad T_{k-1} \le t \le T_k \qquad [k = m+1, m+2, \cdots, n] \tag{6.10}$$

Using the continuity condition at $t = t_1$, we have

$$I_{m-}(t_1) = I_{m+}(t_1) \;\; \text{or, } (1-\lambda)Pt_1 - \sum_{i=1}^{m-1} Q_i = \sum_{i=m}^{n} Q_i \;\; \text{or, } P = \frac{1}{(1-\lambda)t_1} \sum_{i=1}^{n} Q_i \tag{6.11}$$

which gives the relation between the two variables P and $t_1$.
The present value of holding cost for the manufacturer's finished product is

$$HC_p = H_p\Big[\sum_{k=1}^{m-1} L_k + L_{m-} + L_{m+} + \sum_{k=m+1}^{n} L'_k\Big]$$

where,

$$L_k = \int_{T_{k-1}}^{T_k} \Big\{(1-\lambda)Pt - \sum_{i=1}^{k-1} Q_i\Big\} e^{-Rt} dt, \qquad k = 1, 2, \cdots, m-1$$

$$= (1-\lambda)P\Big\{\frac{T_{k-1}e^{-RT_{k-1}} - T_k e^{-RT_k}}{R} + \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R^2}\Big\} - \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R} \sum_{i=1}^{k-1} Q_i$$

$$L_{m-} = \int_{T_{m-1}}^{t_1} \Big\{(1-\lambda)Pt - \sum_{i=1}^{m-1} Q_i\Big\} e^{-Rt} dt$$
$$= (1-\lambda)P\Big\{\frac{T_{m-1}e^{-RT_{m-1}} - t_1 e^{-Rt_1}}{R} + \frac{e^{-RT_{m-1}} - e^{-Rt_1}}{R^2}\Big\} - \frac{e^{-RT_{m-1}} - e^{-Rt_1}}{R}\sum_{i=1}^{m-1} Q_i$$

$$L_{m+} = \int_{t_1}^{T_m} \sum_{i=m}^{n} Q_i e^{-Rt} dt = \frac{e^{-Rt_1} - e^{-RT_m}}{R}\sum_{i=m}^{n} Q_i$$

$$L'_k = \int_{T_{k-1}}^{T_k} \sum_{i=k}^{n} Q_i e^{-Rt} dt = \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R}\sum_{i=k}^{n} Q_i, \qquad k = m+1, m+2, \cdots, n$$

Therefore,

$$HC_p = H_p\Big[(1-\lambda)P\Big\{\frac{1-e^{-Rt_1}}{R^2} - \frac{t_1 e^{-Rt_1}}{R}\Big\} - \sum_{k=1}^{m-1}\frac{e^{-RT_{k-1}} - e^{-RT_k}}{R}\sum_{i=1}^{k-1} Q_i$$
$$-\frac{e^{-RT_{m-1}} - e^{-Rt_1}}{R}\sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1} - e^{-RT_m}}{R}\sum_{i=m}^{n} Q_i$$
$$+\sum_{k=m+1}^{n}\frac{e^{-RT_{k-1}} - e^{-RT_k}}{R}\sum_{i=k}^{n} Q_i\Big] \tag{6.12}$$

The present value of manufacturer's production cost is

$$PC_p = C_p P e^{-Rt} dt = \frac{C_p P(1 - e^{-Rt_1})}{R} \tag{6.13}$$

The present value for transportation cost of perfect products ordered by the market which is reduced at a learning rate $\gamma$ is

$$TC_p = \sum_{i=1}^{n}\{T_{r0} + T_{r1}e^{-\gamma(i-1)}Q_i\}e^{-RT_i} \tag{6.14}$$

The present value of manufacturer's sales revenue is

$$SR_p = S_p \sum_{i=1}^{n} Q_i e^{-RT_i} \tag{6.15}$$

The present value of manufacturer's interest earned is

$$IE_p = \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p\rho Q_i \int_{T_i}^{T_n} e^{-Rt}dt + \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p(1-\rho)Q_i \int_{T_{ei}}^{T_n} e^{-Rt}dt + \sum_{\substack{i=m\\T_i>t_1}}^{n} i_eS_pQ_i \int_{T_i}^{T_n} e^{-Rt}dt$$
$$= \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p\rho Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R} + \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p(1-\rho)Q_i \frac{e^{-RT_{ei}}-e^{-RT_n}}{R} + \sum_{\substack{i=m\\T_i>t_1}}^{n} i_eS_pQ_i \frac{e^{-RT_i}-e^{-RT_n}}{R} \tag{6.16}$$

The present value for manufacturer's total profit is given by $WTP(t_1)$ = *sales revenue + interest earned - set-up cost - raw material's purchase cost - raw material's holding cost - production cost - finished products holding cost - transportation cost - interest payable.* i.e.

$$WTP(t_1) = SR_p + IE_p - Cs_p - PC_r - HC_r - PC_p - HC_p - TC_p - IP_r$$
$$= S_p\sum_{i=1}^{n} Q_ie^{-RT_i} + \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p\rho Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R} + \sum_{\substack{i=1\\T_i\leq t_1}}^{m-1} i_eS_p(1-\rho)Q_i \frac{e^{-RT_{ei}}-e^{-RT_n}}{R}$$
$$+ \sum_{\substack{i=m\\T_i>t_1}}^{n} i_eS_pQ_i \frac{e^{-RT_i}-e^{-RT_n}}{R} - Cs_p - \frac{C_rfP(1-e^{-Rt_1})}{R} - H_rfP\left\{\frac{t_1}{R} - \frac{1-e^{-Rt_1}}{R^2}\right\}$$
$$- \frac{C_pP(1-e^{-Rt_1})}{R} - H_p\Bigg[(1-\lambda)P\left\{\frac{1-e^{-Rt_1}}{R^2} - \frac{t_1e^{-Rt_1}}{R}\right\} - \sum_{k=1}^{m-1} \frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=1}^{k-1} Q_i$$
$$- \frac{e^{-RT_{m-1}}-e^{-Rt_1}}{R}\sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1}-e^{-RT_m}}{R}\sum_{i=m}^{n} Q_i + \sum_{k=m+1}^{n} \frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=k}^{n} Q_i\Bigg]$$
$$- \sum_{i=1}^{n}\{T_{r0} + T_{r1}e^{-\gamma(i-1)}Q_i\}e^{-RT_i} - C_ri_cfP\left\{\frac{(t_1-M)e^{-RM}}{R} - \frac{e^{-RM}-e^{-Rt_1}}{R^2}\right\} \tag{6.17}$$

### 6.3.4 The markets' inventory

The $i^{th}$ market receives its total required quantity $Q_i$ of finished good from the manufacturer at the beginning of its selling season to fulfil the customers' demand rate $d_i$. These markets start their business on or before the production run time $t_1$, by paying $\rho$ portion of the price amount initially and the remaining $(1-\rho)$ portion at the end of his business period. But these markets take the delivery after the production run time $t_1$, by paying the total amount at their business starting time. They pay the initial amount by getting loan from a bank at the rate of interest of $i_c$ per year. Every market earns interest at the rate of $i_d$ by depositing sales revenue continuously. The inventory level $J_i(t)$ for the $i^{th}$ market is governed by the following differential equation (cf. Fig. 6.3):

$$\frac{dJ_i(t)}{dt} = -d_i, \qquad i = 1, 2, 3, \cdots, n. \tag{6.18}$$

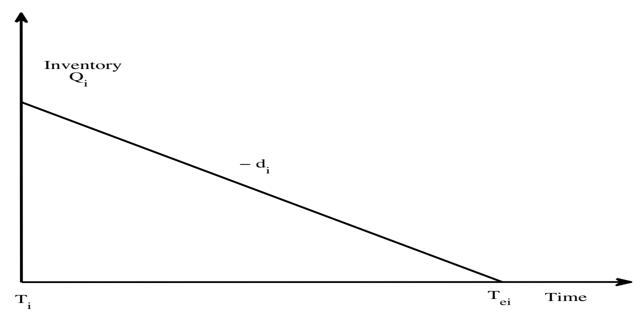



**Figure** 6.3: $i^{th}$ market's inventory vs. time

with the boundary condition $J_i(T_{ei}) = 0$. Using these boundary conditions, the solutions of the Eqs. (6.18) are

$$J_i(t) = d_i(T_{ei} - t) \tag{6.19}$$

The quantity of products received by the each market is

$$Q_i = J_i(T_i) = d_i(T_{ei} - T_i) \tag{6.20}$$

The present value of holding cost for all markets is

$$\begin{aligned} HC_m &= \sum_{i=1}^{n} H_m \int_{T_i}^{T_{ei}} J_i(t) e^{-Rt} dt = \sum_{i=1}^{n} H_m \int_{T_i}^{T_{ei}} d_i(T_{ei} - t) e^{-Rt} dt \\ &= \sum_{i=1}^{n} H_m d_i \left\{ \frac{(T_{ei} - T_i) e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} \end{aligned} \tag{6.21}$$

The present value of all markets' total sales revenue is

$$SR_m = \sum_{i=1}^{n} S_m d_i \int_{T_i}^{T_{ei}} e^{-Rt} dt = \sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} \tag{6.22}$$

The present value of all markets' purchase cost $PC_m$ is equal to present value of sales revenue $SR_p$ of the manufacturer.

The present value of all markets' total interest payable is

$$\begin{aligned} IP_m &= \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \int_{T_i}^{T_{ei}} e^{-Rt} dt + \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_c S_p Q_i \int_{T_i}^{T_{ei}} e^{-Rt} dt \\ &= \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} + \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} \end{aligned} \tag{6.23}$$

The present value of all markets' total interest earned is

$$\begin{aligned} IE_m &= \sum_{i=1}^{n} i_e S_m d_i \int_{T_i}^{T_{ei}} (T_{ei} - t) e^{-Rt} dt \\ &= \sum_{i=1}^{n} i_e S_m d_i \left\{ \frac{(T_{ei} - T_i) e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} \end{aligned} \tag{6.24}$$

Therefore, the total profit of all markets is given by

$$\begin{aligned} MTP(t_1) &= SR_m + IE_m - PC_m - HC_m - IP_m - nCs_m \\ &= \sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} + \sum_{i=1}^{n} i_e S_m d_i \left\{ \frac{(T_{ei} - T_i) e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} \\ &- S_p \sum_{i=1}^{n} Q_i e^{-RT_i} - \sum_{i=1}^{n} H_m d_i \left\{ \frac{(T_{ei} - T_i) e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} \\ &- \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} + \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} - nCs_m \end{aligned} \tag{6.25}$$

### 6.3.5 Model 6.1A : An imperfect production inventory model for a manufacturer-cum-retailer and several seasonal markets

The total profit of the above described model for the integrated system is written as

$$\begin{aligned} ITP(t_1) &= WTP(t_1) + MTP(t_1) \\ &= \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_e S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_n}}{R} + \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_e S_p (1-\rho) Q_i \frac{e^{-RT_{ei}} - e^{-RT_n}}{R} \\ &+ \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_e S_p Q_i \frac{e^{-RT_i} - e^{-RT_n}}{R} - Cs_p - \frac{C_r f P (1 - e^{-Rt_1})}{R} - H_r f P \left\{ \frac{t_1}{R} - \frac{1 - e^{-Rt_1}}{R^2} \right\} \\ &- \frac{C_p P (1 - e^{-Rt_1})}{R} - H_p \left[ (1-\lambda) P \left\{ \frac{1 - e^{-Rt_1}}{R^2} - \frac{t_1 e^{-Rt_1}}{R} \right\} - \sum_{k=1}^{m-1} \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R} \sum_{i=1}^{k-1} Q_i \right. \end{aligned}$$

$$- \frac{e^{-RT_{m-1}} - e^{-Rt_1}}{R} \sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1} - e^{-RT_m}}{R} \sum_{i=m}^{n} Q_i + \sum_{k=m+1}^{n} \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R} \sum_{i=k}^{n} Q_i \Bigg]$$
$$- \sum_{i=1}^{n} \{T_{r0} + T_{r1} e^{-\gamma(i-1)} Q_i\} e^{-RT_i} - C_r i_c f P \left\{ \frac{(t_1 - M)e^{-RM}}{R} - \frac{e^{-RM} - e^{-Rt_1}}{R^2} \right\}$$
$$+ \sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} + \sum_{i=1}^{n} i_e S_m d_i \left\{ \frac{(T_{ei} - T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\}$$
$$- \sum_{i=1}^{n} H_m d_i \left\{ \frac{(T_{ei} - T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} - \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R}$$
$$+ \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} - nCs_m \tag{6.26}$$

Substituting the value of P from Eq. (6.11) to the above integrated profit function (6.26) we have,

$$ITP(t_1) = \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_e S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_n}}{R} + \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_e S_p (1-\rho) Q_i \frac{e^{-RT_{ei}} - e^{-RT_n}}{R}$$
$$+ \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_e S_p Q_i \frac{e^{-RT_i} - e^{-RT_n}}{R} - Cs_p - \frac{C_r f (1 - e^{-Rt_1})}{(1-\lambda) R t_1} \sum_{i=1}^{n} Q_i - \frac{H_r f}{(1-\lambda)} \left\{ \frac{1}{R} - \frac{(1 - e^{-Rt_1})}{R^2 t_1} \right\} \sum_{i=1}^{n} Q_i$$
$$- \frac{C_p (1 - e^{-Rt_1})}{(1-\lambda) R t_1} \sum_{i=1}^{n} Q_i - H_p \Bigg[ \left\{ \frac{1 - e^{-Rt_1}}{R^2 t_1} - \frac{e^{-Rt_1}}{R} \right\} \sum_{i=1}^{n} Q_i - \sum_{k=1}^{m-1} \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R} \sum_{i=1}^{k-1} Q_i$$
$$- \frac{e^{-RT_{m-1}} - e^{-Rt_1}}{R} \sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1} - e^{-RT_m}}{R} \sum_{i=m}^{n} Q_i + \sum_{k=m+1}^{n} \frac{e^{-RT_{k-1}} - e^{-RT_k}}{R} \sum_{i=k}^{n} Q_i \Bigg]$$
$$- \sum_{i=1}^{n} \{T_{r0} + T_{r1} e^{-\gamma(i-1)} Q_i\} e^{-RT_i} - \frac{C_r i_c f}{(1-\lambda)} \left\{ \frac{(t_1 - M)e^{-RM}}{Rt_1} - \frac{e^{-RM} - e^{-Rt_1}}{R^2 t_1} \right\} \sum_{i=1}^{n} Q_i$$
$$+ \sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} + \sum_{i=1}^{n} i_e S_m d_i \left\{ \frac{(T_{ei} - T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\}$$
$$- \sum_{i=1}^{n} H_m d_i \left\{ \frac{(T_{ei} - T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i} - e^{-RT_{ei}}}{R^2} \right\} - \sum_{\substack{i=1 \\ T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R}$$
$$+ \sum_{\substack{i=m \\ T_i > t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i} - e^{-RT_{ei}}}{R} - nCs_m \tag{6.27}$$

### 6.3.6 Model 6.1B: Model 6.1A without inflation

Taking $R \to 0$ in the integrated profit function Eq. (6.27) we get,

$$ITP(t_1) = \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p \rho Q_i (T_n - T_i) + \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p (1-\rho) Q_i (T_n - T_{ei})$$
$$+ \sum_{\substack{i=m\\T_i > t_1}}^{n} i_e S_p Q_i (T_n - T_i) - Cs_p - \frac{C_r f}{(1-\lambda)} \sum_{i=1}^{n} Q_i - \frac{H_r f t_1}{2(1-\lambda)} \sum_{i=1}^{n} Q_i - \frac{C_p}{(1-\lambda)} \sum_{i=1}^{n} Q_i$$
$$-H_p \Bigg[ \frac{1}{2} t_1 \sum_{i=1}^{n} Q_i - \sum_{k=1}^{m-1} (T_k - T_{k-1}) \sum_{i=1}^{k-1} Q_i - (t_1 - T_{m-1}) \sum_{i=1}^{m-1} Q_i + (T_m - t_1) \sum_{i=m}^{n} Q_i$$
$$+ \sum_{k=m+1}^{n} (T_k - T_{k-1}) \sum_{i=k}^{n} Q_i \Bigg] - \sum_{i=1}^{n} \{T_{r0} + T_{r1} e^{-\gamma(i-1)} Q_i\} - \frac{C_r i_c f (t_1 - M)^2}{2(1-\lambda) t_1} \sum_{i=1}^{n} Q_i$$
$$+ \sum_{i=1}^{n} S_m d_i (T_{ei} - T_i) + \sum_{i=1}^{n} \frac{i_e S_m d_i (T_{ei} - T_i)^2}{2} - \sum_{i=1}^{n} \frac{H_m d_i (T_{ei} - T_i)^2}{2}$$
$$- \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_c S_p \rho Q_i (T_{ei} - T_i) + \sum_{\substack{i=m\\T_i > t_1}}^{n} i_c S_p Q_i (T_{ei} - T_i) - nCs_m \tag{6.28}$$

### 6.3.7 Model 6.1C: Model 6.1A without both defective and inflation

Taking $\lambda = 0$ in the Eq. (6.11) and integrated profit function's Eq. (6.28) we get,

$$P = \frac{1}{t_1} \sum_{i=1}^{n} Q_i \tag{6.29}$$

and $$ITP(t_1) = \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p \rho Q_i (T_n - T_i) + \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p (1-\rho) Q_i (T_n - T_{ei})$$
$$+ \sum_{\substack{i=m\\T_i > t_1}}^{n} i_e S_p Q_i (T_n - T_i) - Cs_p - C_r f \sum_{i=1}^{n} Q_i - \frac{H_r f t_1}{2} \sum_{i=1}^{n} Q_i - C_p \sum_{i=1}^{n} Q_i$$
$$-H_p \Bigg[ \frac{1}{2} t_1 \sum_{i=1}^{n} Q_i - \sum_{k=1}^{m-1} (T_k - T_{k-1}) \sum_{i=1}^{k-1} Q_i - (t_1 - T_{m-1}) \sum_{i=1}^{m-1} Q_i + (T_m - t_1) \sum_{i=m}^{n} Q_i$$
$$+ \sum_{k=m+1}^{n} (T_k - T_{k-1}) \sum_{i=k}^{n} Q_i \Bigg] - \sum_{i=1}^{n} \{T_{r0} + T_{r1} e^{-\gamma(i-1)} Q_i\} - \frac{C_r i_c f (t_1 - M)^2}{2 t_1} \sum_{i=1}^{n} Q_i$$

$$+\sum_{i=1}^{n} S_m d_i(T_{ei}-T_i)+\sum_{i=1}^{n}\frac{i_e S_m d_i(T_{ei}-T_i)^2}{2}-\sum_{i=1}^{n}\frac{H_m d_i(T_{ei}-T_i)^2}{2}$$
$$-\sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_c S_p \rho Q_i(T_{ei}-T_i)+\sum_{\substack{i=m\\T_i>t_1}}^{n} i_c S_p Q_i(T_{ei}-T_i)-nCs_m \tag{6.30}$$

Our objective is to find out optimal values of $t_1$ such that the integrated total profit functions $ITP(t_1)$ given by Eqs. (6.27), (6.28) and (6.30) are maximum for the Models-6.1A, 6.1B and 6.1C respectively.

**Lemma 6.1.** *Manufacturer's production run time* $(t_1)$ *must satisfy the condition* $1\le\frac{t_1}{T_k}\le\sum_{i=1}^{n}Q_i/\sum_{i=1}^{k}Q_i$ *for all k=1, 2, $\cdots$, m-1.*

*Proof.* According to our assumption, for k=1, 2, $\cdots$, m-1 we have $T_k\le t_1$ and $I_k(t)=(1-\lambda)Pt-\sum_{i=1}^{k-1}Q_i$. As shortages are not allowed for the system, for each k=1, 2, $\cdots$, m-1 we have

$$I_k(T_k)\ge Q_k \qquad \text{or, } (1-\lambda)PT_k-\sum_{i=1}^{k-1}Q_i\ge Q_k$$
$$\text{or, } \frac{\sum_{i=1}^{n}Q_i}{t_1}T_k\ge\sum_{i=1}^{k-1}Q_i+Q_k \ \text{[Using Eq. (6.11)]}$$
$$\text{or, } T_k\le t_1\le T_k\frac{\sum_{i=1}^{n}Q_i}{\sum_{i=1}^{k}Q_i} \qquad [\text{Since } T_k\le t_1], \qquad \text{or, } 1\le\frac{t_1}{T_k}\le\frac{\sum_{i=1}^{n}Q_i}{\sum_{i=1}^{k}Q_i}$$

□

**Lemma 6.2.** *The integrated profit function* $ITP(t_1)$ *given by Eq. (6.27) is maximum when* $\{C_rfR-H_rf+C_pR+(1-\lambda)H_p\}(1-e^{-Rt_1}-Rt_1e^{-Rt_1})-C_rfi_c(e^{-RM}+RMe^{-RM}-e^{-Rt_1}-Rt_1e^{-Rt_1})=0$ *and* $[\{C_rfR-H_rf+C_pR+(1-\lambda)H_p\}\{1-(1+Rt_1+R^2t_1^2)e^{-Rt_1}\}+C_rfi_c(R^2t_1^2e^{-Rt_1}+Rt_1e^{-Rt_1}-RMe^{-RM}+e^{-RM}-e^{-Rt_1})]>0$.

*Proof.* Taking the first and second order derivatives of $ITP(t_1)$ given by Eq. (6.27) with respect to $t_1$ and setting first derivative equal to 0 gives:

$$\frac{dITP(t_1)}{dt_1}=\Big\{-\frac{C_rf}{(1-\lambda)R}+\frac{H_rf}{(1-\lambda)R^2}-\frac{C_p}{(1-\lambda)R}-\frac{H_p}{R^2}\Big\}\Big(\frac{Re^{-Rt_1}}{t_1}-\frac{1-e^{-Rt_1}}{t_1^2}\Big)\sum_{i=1}^{n}Q_i$$
$$-\frac{C_rfi_c}{(1-\lambda)}\Big(\frac{Me^{-RM}}{Rt_1^2}-\frac{e^{-Rt_1}}{Rt_1}+\frac{e^{-RM}-e^{-Rt_1}}{R^2t_1^2}\Big)\sum_{i=1}^{n}Q_i-H_pe^{-Rt_1}\Big(\sum_{i=1}^{n}Q_i-\sum_{i=1}^{m-1}Q_i-\sum_{i=m}^{n}Q_i\Big)=0$$
$$\text{or, }\{C_rfR-H_rf+C_pR+(1-\lambda)H_p\}(1-e^{-Rt_1}-Rt_1e^{-Rt_1})$$
$$-C_rfi_c(e^{-RM}+RMe^{-RM}-e^{-Rt_1}-Rt_1e^{-Rt_1})=0 \tag{6.31}$$

$$\frac{d^2ITP(t_1)}{dt_1^2} = \Big\{ -\frac{C_rf}{(1-\lambda)R} + \frac{H_rf}{(1-\lambda)R^2} - \frac{C_p}{(1-\lambda)R} - \frac{H_p}{R^2}\Big\}\Big( -\frac{R^2e^{-Rt_1}}{t_1} - \frac{2Re^{-Rt_1}}{t_1^2}$$
$$+ \frac{2(1-e^{-Rt_1})}{t_1^3}\Big)\sum_{i=1}^{n} Q_i - -\frac{C_rfi_c}{(1-\lambda)}\Big( -\frac{2Me^{-RM}}{Rt_1^3} + \frac{2e^{-Rt_1}}{Rt_1^2} - \frac{2(e^{-RM}-e^{-Rt_1})}{R^2t_1^3} + \frac{e^{-Rt_1}}{t_1}\Big)\sum_{i=1}^{n} Q_i$$
$$= -\frac{\sum_{i=1}^{n} Q_i}{(1-\lambda)R^2t_1^3}\Big[\{C_rfR - H_rf + C_pR + (1-\lambda)H_p\}(1-e^{-Rt_1} - Rt_1e^{-Rt_1}) - C_rfi_c(e^{-RM}$$
$$+ RMe^{-RM} - e^{-Rt_1} - Rt_1e^{-Rt_1}) + \{C_rfR - H_rf + C_pR + (1-\lambda)H_p\}(1-e^{-Rt_1} - Rt_1e^{-Rt_1}$$
$$- R^2t_1^2e^{-Rt_1}) + C_rfi_c(R^2t_1^2e^{-Rt_1} + Rt_1e^{-Rt_1} - RMe^{-RM} + e^{-RM} - e^{-Rt_1})\Big]$$
$$= -\frac{\sum_{i=1}^{n} Q_i}{(1-\lambda)R^2t_1^3}\Big[\{C_rfR - H_rf + C_pR + (1-\lambda)H_p\}\{1-(1+Rt_1+R^2t_1^2)e^{-Rt_1}\}$$
$$+ C_rfi_c(R^2t_1^2e^{-Rt_1} + Rt_1e^{-Rt_1} - RMe^{-RM} + e^{-RM} - e^{-Rt_1})\Big] \text{ [Using Eq. (6.31)]} \tag{6.32}$$

For the maximum value of $ITP(t_1)$, we have $\frac{d^2ITP(t_1)}{dt_1^2} < 0$. Hence, above expression of $\frac{d^2ITP(t_1)}{dt_1^2}$ is negative if $[\{C_rfR - H_rf + C_pR + (1-\lambda)H_p\}\{1-(1+Rt_1+R^2t_1^2)e^{-Rt_1}\} + C_rfi_c(R^2t_1^2e^{-Rt_1} + Rt_1e^{-Rt_1} - RMe^{-RM} + e^{-RM} - e^{-Rt_1})] > 0$. □

**Lemma 6.3.** *The integrated profit function $ITP(t_1)$ given by Eq. (6.28) is maximum at $t_1^* = M\sqrt{\frac{C_rfi_c}{H_rf + C_rfi_c - (1-\lambda)H_p}}$ and $t_1^*$ is feasible if $H_rf < (1-\lambda)H_p < H_rf + C_rfi_c$.*

*Proof.* Taking the first and second order derivatives of $ITP(t_1)$ given by Eq. (6.28) with respect to $t_1$ and setting first derivative equal to 0 gives:

$$\frac{dITP(t_1)}{dt_1} = -\frac{H_rf}{2(1-\lambda)}\sum_{i=1}^{n} Q_i - \frac{C_rfi_c}{2(1-\lambda)}(1 - \frac{M^2}{t_1^2})\sum_{i=1}^{n} Q_i$$
$$- H_p(\frac{1}{2}\sum_{i=1}^{n} Q_i - \sum_{i=1}^{m-1} Q_i - \sum_{i=m}^{n} Q_i) = 0$$
$$\text{or, } t_1 = M\sqrt{\frac{C_rfi_c}{H_rf + C_rfi_c - (1-\lambda)H_p}} \tag{6.33}$$

$$\text{Again, } \frac{d^2ITP(t_1)}{dt_1^2} = -\frac{C_rfi_c}{(1-\lambda)t_1^3}\sum_{i=1}^{n} Q_i < 0, \qquad \text{for all } t_1 > 0.$$

Therefore, $ITP(t_1)$ attends maximum at $t_1^* = M\sqrt{\frac{C_rfi_c}{H_rf + C_rfi_c - (1-\lambda)H_p}}$. Now, according to our assumption, the value $t_1$ obtained from Eq. (6.33), is feasible
if $t_1 < M$ and $H_rf + C_rfi_c - (1-\lambda)H_p > 0$

or, if $M\sqrt{\frac{C_rfi_c}{H_rf+C_rfi_c-(1-\lambda)H_p}} < M$ and $(1-\lambda)H_p < H_rf + C_rfi_c$ [Using Eq. (6.33)]
or, if $H_rf < (1-\lambda)H_p$ and $(1-\lambda)H_p < H_rf + C_rfi_c$
or, if $H_rf < (1-\lambda)H_p < H_rf + C_rfi_c$. □

**Lemma 6.4.** *The integrated profit function $ITP(t_1)$ given by Eq. (6.30) is maximum at $t_1^* = M\sqrt{\frac{C_rfi_c}{H_rf+C_rfi_c-H_p}}$ and $t_1^*$ is feasible if $H_rf < H_p < H_rf + C_rfi_c$.*

*Proof.* The proof is similar as Lemma 6.3 with taking $\lambda = 0$ and corresponding optimum value of $t_1$ is

$$t_1 = M\sqrt{\frac{C_rfi_c}{H_rf + C_rfi_c - H_p}} \tag{6.34}$$

□

### 6.3.8 Mathematical Models Formulation with fuzzy credit period

Let us consider that the raw material supplier gives an opportunity to the manufacturer-cum-retailer by offering a fuzzy credit period $(\widetilde{M})$. Here, the credit period $\widetilde{M}$ is represented as TFN and TrFN. So due to fuzzy credit period $(\widetilde{M})$, the optimum values of integrated profit function $ITP(t_1)$ in Eq. (6.27) will be different for various values of M with some degree of belongingness. Therefore in such situation, the profit function will be fuzzy in nature and it is denoted by $\widetilde{ITP}(t_1)$, where

$$\begin{aligned}
\widetilde{ITP}(t_1) &= \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_eS_p\rho Q_i\frac{e^{-RT_i}-e^{-RT_n}}{R} + \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_eS_p(1-\rho)Q_i\frac{e^{-RT_{ei}}-e^{-RT_n}}{R}\\
&+ \sum_{\substack{i=m\\T_i>t_1}}^{n} i_eS_pQ_i\frac{e^{-RT_i}-e^{-RT_n}}{R} - Cs_p - \frac{C_rf(1-e^{-Rt_1})}{(1-\lambda)Rt_1}\sum_{i=1}^{n}Q_i - \frac{H_rf}{(1-\lambda)}\left\{\frac{1}{R}-\frac{(1-e^{-Rt_1})}{R^2t_1}\right\}\sum_{i=1}^{n}Q_i\\
&- \frac{C_p(1-e^{-Rt_1})}{(1-\lambda)Rt_1}\sum_{i=1}^{n}Q_i - H_p\left[\left\{\frac{1-e^{-Rt_1}}{R^2t_1}-\frac{e^{-Rt_1}}{R}\right\}\sum_{i=1}^{n}Q_i - \sum_{k=1}^{m-1}\frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=1}^{k-1}Q_i\right.\\
&\left.- \frac{e^{-RT_{m-1}}-e^{-Rt_1}}{R}\sum_{i=1}^{m-1}Q_i + \frac{e^{-Rt_1}-e^{-RT_m}}{R}\sum_{i=m}^{n}Q_i + \sum_{k=m+1}^{n}\frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=k}^{n}Q_i\right]\\
&- \sum_{i=1}^{n}\{T_{r0}+T_{r1}e^{-\gamma(i-1)}Q_i\}e^{-RT_i} - \frac{C_ri_cf}{(1-\lambda)}\left\{\frac{(t_1-\widetilde{M})e^{-R\widetilde{M}}}{Rt_1}-\frac{e^{-R\widetilde{M}}-e^{-Rt_1}}{R^2t_1}\right\}\sum_{i=1}^{n}Q_i\\
&+ \sum_{i=1}^{n}S_md_i\frac{e^{-RT_i}-e^{-RT_{ei}}}{R} + \sum_{i=1}^{n}i_eS_md_i\left\{\frac{(T_{ei}-T_i)e^{-RT_i}}{R}-\frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2}\right\}\\
&- \sum_{i=1}^{n}H_md_i\left\{\frac{(T_{ei}-T_i)e^{-RT_i}}{R}-\frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2}\right\} - \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_cS_p\rho Q_i\frac{e^{-RT_i}-e^{-RT_{ei}}}{R}
\end{aligned}$$

$$+\sum_{\substack{i=m\\T_i>t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R} - nCs_m$$

The expected value of the DV $t_1$ is obtained from the following equation

$$\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}(1-e^{-Rt_1}-Rt_1 e^{-Rt_1}) \\ - C_r f i_c(e^{-RE[M]} + RE[M]e^{-RE[M]} - e^{-Rt_1} - Rt_1 e^{-Rt_1}) = 0 \quad (6.35)$$

and the corresponding concavity condition is reduced to

$$[\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}\{1-(1+Rt_1+R^2t_1^2)e^{-Rt_1}\} \\ + C_r f i_c(R^2t_1^2 e^{-Rt_1} + Rt_1 e^{-Rt_1} - RE[M]e^{-RE[M]} + e^{-RE[M]} - e^{-Rt_1})] > 0 \quad (6.36)$$

Similarly, $\widetilde{ITP}(t_1)$ for the model without inflation (Model 6.1B) and the model without both inflation and defective (Model 6.1C) are obtained by substituting $M$ by $\widetilde{M}$ in the Eqs. (6.28) and (6.30) respectively.

### 6.3.9 Defuzzification algorithm to get the optimum value of $t_1$

To get the optimum value of the production run time $(t_1)$ in the proposed integrated models with fuzzy credit period, the following steps are used.

Step 1: At first to get the expression of fuzzy integrated profit $\widetilde{ITP}(t_1)$, $M$ is replaced by $\widetilde{M}$.

Step 2: Calculate the expected value of $E[\widetilde{ITP}(t_1)]$ for the fuzzy credit period $\widetilde{M}$ by using the fuzzy extension principle and other necessary expressions (Eqs. (6.35) and (6.36)).

Step 3: To get the optimal value of $t_1$, solve modified expected Eq. (6.35) through the standard LINGO software.

Step 4: Putting the value of $t_1$, check the Lemma 6.1 and concavity condition of objective function presented by inequality (6.36)

Step 5: If both Lemma 6.1 and concavity condition (6.36) are satisfied by $t_1$, putting the value of $t_1$ in the expected profit function to obtain the maximum profit.

Similar procedure is applied to get the optimal solution for the model without inflation (Model 6.1B) and the model without both inflation and defective rate (Model 6.1C).

### 6.3.10 Numerical Experiments and Results

To illustrate the proposed models, three different types of trade credit periods are considered. In **Experiment 1**, deterministic model and in other two experiments **Experiment 2** and **Experiment 3**, fuzzy models with as triangular and trapezoidal fuzzy trade credit period are considered respectively. The solutions to these experiments are obtained by using the GRG method (LINGO 14.0) and the corresponding figures (Figs. 6.4, 6.5 and 6.6) are drawn by

Mathematica software.

**Experiment 1:** Let a manufacturer-cum-retailer sells the finished perfect product to three different markets in different selling seasons. The required raw-material is procured by the supplier. Here, the supplier offers a crisp credit period to the manufacturer to settle the account. We consider such a supply chain situation with the following data:
$M = 0.25$year, $f = 1.75$units, $H_r =$ \$2.00/unit, $C_r =$ \$5.00/unit, $Cs_p =$ \$5000.00/cycle, $C_p =$ \$1.50/unit, $H_p =$ \$4.0/unit/year, $i_c = 0.15$/\$/unit, $i_e = 0.10$/\$/unit, $R = 0.10$, $\lambda = 0.10$, $\rho = 0.50$, $S_p =$ \$30/unit, $T_{r0} =$ \$100.00/transport, $T_{r1} =$ \$2.50/unit, $\gamma = 0.25$, $T_1 = 0.20$year, $T_2 = 0.45$year, $T_3 = 1.00$year, $T_{e1} = 0.60$year, $T_{e2} = 0.95$year, $T_{e3} = 1.40$year, $Cs_m =$ \$2000/order, $S_m =$ \$40/unit, $H_m =$ \$6.0/unit/year, $d_1 = 2000$unit/year, $d_2 = 4000$unit/year, $d_3 = 4000$unit/year.

**Table** 6.3: Optimum results for Models 6.1A, 6.1B and 6.1C under Experiment 1

| Model | $t_1^*$ | $ITP^*$ | $P^*$ | $IP_r^*$ | $IE_p^*$ | $PC_r^*$ | $HC_r^*$ | $PC_p^*$ | $HC_p^*$ | $TR_p^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 6.1A | 0.6707 | 79437 | 7288 | 814 | 3018 | 41374 | 5612 | 7093 | 4471 | 8162 |
| 6.1B | 0.2601 | 90371 | 18796 | 1 | 4740 | 42778 | 2225 | 7333 | 8351 | 8620 |
| 6.1C | 0.3177 | 95628 | 13847 | 42 | 4740 | 38500 | 2446 | 6600 | 7843 | 8620 |

Now, the manufacturer-cum-retailer is interested to find out the optimal profits jointly with the markets along with optimal production run time. The optimal solution are presented in Table 6.3 and the concavity property of the models are graphically shown in Figs. 6.4-6.6.

**Experiment 2:** Let supplier's offered trade credit period be a triangular fuzzy number & is defined as $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$, where $M_0 = 0.25$years, $0 < \Delta_1 < M_0$ and $0 < \Delta_2$. For different values of $\Delta_1$ and $\Delta_2$, Model 6.1A, 6.1B and 6.1C are optimized for the same input data as in Experiment 1 and optimum results are presented in Table 6.4.

**Table** 6.4: Optimum results for Models 6.1A, 6.1B and 6.1C under Experiment 2

| | | Model 6.1A | | | Model 6.1B | | | Model 6.1C | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta_1$ | $\Delta_2$ | $t_1^*$ | $ITP^*$ | $P^*$ | $t_1^*$ | $ITP^*$ | $P^*$ | $t_1^*$ | $ITP^*$ | $P^*$ |
| 0.01 | 0.05 | 0.6980 | 79477 | 7004 | 0.2705 | 90374 | 18073 | 0.3304 | 95640 | 13315 |
| 0.02 | 0.04 | 0.6843 | 79457 | 7144 | 0.2653 | 90372 | 18427 | 0.3241 | 95634 | 13576 |
| 0.03 | 0.03 | 0.6707 | 79437 | 7288 | 0.2601 | 90371 | 18796 | 0.3177 | 95627 | 13847 |
| 0.04 | 0.02 | 0.6571 | 79418 | 7439 | 0.2549 | 90370 | 19179 | 0.3114 | 95621 | 14130 |
| 0.05 | 0.01 | 0.6435 | 79398 | 7597 | 0.2497 | 90368 | 19579 | 0.3050 | 95615 | 14424 |

**Experiment 3:** This experiment's input are similar as Experiment 1 with considering

supplier's offered trade credit period be a trapezoidal fuzzy number which is represented as $\widetilde{M} = (M_0 - \Delta_1, M_0 - \Delta_2, M_0 + \Delta_3, M_0 + \Delta_4)$, where $M_0 = 0.25$years, $0 < \Delta_2 < \Delta_1 < M_0$ and $0 < \Delta_3 < \Delta_4$. For different values of $\Delta_1$, $\Delta_2$, $\Delta_3$ and $\Delta_4$, Model 6.1A, 6.1B and 6.1C are optimized and optimum results are presented in Table 6.5.

**Table** 6.5: Optimum results for Models 6.1A, 6.1B and 6.1C under Experiment 3

| | | | | Model 6.1A | | | Model 6.1B | | | Model 6.1C | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\Delta_4$ | $t_1^*$ | $ITP^*$ | $P^*$ | $t_1^*$ | $ITP^*$ | $P^*$ | $t_1^*$ | $ITP^*$ | $P^*$ |
| 0.01 | 0.006 | 0.010 | 0.05 | 0.7006 | 79481 | 6977 | 0.2715 | 90374 | 18004 | 0.3317 | 95641 | 13264 |
| 0.02 | 0.007 | 0.009 | 0.04 | 0.6857 | 79459 | 7129 | 0.2658 | 90372 | 18391 | 0.3247 | 95634 | 13549 |
| 0.03 | 0.008 | 0.008 | 0.03 | 0.6707 | 79437 | 7288 | 0.2601 | 90371 | 19796 | 0.3177 | 95628 | 13847 |
| 0.04 | 0.009 | 0.007 | 0.02 | 0.6558 | 79416 | 7455 | 0.2543 | 90370 | 19219 | 0.3107 | 95620 | 14159 |
| 0.05 | 0.010 | 0.006 | 0.01 | 0.6408 | 79394 | 7629 | 0.2486 | 90368 | 19661 | 0.3037 | 95614 | 14485 |

### 6.3.11 Sensitivity Analysis

Here the effect of the parameters duration of credit $M$, rate of defective $\lambda$, rate of inflation $R$, interest paid amount $\rho$, interest charged $i_c$ and interest earned $i_d$ have been shown in Table 6.6.

**Table** 6.6: Sensitivity analysis on parameters for Model 6.1A

| On trade credit (M) | | | | On defective rate ($\lambda$) | | | | On inflation (R) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $t_1^*$ | $ITP^*$ | $P^*$ | $\lambda$ | $t_1^*$ | $ITP^*$ | $P^*$ | $R$ | $t_1^*$ | $ITP^*$ | $P^*$ |
| 0.15 | 0.4001 | 79044 | 12217 | 0.11 | 0.6076 | 78824 | 8136 | 0.05 | 0.3428 | 85032 | 14259 |
| 0.17 | 0.4540 | 79122 | 10767 | 0.12 | 0.5594 | 78202 | 8937 | 0.06 | 0.3714 | 83866 | 13162 |
| 0.19 | 0.5080 | 79201 | 9623 | 0.13 | 0.5211 | 77569 | 9704 | 0.07 | 0.4086 | 82719 | 11965 |
| 0.21 | 0.5621 | 79280 | 8697 | 0.14 | 0.4897 | 76925 | 10447 | 0.08 | 0.4597 | 81596 | 10634 |
| 0.23 | 0.6163 | 79359 | 7932 | 0.15 | 0.4633 | 76270 | 11171 | 0.09 | 0.5364 | 80499 | 9114 |
| On paid amount in % ($\rho$) | | | | On Interest charge ($i_c$) | | | | On Interest earned ($i_d$) | | | |
| $\rho$ | $t_1^*$ | $ITP^*$ | $P^*$ | $i_c$ | $t_1^*$ | $ITP^*$ | $P^*$ | $i_d$ | $t_1^*$ | $ITP^*$ | $P^*$ |
| 0.20 | 0.6707 | 79996 | 7288 | 0.16 | 0.5701 | 79036 | 8575 | 0.05 | 0.6707 | 76107 | 7288 |
| 0.30 | 0.6707 | 79809 | 7288 | 0.17 | 0.5108 | 78648 | 9570 | 0.06 | 0.6707 | 76773 | 7288 |
| 0.40 | 0.6707 | 79623 | 7288 | 0.18 | 0.4712 | 78268 | 10376 | 0.07 | 0.6707 | 77439 | 7288 |
| 0.60 | 0.6707 | 79252 | 7288 | 0.19 | 0.4425 | 77892 | 11048 | 0.08 | 0.6707 | 78105 | 7288 |
| 0.70 | 0.6707 | 79066 | 7288 | 0.20 | 0.4207 | 77520 | 11621 | 0.09 | 0.6707 | 78771 | 7288 |

### 6.3.12 Managerial Insights

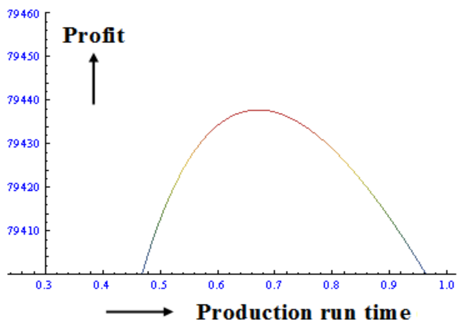



**Figure** 6.4: Integrated profit against production run time for Model 6.1A

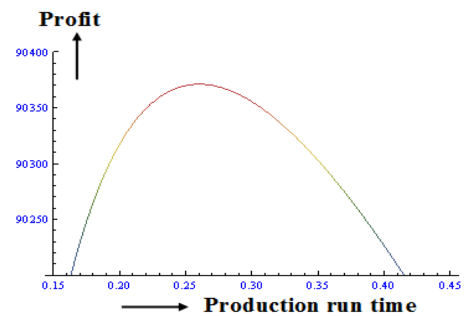



**Figure** 6.5: Integrated profit against production run time for Model 6.1B

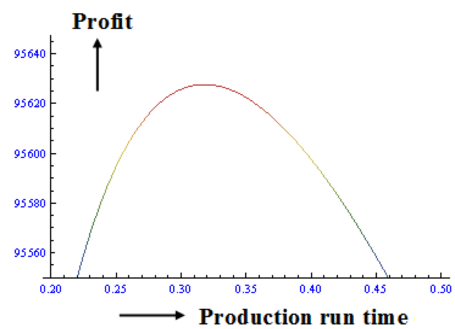



**Figure** 6.6: Integrated profit against production run time for Model 6.1C

(i) Table 6.3 reveals that Model 6.1C gives the highest profit i.e. model without both defective and inflation is more profitable then the other. This is as per our expectation. We have observed that the raw material's purchase cost, production cost and holding cost of finished products decrease for this model without inflation while interest paid and holding cost of raw material increase. Also, interest earned and transportation cost remain same whilst the total decreasing cost value dominates all other increasing costs. For the obvious reason, model with defective gives the worse profit than the model without defective. Comparing the optimal results of Model 6.1A with Models 6.1B and 6.1C, we observe that production run time is largest in Model 6.1A and all the costs and revenue are less than those of Models 6.1B and 6.1C. Thus, the profit is also less due to the presence of inflation.

(ii) It is observed from the Figs. 6.4, 6.5 and 6.6 that for a fixed value of credit period (M), the integrated total profits (ITP) initially increase with the increase of production run time $(t_1)$. As it approaches to its peak level of production run-time, it decreases. Thus integrated profit is concave in nature.

(iii) By taking impreciseness in trade credit, decision-makers absorb all the turbulence in the costs due to market fluctuation. From Tables 6.4 and 6.5, it is also concluded that, the trade credit period is proportional to the production run time rate $(t_1)$ and inversely proportional to production rate $P$. Therefore, results of the numerical experiments 2 and 3 presented in Tables 6.4 and 6.5, confirm that the fuzzy trade credit, in general, has a positive role on improving the integrated supply chain's performance.

(iv) Considering the results of sensitivity, it is clear that, as the trade credit period $(M)$ increases, the replenishment cycle length $t_1$ increases significantly (cf. Table 6.6), but the total profit strictly increases. Thus, from these results, we can conclude that the trade credit strategy has a positive impact on reducing the total cost and improving the supply chain's performance which is as per our expectation.

(v) Through Table 6.6, we have tried to analyse the effect of the defective rate $\lambda$ on $t_1$, $P$ and on the profit $ITP$ of the system. It is observed that as the defective rate $\lambda$ increases, profit $ITP$ and production run time $t_1$ decrease with the increase in production rate $P$. Hence as per manufacturer point of view, decreasing the production run time, the manufacturer tries to decrease the defective quantity and increases his profit. Hence, if the manufacturer can effectively reduce the defective rate of the product by improving the production process, the profit earned by the manufacturer will automatically increase.

(vi) Also, it is found from the literature that inflation rate and credit period have significant decisive effects on the production time, production rate and profit. From the analysis of Table 6.6, it is found correct. Inflation is the state of a continuous increase in the price of goods and service. Hence, it is obvious that the increase in the rate of inflation causes the total profit of the system to go down.

(vii) These markets start their business on or before the production run time, initially pay a part of total payment against the purchase of products. From Table 6.6, we observe that increase of $\rho$ does not change the value of $t_1$ (DV) but the profit decreases due to the increase of interest charged by the bank at market's level.

(viii) From Table 6.6, it is again observed that as the interest paid per unit item increases from 0.16 to 0.20, then the production time and profit of the system decrease. Hence from a managerial viewpoint, it implies that when the interest paid is high, then the production manager should produce less amount of goods.

(ix) From Table 6.6, we see that there is no change in optimum decision $t_1$ and $P$ against the increase of $i_d$. Rather profit of the integrated system increases. This is because, more interest is earned from sales revenue by manufacturer and markets' retailer.

## 6.4 Model-6.2 : A fuzzy imperfect EPL model with dynamic demand under bi-level trade credit policy [2]

### 6.4.1 Assumptions and Notations

*The following notations are used to develop the proposed model:*

| | |
|---|---|
| $\widetilde{I}(t)$ | Fuzzy inventory level at time t |
| P | Production rate per year |
| $\widetilde{\lambda}$ | Fuzzy defective rate |
| Cs | Set up cost per order |
| r | Unit purchasing price of raw materials |
| s | Unit selling price |
| h | Unit stock holding cost per item per year excluding interest charges |
| $i_e$ | interest earned per order quantity per year by the the retailer |
| $i_p$ | interest charged per order quantity in stocks per year by the supplier |
| M | Retailer's trade credit period offered by supplier in years |
| N | Customers trade credit period offered by retailer |
| $\widetilde{D}(t)$ | Fuzzy demand rate per year |
| $\widetilde{T}$ | Fuzzy cycle length in years |
| $t_1$ | Length of time up to stop the production. It is taken as DV |
| $\widetilde{TP}_{ij}$ | ( for i=1,2 ; j=1,2,...,6) Fuzzy annual total profit, which is a function of $t_1$, where annual total Profit=Sales revenue - Purchasing cost - Ordering cost - Holding cost - Interest to be paid + Interest earned |

*The following assumptions are used to develop the proposed model:*

(i) Demand rate $\widetilde{D}(t)$ increases with time during credit period of customers and is of the form $\widetilde{D}(t) = \widetilde{a} - \widetilde{b}e^{-ct}$ if $0 < t \leq N$ and $\widetilde{a} - \widetilde{b}e^{-cN}$ if $N \leq t \leq T$, where $\widetilde{a}$, $\widetilde{b}$, c are positive constants.

(ii) Shortages are not allowed and lead time is negligible.

(iii) Time horizon is infinite.

(iv) Interest charged per order quantity ($i_p$) is greater than or equal to interest earned per order quantity ($i_e$) i.e., ($i_p \geq i_e$). Selling price (s) is greater than or equal to purchasing price (r) of unit raw materials. i.e., $s \geq r$.

(v) Production rate P, is known and constant.

(vi) Production run time ($t_1$) is taken as DV.

(vii) When $N < M$, the retailer can accumulate revenue and earn interest during the period N to M with rate $i_e$ under the condition of trade credit. When $N \geq M$, retailer does not earn any interest.

[2]This model has been communicated in **Applied Mathematical Modelling**, ELSEVIER.

### 6.4.2 Mathematical Model Development

Here a fuzzy imperfect production inventory model of an item is developed where the raw materials of the item are supplied to the manufacturer-cum-retailer for production. The supplier offers the retailer a delay period (M) for payment and the manufacturer also offers his/her customers a delay period (N) for payment to stimulate his/her customer's demand. i.e., item purchased by the customers during this period has to pay final payment at time N. As effective credit period of a customer purchasing item at t $(0 < t < N)$ is (N-t), it is assumed that demand increases at a decreasing rate during [0, N]. After the trade credit period [0, N], demand at t=N, prevails for the rest of the period. Inventory builds up to $t_1$ (maximum stock level) at the rate $(1 - \widetilde{\lambda})P - \widetilde{D}(t)$ and then it reaches to zero at $\widetilde{T}$. Depending on different values of M, N, $t_1$ and $\widetilde{T}$, twelve scenarios are observed to develop the model in conventional approach. For all the scenarios, production, screening cost and total sale revenue are same as described below:

**Production-cum-Purchasing cost:** The purchasing cost of the raw materials during $[0, t_1]$ is $\widetilde{PC} = r(1 - \widetilde{\lambda})\int_0^{t_1} Pdt$. Let $\alpha$-cut set of the above said cost is $\widetilde{PC}[\alpha] = [PC_L(\alpha), PC_R(\alpha)] = [r(1-\lambda_R)Pt_1, r(1-\lambda_L)Pt_1]$.

**Screening cost:** The screening cost to find the defective product for one cycle is $SC = u.Pt_1$

**Total Sale Revenue:** Total sales revenue of perfect products is $\widetilde{SP}[\alpha]$. Then $\alpha$-cut set of the above said revenue is $\widetilde{SP}[\alpha] = [SP_L(\alpha), SP_R(\alpha)] = [s(1-\lambda_R)Pt_1, s(1-\lambda_L)Pt_1]$.

**Conventional Approach**

Depending on the values of M and N two cases arise- **Case-1:** $N < M$ and **Case-2:** $N \geq M$. Under these two cases, six subcases may arise which are presented below:

| Case-1: $N < M$ | | Case-2: $N \geq M$ | |
|---|---|---|---|
| Subcases | Criteria | Subcases | Criteria |
| 1.1 | $N < M \leq t_1 < \widetilde{T}$ | 2.1 | $M < N \leq t_1 < \widetilde{T}$ |
| 1.2 | $N \leq t_1 < M < \widetilde{T}$ | 2.2 | $M \leq t_1 < N < \widetilde{T}$ |
| 1.3 | $N \leq t_1 < \widetilde{T} < M$ | 2.3 | $M \leq t_1 < \widetilde{T} < N$ |
| 1.4 | $t_1 \leq N < M \leq \widetilde{T}$ | 2.4 | $t_1 \leq M < N \leq \widetilde{T}$ |
| 1.5 | $t_1 \leq N \leq \widetilde{T} < M$ | 2.5 | $t_1 \leq M \leq \widetilde{T} < N$ |
| 1.6 | $t_1 < \widetilde{T} \leq N < M$ | 2.6 | $t_1 < \widetilde{T} \leq M < N$ |

**Subcase-1.1:** $N < M \leq t_1 < \widetilde{T}$

For a single item imperfect production process, the governing differential equations are:

$$\frac{d\widetilde{I}(t)}{dt} = \begin{cases} (1-\widetilde{\lambda})P - (\widetilde{a} - \widetilde{b}e^{-ct}), & 0 \leq t \leq N \\ (1-\widetilde{\lambda})P - (\widetilde{a} - \widetilde{b}e^{-cN}), & N \leq t \leq t_1 \\ -(\widetilde{a} - \widetilde{b}e^{-cN}), & t_1 \leq t \leq \widetilde{T} \end{cases} \tag{6.37}$$

with the boundary conditions $\widetilde{I}(0) = 0 = \widetilde{I}(\widetilde{T})$ and maximum inventory occur at $t = t_1$. To solve the above FDE (6.37), we use the solution procedure of Chalco-Cano [39] as describe in the preliminaries section. Since $I(t) \geq 0$ for all t, $\frac{d\widetilde{I}}{dt}$ is the first form (I) of H-derivative. Thus, the corresponding crisp differential equations in $\alpha$-cut form are

$$\frac{dI_L(t)}{dt} = \begin{cases} (1-\lambda_R)P - (a_R - b_L e^{-ct}), & 0 \leq t \leq N \\ (1-\lambda_R)P - (a_R - b_L e^{-cN}), & N \leq t \leq t_1 \\ -(a_R - b_L e^{-cN}), & t_1 \leq t \leq T_L \end{cases} \tag{6.38}$$

$$\frac{dI_R(t)}{dt} = \begin{cases} (1-\lambda_L)P - (a_L - b_R e^{-ct}), & 0 \leq t \leq N \\ (1-\lambda_L)P - (a_L - b_R e^{-cN}), & N \leq t \leq t_1 \\ -(a_L - b_R e^{-cN}), & t_1 \leq t \leq T_R \end{cases} \tag{6.39}$$

In $0 \leq t \leq N$ with initial conditions $I_L(0) = 0 = I_R(0)$, the solutions are

$$\begin{cases} I_L(t) = \{(1-\lambda_R)P - a_R\}t + \frac{b_L}{c}(1 - e^{-ct}) \\ I_R(t) = \{(1-\lambda_L)P - a_L\}t + \frac{b_R}{c}(1 - e^{-ct}) \end{cases} \tag{6.40}$$

Now $I_R(t) - I_L(t) = (\lambda_R - \lambda_L)Pt + (a_R - a_L)t + \frac{b_R - b_L}{c}(1 - e^{-ct}) = (1-\alpha)\{(\lambda_3 - \lambda_1)Pt + (a_3 - a_1)t + \frac{(b_3 - b_1)}{c}(1 - e^{-ct})\} \geq 0$ for all $0 \leq \alpha \leq 1$. As $I_L(t) \leq I_R(t)$ for all $0 \leq t \leq N$, the solutions (6.40) are valid.

In $N \leq t \leq t_1$ with initial conditions $I_L(N) = \{(1-\lambda_R)P - a_R\}N + \frac{b_L}{c}(1 - e^{-cN})$ and $I_R(N) = \{(1-\lambda_L)P - a_L\}N + \frac{b_R}{c}(1 - e^{-cN})$, the solutions are

$$\begin{cases} I_L(t) = \{(1-\lambda_R)P - a_R\}t + b_L e^{-cN}(t - N) + \frac{b_L}{c}(1 - e^{-cN}) \\ I_R(t) = \{(1-\lambda_L)P - a_L\}t + b_R e^{-cN}(t - N) + \frac{b_R}{c}(1 - e^{-cN}) \end{cases} \tag{6.41}$$

Now
$I_R(t) - I_L(t) = (\lambda_R - \lambda_L)Pt + (a_R - a_L)t + (b_R - b_L)e^{-cN}(t - N) + \frac{b_R - b_L}{c}(1 - e^{-cN}) = (1-\alpha)\{(\lambda_3 - \lambda_1)Pt + (a_3 - a_1)t + (b_3 - b_1)e^{-cN}(t - N) + \frac{(b_3 - b_1)}{c}(1 - e^{-cN})\} \geq 0$ for all $0 \leq \alpha \leq 1$ and $N \leq t \leq t_1$. As $I_L(t) \leq I_R(t)$, the solutions (6.41) are valid.

In $t_1 \leq t \leq \widetilde{T}$ with initial conditions $I_L(t_1) = \{(1-\lambda_R)P - a_R\}t_1 + b_L e^{-cN}(t_1 - N) + \frac{b_L}{c}(1 - e^{-cN})$ and $I_R(t_1) = \{(1-\lambda_L)P - a_L\}t_1 + b_R e^{-cN}(t_1 - N) + \frac{b_R}{c}(1 - e^{-cN})$, the corresponding solutions are

$$\begin{cases} I_L(t) = (1-\lambda_R)Pt_1 + (-a_R + b_L e^{-cN})t - b_L N e^{-cN} + \frac{b_L}{c}(1 - e^{-cN}) \\ I_R(t) = (1-\lambda_L)Pt_1 + (-a_L + b_R e^{-cN})t - b_R N e^{-cN} + \frac{b_R}{c}(1 - e^{-cN}) \end{cases} \tag{6.42}$$

Now $I_R(t) - I_L(t) = (\lambda_R - \lambda_L)Pt_1 + (a_R - a_L)t + (b_R - b_L)e^{-cN}(t - N) + \frac{b_R - b_L}{c}(1 - e^{-cN}) = (1-\alpha)\{(\lambda_3 - \lambda_1)Pt_1 + (a_3 - a_1)t + (b_3 - b_1)e^{-cN}(t - N) + \frac{(b_3 - b_1)}{c}(1 - e^{-cN})\} \geq 0$ for

all $0 \leq \alpha \leq 1$ and $t_1 \leq t \leq \widetilde{T}$. As $I_L(t) \leq I_R(t)$, the solutions (6.42) are valid.
Applying the boundary conditions $I_L(T_L) = 0 = I_R(T_R)$ we get,

$$\begin{cases} T_L = \frac{(1-\lambda_R)Pt_1 - b_L N e^{-cN} + \frac{b_L}{c}(1-e^{-cN})}{a_R - b_L e^{-cN}} \\ T_R = \frac{(1-\lambda_L)Pt_1 - b_R N e^{-cN} + \frac{b_R}{c}(1-e^{-cN})}{a_L - b_R e^{-cN}} \end{cases} \tag{6.43}$$

**Holding cost:** Holding cost during $[0, \widetilde{T}]$ is
$\widetilde{HC}_{11} = h\int_0^{\widetilde{T}} \widetilde{I}(t)dt = h[\int_0^N \widetilde{I}(t)dt + \int_N^{t_1} \widetilde{I}(t)dt + \int_{t_1}^{\widetilde{T}} \widetilde{I}(t)dt]$. Let $\alpha$-cut set of total holding cost $\widetilde{HC}_{11}$ is $\widetilde{HC}_{11}[\alpha] = [HC_{11L}(\alpha), HC_{11R}(\alpha)]$. Then

$$\begin{cases} HC_{11L}(\alpha) = h[\int_0^N I_L(\alpha,t)dt + \int_N^{t_1} I_L(\alpha,t)dt + \int_{t_1}^{T_L} I_L(\alpha,t)dt] \\ = h\Big[\frac{1}{2}\{(1-\lambda_R)P - a_R\}t_1^2 + \frac{b_L N}{c} - \frac{b_L}{c^2}(1-e^{-cN}) + \frac{1}{2}b_L e^{-cN}(t_1-N)^2 \\ +\frac{b_L}{c}(1-e^{-cN})(t_1-N) + \frac{1}{2}(-a_R + b_L e^{-cN})(T_L^2 - t_1^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} \\ +\frac{b_L}{c}(1-e^{-cN})\}(T_L - t_1)\Big] \\ HC_{11R}(\alpha) = h\Big[\frac{1}{2}\{(1-\lambda_L)P - a_L\}t_1^2 + \frac{b_R N}{c} - \frac{b_R}{c^2}(1-e^{-cN}) + \frac{1}{2}b_R e^{-cN}(t_1-N)^2 \\ +\frac{b_R}{c}(1-e^{-cN})(t_1-N) + \frac{1}{2}(-a_L + b_R e^{-cN})(T_R^2 - t_1^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ +\frac{b_R}{c}(1-e^{-cN})\}(T_R - t_1)\Big] \end{cases}$$

**Interest Calculation:** Interest to be paid during $[M, \widetilde{T}]$ is given by $\widetilde{IP}_{11} = r.i_p \int_M^{\widetilde{T}} \widetilde{I}(t)dt = r.i_p[\int_M^{t_1} \widetilde{I}(t)dt + \int_{t_1}^{\widetilde{T}} \widetilde{I}(t)dt]$. Let $\alpha$-cut set of interest to be paid $\widetilde{IP}_{11}$ is $\widetilde{IP}_{11}[\alpha] = [IP_{11L}(\alpha), IP_{11R}(\alpha)]$. Then

$$\begin{cases} IP_{11L}(\alpha) = r.i_p[\int_M^{t_1} I_L(\alpha,t)dt + \int_{t_1}^{T_L} I_L(\alpha,t)dt] \\ = r.i_p\Big[\frac{1}{2}\{(1-\lambda_R)P - a_R + b_L e^{-cN}\}(t_1^2 - M^2) + \{\frac{b_L}{c}(1-e^{-cN}) - b_L N e^{-cN}\}(t_1 - M) \\ +\frac{1}{2}(-a_R + b_L e^{-cN})(T_L^2 - t_1^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} + \frac{b_L}{c}(1-e^{-cN})\}(T_L - t_1)\Big] \\ IP_{11RL}(\alpha) = r.i_p[\int_M^{t_1} I_R(\alpha,t)dt + \int_{t_1}^{T_R} I_R(\alpha,t)dt] \\ = r.i_p\Big[\frac{1}{2}\{(1-\lambda_L)P - a_L + b_R e^{-cN}\}(t_1^2 - M^2) + \{\frac{b_R}{c}(1-e^{-cN}) - b_R N e^{-cN}\}(t_1 - M) \\ +\frac{1}{2}(-a_L + b_R e^{-cN})(T_R^2 - t_1^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} + \frac{b_R}{c}(1-e^{-cN})\}(T_R - t_1)\Big] \end{cases}$$

Interest earned during $[0, M]$ is given by
$\widetilde{IE}_{11} = s.i_e \int_0^M \widetilde{D}(t)dt = s.i_e[\int_0^N \widetilde{D}(t)dt + \int_N^M \widetilde{D}(t)dt]$. Let $\alpha$-cut set of interest earned

is $\widetilde{IE}_{11}$ is $\widetilde{IE}_{11}[\alpha] = [IE_{11L}(\alpha), IE_{11R}(\alpha)]$. Then

$$\begin{cases} IE_{11L}(\alpha) = s.i_e[\int_0^N D_L(\alpha,t)(M-N)dt + \int_N^M D_L(\alpha,t)(M-t)dt] \\ = s.i_e\Big[\{a_L N - \frac{b_R}{c}(1-e^{-cN})\}(M-N) + \frac{1}{2}(a_L - b_R e^{-cN})(M-N)^2\Big] \\ IE_{11R}(\alpha) = s.i_e\Big[\{a_R N - \frac{b_L}{c}(1-e^{-cN})\}(M-N) + \frac{1}{2}(a_R - b_L e^{-cN})(M-N)^2\Big] \end{cases} \tag{6.44}$$

**Subcase-1.2:** $N \le t_1 < M < \widetilde{T}$

In this case, inventory level $\widetilde{I}(t)$, holding cost $\widetilde{HC}_{12}$, interest earned $\widetilde{IE}_{12}$ are same as the expressions of subcase-1.1. Interest paid is given as $\widetilde{IP}_{12} = r.i_p \int_M^{\widetilde{T}} \widetilde{I}(t)dt$. i.e.,

$$\begin{cases} IP_{12L}(\alpha) = r.i_p\Big[\frac{1}{2}(-a_R + b_L e^{-cN})(T_L^2 - M^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} \\ \qquad + \frac{b_L}{c}(1-e^{-cN})\}(T_L - M)\Big] \\ IP_{12RL}(\alpha) = r.i_p\Big[\frac{1}{2}(-a_L + b_R e^{-cN})(T_R^2 - M^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ \qquad + \frac{b_R}{c}(1-e^{-cN})\}(T_R - M)\Big] \end{cases} \tag{6.45}$$

**Subcase-1.3:** $N \le t_1 < \widetilde{T} \le M$

In this case, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{13}$ are same as the expressions of subcase-1.1. Interest paid is $\widetilde{IP}_{13} = 0$. Interest earned $\widetilde{IE}_{13} = s.i_e\Big[\int_0^N \widetilde{D}(t)dt(\widetilde{T} - N) + \int_N^{\widetilde{T}} \widetilde{D}(t)dt(\widetilde{T} - t) + \int_0^{\widetilde{T}} \widetilde{D}(t)dt(M - \widetilde{T})$. i.e.,

$$\begin{cases} IE_{13L}(\alpha) = s.i_e\Big[\{a_L N - \frac{b_R}{c}(1-e^{-cN})\}\{(T_L - N) + (M - T_R)\} \\ \qquad + (a_L - b_R e^{-cN})\{\frac{(T_L-N)^2}{2} + (T_L - N)\}\Big] \\ IE_{13R}(\alpha) = s.i_e\Big[\{a_R N - \frac{b_L}{c}(1-e^{-cN})\}\{(T_R - N) + (M - T_L)\} \\ \qquad + (a_R - b_L e^{-cN})\{\frac{(T_R-N)^2}{2} + (T_R - N)\}\Big] \end{cases} \tag{6.46}$$

**Subcase-1.4:** $t_1 \le N < M \le \widetilde{T}$

The governing differential equations are:

$$\frac{d\widetilde{I}(t)}{dt} = \begin{cases} (1-\widetilde{\lambda})P - (\widetilde{a} - \widetilde{b}e^{-ct}), \quad 0 \le t \le t_1 \\ -(\widetilde{a} - \widetilde{b}e^{-ct}), \quad t_1 \le t \le N \\ -(\widetilde{a} - \widetilde{b}e^{-cN}), \quad N \le t \le \widetilde{T} \end{cases} \tag{6.47}$$

with the boundary conditions $\widetilde{I}(0) = 0 = \widetilde{I}(\widetilde{T})$ and maximum inventory occur at $t = t_1$. According to Chalco-Cano's [39] solution technique, the corresponding crisp differential equations in $\alpha$-cut form are

$$\frac{dI_L(t)}{dt} = \begin{cases} (1-\lambda_R)P - (a_R - b_L e^{-ct}), & 0 \le t \le t_1 \\ -(a_R - b_L e^{-ct}), & t_1 \le t \le N \\ -(a_R - b_L e^{-cN}), & N \le t \le T_L \end{cases} \tag{6.48}$$

$$\frac{dI_R(t)}{dt} = \begin{cases} (1-\lambda_L)P - (a_L - b_R e^{-ct}), & 0 \le t \le t_1 \\ -(a_L - b_R e^{-ct}), & t_1 \le t \le N \\ -(a_L - b_R e^{-cN}), & N \le t \le T_R \end{cases} \tag{6.49}$$

In $0 \le t \le t_1$ with initial conditions $I_L(0) = 0 = I_R(0)$, The solutions are

$$\begin{cases} I_L(t) = \{(1-\lambda_R)P - a_R\}t + \frac{b_L}{c}(1 - e^{-ct}) \\ I_R(t) = \{(1-\lambda_L)P - a_L\}t + \frac{b_R}{c}(1 - e^{-ct}) \end{cases} \tag{6.50}$$

In $t_1 \le t \le N$ with initial conditions $I_L(t_1) = \{(1-\lambda_R)P - a_R\}t_1 + \frac{b_L}{c}(1 - e^{-ct_1})$ and $I_R(t_1) = \{(1-\lambda_L)P - a_L\}t_1 + \frac{b_R}{c}(1 - e^{-ct_1})$, the solutions are

$$\begin{cases} I_L(t) = (1-\lambda_R)Pt_1 - a_R t + \frac{b_L}{c}(1 - e^{-ct}) \\ I_R(t) = (1-\lambda_L)Pt_1 - a_L t + \frac{b_R}{c}(1 - e^{-ct}) \end{cases} \tag{6.51}$$

In $N \le t \le \widetilde{T}$ with initial conditions $I_L(N) = (1-\lambda_R)Pt_1 - a_R N + \frac{b_L}{c}(1 - e^{-cN})$ and $I_R(N) = (1-\lambda_L)Pt_1 - a_L N + \frac{b_R}{c}(1 - e^{-cN})$, the corresponding solutions are

$$\begin{cases} I_L(t) = (1-\lambda_R)Pt_1 + (-a_R + b_L e^{-cN})t - b_L N e^{-cN} + \frac{b_L}{c}(1 - e^{-cN}) \\ I_R(t) = (1-\lambda_L)Pt_1 + (-a_L + b_R e^{-cN})t - b_R N e^{-cN} + \frac{b_R}{c}(1 - e^{-cN}) \end{cases} \tag{6.52}$$

The above solutions (6.50)-(6.52) of FDE (6.47) are valid solutions. The validities are checked as in subcase-1.1.

Applying the boundary conditions $I_L(T_L) = 0 = I_R(T_R)$ we get, the expressions of $[T_L, T_R]$ and which are given by (6.43) as obtained in subcase-1.1.

**Holding cost:** Holding cost during $[0, \widetilde{T}]$ is
$\widetilde{HC}_{14} = h\int_0^{\widetilde{T}} \widetilde{I}(t)dt = h[\int_0^{t_1} \widetilde{I}(t)dt + \int_{t_1}^{N} \widetilde{I}(t)dt + \int_N^{\widetilde{T}} \widetilde{I}(t)dt]$. Let $\alpha$-cut set of total

holding cost $\widetilde{HC}_{11}$ is $\widetilde{HC}_{14}[\alpha] = [HC_{14L}(\alpha), HC_{14R}(\alpha)]$. Then

$$\left\{\begin{array}{l} HC_{14L}(\alpha) = h[\int_0^{t_1} I_L(\alpha,t)dt + \int_{t_1}^{N} I_L(\alpha,t)dt + \int_{N}^{T_L} I_L(\alpha,t)dt] \\ = h\Big[\frac{1}{2}\{(1-\lambda_R)P - a_R\}t_1^2 + \frac{b_L t_1}{c} - \frac{b_L}{c^2}(1-e^{-ct_1}) + (1-\lambda_R)Pt_1(N-t_1) \\ -\frac{a_R}{2}(N^2-t_1^2) + \frac{b_L}{2}(N-t_1) + \frac{b_L}{c^2}(e^{-cN}-e^{-ct_1}) \\ +\frac{1}{2}(-a_R + b_L e^{-cN})(T_L^2 - N^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} \\ +\frac{b_L}{c}(1-e^{-cN})\}(T_L - N)\Big] \\ HC_{14R}(\alpha) = h\Big[\frac{1}{2}\{(1-\lambda_L)P - a_L\}t_1^2 + \frac{b_R t_1}{c} - \frac{b_R}{c^2}(1-e^{-ct_1}) \\ +(1-\lambda_L)Pt_1(N-t_1) - \frac{a_L}{2}(N^2-t_1^2) + \frac{b_R}{2}(N-t_1) + \frac{b_R}{c^2}(e^{-cN}-e^{-ct_1}) \\ +\frac{1}{2}(-a_L + b_R e^{-cN})(T_R^2 - N^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ +\frac{b_R}{c}(1-e^{-cN})\}(T_R - N)\Big] \end{array}\right. \tag{6.53}$$

**Interest Calculation:** Interest to be paid during $[0, M]$ is given by $\widetilde{IP}_{14} = r.i_p \int_M^{\widetilde{T}} \widetilde{I}(t)dt$. Let $\alpha$-cut set of interest to be paid $\widetilde{IP}_{14}$ is $\widetilde{IP}_{14}[\alpha] = [IP_{14L}(\alpha), IP_{14R}(\alpha)]$. Then

$$\left\{\begin{array}{l} IP_{14L}(\alpha) = r.i_p \int_M^{T_L} I_L(\alpha,t)dt \\ = r.i_p\Big[\frac{1}{2}\{-a_R + b_L e^{-cN}\}(T_L^2 - M^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} \\ +\frac{b_L}{c}(1-e^{-cN})\}(T_L - M)\Big] \\ IP_{14RL}(\alpha) = r.i_p\Big[\frac{1}{2}\{-a_L + b_R e^{-cN}\}(T_R^2 - M^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ +\frac{b_R}{c}(1-e^{-cN})\}(T_R - M)\Big] \end{array}\right. \tag{6.54}$$

Interest earned during $[0, M]$ is given by $\widetilde{IE}_{14}$ which is same as in subcase-1.1.

**Subcase-1.5:** $t_1 \leq N < \widetilde{T} \leq M$

In this case, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{15}$ are same as the expressions of subcase-1.4. Interest paid is $\widetilde{IP}_{15} = 0$. Interest earned $\widetilde{IE}_{15}$ is same as in subcase-1.3.

**Subcase-1.6:** $t_1 \leq \widetilde{T} \leq N < M$

The governing differential equations are:

$$\frac{d\widetilde{I}(t)}{dt} = \left\{\begin{array}{ll} (1-\widetilde{\lambda})P - (\widetilde{a} - \widetilde{b}e^{-ct}), & 0 \leq t \leq t_1 \\ -(\widetilde{a} - \widetilde{b}e^{-ct}), & t_1 \leq t \leq \widetilde{T} \end{array}\right. \tag{6.55}$$

with the boundary conditions $\widetilde{I}(0) = 0 = \widetilde{I}(\widetilde{T})$ and maximum inventory occur at $t = t_1$. The corresponding crisp differential equations in $\alpha$-cut form are

$$\frac{dI_L(t)}{dt} = \begin{cases} (1-\lambda_R)P - (a_R - b_L e^{-ct}), & 0 \le t \le t_1 \\ -(a_R - b_L e^{-ct}), & t_1 \le t \le T_L \end{cases} \tag{6.56}$$

$$\frac{dI_R(t)}{dt} = \begin{cases} (1-\lambda_L)P - (a_L - b_R e^{-ct}), & 0 \le t \le t_1 \\ -(a_L - b_R e^{-ct}), & t_1 \le t \le T_R \end{cases} \tag{6.57}$$

In $0 \le t \le t_1$ with initial conditions $I_L(0) = 0 = I_R(0)$, The solutions are

$$\begin{cases} I_L(t) = \{(1-\lambda_R)P - a_R\}t + \frac{b_L}{c}(1-e^{-ct}) \\ I_R(t) = \{(1-\lambda_L)P - a_L\}t + \frac{b_R}{c}(1-e^{-ct}) \end{cases} \tag{6.58}$$

In $t_1 \le t \le \widetilde{T}$ with initial conditions $I_L(t_1) = \{(1-\lambda_R)P - a_R\}t_1 + \frac{b_L}{c}(1-e^{-ct_1})$ and $I_R(t_1) = \{(1-\lambda_L)P - a_L\}t_1 + \frac{b_R}{c}(1-e^{-ct_1})$, the solutions are

$$\begin{cases} I_L(t) = (1-\lambda_R)Pt_1 - a_R t + \frac{b_L}{c}(1-e^{-ct}) \\ I_R(t) = (1-\lambda_L)Pt_1 - a_L t + \frac{b_R}{c}(1-e^{-ct}) \end{cases} \tag{6.59}$$

The validities of above solutions (6.58) and (6.59) are checked by same procedure as present in subcase-1.1. Now applying the boundary conditions $I_L(T_L) = 0 = I_R(T_R)$ we get, the expressions of $[T_L, T_R]$, which are given by

$$\begin{cases} a_R T_L + \frac{b_L}{c}e^{-cT_L} = (1-\lambda_R)Pt_1 + \frac{b_L}{c} \\ a_L T_R + \frac{b_R}{c}e^{-cT_R} = (1-\lambda_L)Pt_1 + \frac{b_R}{c} \end{cases} \tag{6.60}$$

**Holding cost:** Holding cost during $[0, \widetilde{T}]$ is
$\widetilde{HC}_{16} = h\int_0^{\widetilde{T}} \widetilde{I}(t)dt = h[\int_0^{t_1} \widetilde{I}(t)dt + \int_{t_1}^{\widetilde{T}} \widetilde{I}(t)dt]$. Let $\alpha$-cut set of total holding cost $\widetilde{HC}_{16}$ is $\widetilde{HC}_{16}[\alpha] = [HC_{16L}(\alpha), HC_{16R}(\alpha)]$. Then

$$\begin{cases} HC_{16L}(\alpha) = h[\int_0^{t_1} I_L(\alpha,t)dt + \int_{t_1}^{T_L} I_L(\alpha,t)dt] \\ = h\Big[\frac{1}{2}\{(1-\lambda_R)P - a_R\}t_1^2 + \frac{b_L t_1}{c} - \frac{b_L}{c^2}(1-e^{-ct_1}) \\ +\{(1-\lambda_R)Pt_1 + \frac{b_L}{c}\}(T_L - t_1) - \frac{a_R}{2}(T_L^2 - t_1^2) + \frac{b_L}{c^2}(e^{-cT_L} - e^{-ct_1})\Big] \\ HC_{16R}(\alpha) = h\Big[\frac{1}{2}\{(1-\lambda_L)P - a_L\}t_1^2 + \frac{b_R t_1}{c} - \frac{b_R}{c^2}(1-e^{-ct_1}) \\ +\{(1-\lambda_L)Pt_1 + \frac{b_R}{c}\}(T_R - t_1) - \frac{a_L}{2}(T_R^2 - t_1^2) + \frac{b_R}{c^2}(e^{-cT_R} - e^{-ct_1})\Big]] \end{cases} \tag{6.61}$$

**Interest Calculation:** Interest to be paid during $[0, M]$ is given by $\widetilde{IP}_{16} = 0$. Interest earned is given by

$$\begin{cases} IE_{16L}(\alpha) = s.i_e \int_0^{T_L} D_L(t)dt(M-N) = s.i_e(M-N)\{a_L T_L - \frac{b_R}{c}(1-e^{-cT_L})\} \\ IE_{16L}(\alpha) = s.i_e \int_0^{T_R} D_R(t)dt(M-N) = s.i_e(M-N)\{a_R T_R - \frac{b_L}{c}(1-e^{-cT_R})\} \end{cases} \tag{6.62}$$

**Subcase-2.1:** $M < N \leq t_1 < \widetilde{T}$

In this case, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{21}$ are same as the expressions of subcase-1.1. Interest earned is $\widetilde{IE}_{21} = 0$ and interest paid $\widetilde{IP}_{21}$ is given by

$$\begin{cases} IP_{21L}(\alpha) = r.i_p[(1-\lambda_R)PM(N-M) + \int_M^N P(N-t)dt \\ + \int_N^{t_1} I_L(\alpha,t)dt + \int_{t_1}^{T_L} I_L(\alpha,t)dt] \\ = r.i_p\Big[(1-\lambda_R)PM(N-M) + \frac{1}{2}(1-\lambda_R)P(N-M)^2 + \frac{(1-\lambda_R)P-a_R}{2}(t_1^2-N^2) \\ + \frac{b_L e^{-cN}}{2}(t_1-N)^2 + \frac{b_L}{c}(1-e^{-cN})(t_1-N) + \frac{1}{2}(-a_R+b_L e^{-cN})(T_L^2-t_1^2) \\ + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} + \frac{b_L}{c}(1-e^{-cN})\}(T_L-t_1)\Big] \\ IP_{21R}(\alpha) = r.i_p\Big[(1-\lambda_L)PM(N-M) + \frac{1}{2}(1-\lambda_L)P(N-M)^2 \\ + \frac{(1-\lambda_L)P-a_L}{2}(t_1^2-N^2) + \frac{b_R e^{-cN}}{2}(t_1-N)^2 + \frac{b_R}{c}(1-e^{-cN})(t_1-N) \\ + \frac{1}{2}(-a_L+b_R e^{-cN})(T_R^2-t_1^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ + \frac{b_R}{c}(1-e^{-cN})\}(T_R-t_1)\Big] \end{cases} \tag{6.63}$$

**Subcase-2.2:** $M \leq t_1 \leq N < \widetilde{T}$

Here, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{22}$ are same as the expressions of subcase-1.4. Interest earned is $\widetilde{IE}_{22} = 0$ and interest paid $\widetilde{IP}_{22}$ is given by

$$\begin{cases} IP_{22L}(\alpha) = r.i_p[(1-\lambda_R)PM(N-M) + \int_M^{t_1} P(t_1-t)dt + (1-\lambda_R)P(t_1-M)(N-t_1) \\ + \int_N^{T_L} I_L(\alpha,t)dt] \\ = r.i_p\Big[(1-\lambda_R)PM(N-M) + \frac{1}{2}(1-\lambda_R)P(t_1-M)^2 + (1-\lambda_R)P(t_1-M)(N-t_1) \\ + \frac{1}{2}(-a_R+b_L e^{-cN})(T_L^2-N^2) + \{(1-\lambda_R)Pt_1 - b_L N e^{-cN} + \frac{b_L}{c}(1-e^{-cN})\}(T_L-N)\Big] \\ IP_{22R}(\alpha) = r.i_p\Big[(1-\lambda_L)PM(N-M) + \frac{1}{2}(1-\lambda_L)P(t_1-M)^2 \\ +(1-\lambda_L)P(t_1-M)(N-t_1) + \frac{1}{2}(-a_L+b_R e^{-cN})(T_R^2-N^2) + \{(1-\lambda_L)Pt_1 - b_R N e^{-cN} \\ + \frac{b_R}{c}(1-e^{-cN})\}(T_R-N)\Big] \end{cases}$$

**Subcase-2.3:** $M \leq t_1 < \widetilde{T} < N$

Here, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{23}$ are same as the expressions of subcase-1.6. Interest earned is $\widetilde{IE}_{23} = 0$ and interest paid $\widetilde{IP}_{23}$ is given by

$$\begin{cases} IP_{23L}(\alpha) = r.i_p[(1-\lambda_R)PM(N-M) + \int_M^{t_1} P(t_1-t)dt + P(t_1-M)(N-t_1)] \\ = r.i_p\Big[(1-\lambda_R)PM(N-M) + \frac{1}{2}(1-\lambda_R)P(t_1-M)^2 + P(t_1-M)(N-t_1)\Big] \\ IP_{23R}(\alpha) = r.i_p\Big[(1-\lambda_L)PM(N-M) + \frac{1}{2}(1-\lambda_L)P(t_1-M)^2 \\ +P(t_1-M)(N-t_1)\Big] \end{cases} \tag{6.64}$$

**Subcase-2.4:** $t_1 \leq M < N \leq \widetilde{T}$

Here, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{24}$ are same as the expressions of subcase-1.4. Interest earned is $\widetilde{IE}_{24} = 0$ and interest paid $\widetilde{IP}_{24}$ is given by

$$\begin{cases} IP_{24L}(\alpha) = r.i_p[(1-\lambda_R)Pt_1(N-M) + \int_N^{T_L} PI_L(t)dt] \\ = r.i_p\Big[(1-\lambda_R)Pt_1(N-M) + \frac{1}{2}(-a_R + b_L e^{-cN})(T_L^2 - N^2) \\ +\{(1-\lambda_R)Pt_1 - b_L N e^{-cN} + \frac{b_L}{c}(1-e^{-cN})\}(T_L - N)\Big] \\ IP_{24R}(\alpha) = r.i_p\Big[(1-\lambda_L)Pt_1(N-M) + \frac{1}{2}(-a_L + b_R e^{-cN})(T_R^2 - N^2) \\ +\{(1-\lambda_L)Pt_1 - b_R N e^{-cN} + \frac{b_R}{c}(1-e^{-cN})\}(T_R - N)\Big] \end{cases} \tag{6.65}$$

**Subcase-2.5:** $t_1 \leq M \leq \widetilde{T} < N$

Here, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{25}$ are same as the expressions of subcase-1.6. Interest earned is $\widetilde{IE}_{25} = 0$ and interest paid $\widetilde{IP}_{25}$ is given by

$$\begin{cases} IP_{25L}(\alpha) = r.i_p(1-\lambda_R)Pt_1(N-M) \\ IP_{25R}(\alpha) = r.i_p(1-\lambda_L)Pt_1(N-M) \end{cases} \tag{6.66}$$

**Subcase-2.6:** $t_1 < \widetilde{T} \leq M < N$

Here, inventory level $\widetilde{I}(t)$ and holding cost $\widetilde{HC}_{26}$ are same as the expressions of subcase-1.6. Interest earned is $\widetilde{IE}_{26} = 0$ and interest paid $\widetilde{IP}_{26}$ is given by

$$\begin{cases} IP_{26L}(\alpha) = r.i_p(1-\lambda_R)Pt_1(N-M) \\ IP_{26R}(\alpha) = r.i_p(1-\lambda_L)Pt_1(N-M) \end{cases} \tag{6.67}$$

### 6.4.3 Optimization Problems

Total profit=Sales revenue + Interest earned - Set-up cost - Purchasing cost - Screening cost - Holding cost - Interest paid. From the above discussion, the $\alpha$-cut set of average profit in j-th subcase of i-th case is given by

$$\begin{array}{l} \widetilde{TP}_{ij} = \widetilde{SP} + \widetilde{IE}_{ij} - Cs - \widetilde{PC} - SC - \widetilde{HC}_{ij} - \widetilde{IP}_{ij} \\ TP_{ijL} = SP_L + IE_{ijL} - Cs - PC_R - SC - HC_{ijR} - IP_{ijR} \\ TP_{ijR} = SP_R + IE_{ijR} - Cs - PC_L - SC - HC_{ijL} - IP_{ijL} \end{array} \tag{6.68}$$

and average total profit $\widetilde{ATP}_{ij} = \frac{\widetilde{TP}_{ij}}{\widetilde{T}}$, which is expressed as

$$[ATP_{ijL}, ATP_{ijR}] = [\frac{TP_{ijL}}{T_R}, \frac{TP_{ijR}}{T_L}] \tag{6.69}$$

Now our problem is to determine optimal value of $t_1$ to maximize the average profit $\widetilde{ATP}_{ij}$ (i=1,2; j=1,2,...,6).

**New Approach**

Let $\widetilde{T}'$ be the time for final payment for the remaining dues after the credit period (M) by the retailer to the supplier. Now we formulate two models as described in the following subcases.

**Subcase 3.1:** $N < M \leq \widetilde{T}' < \widetilde{T}$

**During [0,M]:** Revenue earned from sale is $\widetilde{RE}_{31}^{(1)} = s[\int_0^N(\widetilde{a}-\widetilde{b}e^{-ct})dt+\int_N^M(\widetilde{a}-\widetilde{b}e^{-cN})dt]$. Then we have,

$$\begin{aligned} RE_{31L}^{(1)} &= s\{a_LM - \tfrac{b_R}{c}(1-e^{-cN}) - (M-N)b_Re^{-cN}\} \\ RE_{31R}^{(1)} &= s\{a_RM - \tfrac{b_L}{c}(1-e^{-cN}) - (M-N)b_Le^{-cN}\} \end{aligned} \tag{6.70}$$

Interest earned on this revenue is $\widetilde{IER}_{31}^{(1)} = s.i_e[(M-N)\int_0^N(\widetilde{a}-\widetilde{b}e^{-ct})dt + \int_N^M(\widetilde{a}-\widetilde{b}e^{-cN})(M-t)dt]$. Then

$$\begin{aligned} IER_{31L}^{(1)} &= s.i_e[(M-N)\{a_LN - \tfrac{b_R}{c}(1-e^{-cN})\} + \tfrac{1}{2}(a_L - b_Re^{-cN})(M-N)^2 \\ IER_{31R}^{(1)} &= s.i_e[(M-N)\{a_RN - \tfrac{b_L}{c}(1-e^{-cN})\} + \tfrac{1}{2}(a_R - b_Le^{-cN})(M-N)^2 \end{aligned} \tag{6.71}$$

At t=M, the due/remaining payment is $\widetilde{DP}_{31} = r(1-\widetilde{\lambda})Pt_1 - (\widetilde{RE}_{31}^{(1)} + \widetilde{IER}_{31}^{(1)})$. i.e.,

$$\begin{aligned} DP_{31L} &= r(1-\lambda_R)Pt_1 - (RE_{31R}^{(1)} + IER_{31R}^{(1)}) \\ DP_{31R} &= r(1-\lambda_L)Pt_1 - (RE_{31L}^{(1)} + IER_{31L}^{(1)}) \end{aligned} \tag{6.72}$$

Amount of due payment with interest at $\widetilde{T}'$ paid to the supplier is $\widetilde{ADP}_{31} = \widetilde{DP}_{31} + i_p.\widetilde{DP}_{31}(\widetilde{T}' - M)$. The equivalent $\alpha$-cut set is

$$\begin{aligned} ADP_{31L} &= DP_{31L}\{1 + i_p(T'_L - M)\} \\ ADP_{31R} &= DP_{31R}\{1 + i_p(T'_R - M)\} \end{aligned} \tag{6.73}$$

**During [M, $\widetilde{T}'$]:** Revenue earned from sale is $\widetilde{RE}_{31}^{(2)} = s\int_M^{\widetilde{T}'}(\widetilde{a}-\widetilde{b}e^{-cN})dt$. Then,

$$\begin{aligned} RE_{31L}^{(2)} &= s(a_L - b_Re^{-cN})(T'_L - M) \\ RE_{31R}^{(2)} &= s(a_R - b_Le^{-cN})(T'_R - M) \end{aligned} \tag{6.74}$$

Interest earned on this revenue is $\widetilde{IER}_{31}^{(2)} = s.i_e\int_M^{\widetilde{T}'}(\widetilde{a}-\widetilde{b}e^{-cN})(\widetilde{T}'-t)dt]$. Then

$$\begin{aligned} IER_{31L}^{(2)} &= \tfrac{1}{2}s.i_e(a_L - b_Re^{-cN})(T'_L - M)^2 \\ IER_{31R}^{(1)} &= \tfrac{1}{2}s.i_e(a_R - b_Le^{-cN})(T'_R - M)^2 \end{aligned} \tag{6.75}$$

According to our assumption

$$\begin{aligned} ADP_{31L} &= RE_{31L}^{(2)} + IER_{31L}^{(2)} \\ ADP_{31R} &= RE_{31R}^{(2)} + IER_{31R}^{(2)} \end{aligned} \tag{6.76}$$

Profit of the system $\widetilde{TP}_{31}$= Revenue during $[\widetilde{T}', \widetilde{T}]$+Interest earned on it - Holding cost - Set up cost -Screening cost= $s\int_{\widetilde{T}'}^{\widetilde{T}}(\widetilde{a}-\widetilde{b}e^{-cN})dt + s.i_e\int_{\widetilde{T}'}^{\widetilde{T}}(\widetilde{a}-\widetilde{b}e^{-cN})(\widetilde{T}-t)dt$. It is reduced as

$$\begin{aligned} &TP_{31L} = s(a_L - b_R e^{-cN})(T_L - T'_R) + \tfrac{1}{2}s.i_e(a_L - b_R e^{-cN})(T_L - T'_R)^2 \\ &-HC_{11R} - Cs - u.Pt_1 \\ &TP_{31R} = s(a_R - b_L e^{-cN})(T_R - T'_L) + \tfrac{1}{2}s.i_e(a_R - b_L e^{-cN})(T_R - T'_L)^2 \\ &-HC_{11L} - Cs - u.Pt_1 \end{aligned} \tag{6.77}$$

Now our problem is to determine optimal value of $t_1$ to maximize the average profit $\widetilde{ATP}_{31} = \frac{\widetilde{TP}_{31}}{\widetilde{T}}$ with respect to constraints (6.76) and (6.43).

**Subcase 3.2:** $M < N \leq \widetilde{T}' < \widetilde{T}$

**During [0,N]:** Revenue earned from sale is $\widetilde{RE}_{32}^{(1)} = s[\int_0^N(\widetilde{a}-\widetilde{b}e^{-ct})dt]$. Then we have,

$$\begin{aligned} RE_{32L}^{(1)} &= s\{a_L N - \tfrac{b_R}{c}(1-e^{-cN})\} \\ RE_{32R}^{(1)} &= s\{a_R N - \tfrac{b_L}{c}(1-e^{-cN})\} \end{aligned} \tag{6.78}$$

At t=N, the due/remaining payment is $\widetilde{DP}_{32} = r(1-\widetilde{\lambda})Pt_1 + r(1-\widetilde{\lambda})PMi_p(N-M) - \widetilde{RE}_{32}^{(1)}$. i.e.,

$$\begin{aligned} DP_{32L} &= r(1-\lambda_R)Pt_1 + r(1-\lambda_R)PMi_p(N-M) - RE_{32R}^{(1)} \\ DP_{32R} &= r(1-\lambda_L)Pt_1 + r(1-\lambda_L)PMi_p(N-M) - RE_{32L}^{(1)} \end{aligned} \tag{6.79}$$

Amount of due payment with interest at $\widetilde{T}'$ paid to the supplier is $\widetilde{ADP}_{32} = \widetilde{DP}_{32} + i_p.\widetilde{DP}_{32}(\widetilde{T}'-N)$. The equivalent $\alpha$-cut set is

$$\begin{aligned} ADP_{32L} &= DP_{32L}\{1 + i_p(T'_L - N)\} \\ ADP_{32R} &= DP_{32R}\{1 + i_p(T'_R - N)\} \end{aligned} \tag{6.80}$$

**During [N, $\widetilde{T}'$]:** Revenue earned from sale is $\widetilde{RE}_{32}^{(2)} = s\int_N^{\widetilde{T}'}(\widetilde{a}-\widetilde{b}e^{-cN})dt$. Then,

$$\begin{aligned} RE_{32L}^{(2)} &= s(a_L - b_R e^{-cN})(T'_L - N) \\ RE_{32R}^{(2)} &= s(a_R - b_L e^{-cN})(T'_R - N) \end{aligned} \tag{6.81}$$

Interest earned on this revenue is $\widetilde{IER}_{32}^{(2)} = s.i_e \int_N^{\widetilde{T'}} (\widetilde{a} - \widetilde{b}e^{-cN})(\widetilde{T'} - t)dt]$. Then

$$\begin{aligned} IER_{32L}^{(2)} &= \tfrac{1}{2}s.i_e(a_L - b_R e^{-cN})(T'_L - N)^2 \\ IER_{32R}^{(1)} &= \tfrac{1}{2}s.i_e(a_R - b_L e^{-cN})(T'_R - N)^2 \end{aligned} \tag{6.82}$$

According to our assumption

$$\begin{aligned} ADP_{32L} &= RE_{32L}^{(2)} + IER_{32L}^{(2)} \\ ADP_{32R} &= RE_{32R}^{(2)} + IER_{32R}^{(2)} \end{aligned} \tag{6.83}$$

Profit of the system $\widetilde{TP}_{32}$ is same as equations (6.77). Now our problem is to determine optimal value of $t_1$ to maximize the average profit $\widetilde{ATP}_{32} = \frac{\widetilde{TP}_{32}}{\widetilde{T}}$ with respect to constraints (6.83) and (6.43).

### 6.4.4 Solution Methodology

The profit maximization problems are multi-objective optimization problems. To convert it as a single objective optimization problem we use weighted sum method. The problem reduces to:
Maximize $ATP_{ij}$=$\delta_1 ATP_{ijL} + \delta_2 ATP_{ijR}$ subject to $\delta_1 + \delta_2 = 1$.

The above non-linear optimization problems are solved by TLBO (cf.§ 2.2.3.3) for particular sets of data.

### 6.4.5 Numerical Experiments and Results

In this section, we perform numerical experiments and present the results. For the conventional approach, Experiment-1 is performed for twelve different subcases and Experiment-2 is done using the new approach.

**Experiment-1:** We consider the following input data:
$\alpha = 0.750$, $\widetilde{\lambda} = (0.25, 0.30, 0.35)$, $\widetilde{a} = (1000, 1200, 1400)$, $\widetilde{a} = (200, 300, 400)$, $c = 2.50$, $s = 50$, $r = 25$,$u = 05$, $h = 0.750$, $i_e = 0.10$, $i_p = 0.15$, $P = 2000$, $\delta_1 = 0.50$ and $\delta_2 = 0.50$. For twelve different subcases, the inputs of M and N and the corresponding optimum production run times, optimum profits and cycle times are presented in Table 6.7.

**Experiment-2:** Considering same input data of Experiment-1, we obtain the optimum results of subcases-3.1 and 3.2 only, which are presented in Table 6.8.

**Table** 6.7: Optimum results of different subcases for conventional approach

| Subcases | Input | | Optimum results | | | | | |
|---|---|---|---|---|---|---|---|---|
| | M | N | $t_1^*$ | $[ATP_{ijL}^*, ATP_{ijR}^*]$ | $[T_L^*, T_R^*]$ | Avg.Profit | Avg. Int. paid | Avg. Int. earned |
| 1.1 | 0.50 | 0.25 | 0.9980 | [14024, 19990] | [1.2573, 1.4744] | 17007 | 613 | 346 |
| 1.2 | 0.75 | 0.25 | 0.6477 | [15014, 20935] | [0.8205, 0.9630] | 17975 | 51 | 1428 |
| 1.3 | 1.25 | 0.25 | 0.6387 | [16547, 24893] | [0.8093, 0.9499] | 20720 | – | 4152 |
| 1.4 | 1.85 | 1.75 | 1.7113 | [17091, 22971] | [1.9699, 2.2338] | 17091 | 88 | 489 |
| 1.5 | 1.20 | 0.65 | 0.6500 | [16570, 24215] | [0.7918, 0.9107] | 20393 | – | 2972 |
| 1.6 | 0.90 | 0.70 | 0.4895 | [14792, 20431] | [0.6072, 0.7000] | 17612 | – | 1071 |
| 2.1 | 0.25 | 0.50 | 1.7399 | [14490, 20692] | [2.0760, 2.3896] | 17591 | 1248 | – |
| 2.2 | 0.85 | 1.25 | 1.2500 | [14729, 20298] | [1.4611, 1.6621] | 17513 | 1865 | – |
| 2.3 | 0.75 | 1.60 | 1.2016 | [12931, 18104] | [1.4072, 1.6000] | 15518 | 3497 | – |
| 2.4 | 0.85 | 0.90 | 0.8500 | [15162, 20666] | [1.0164, 1.1622] | 17914 | 290 | – |
| 2.5 | 1.00 | 2.00 | 1.0000 | [11914, 16987] | [1.1834, 1.3482] | 14451 | 4169 | – |
| 2.6 | 1.75 | 1.90 | 1.3222 | [15809, 21310] | [1.5405, 1.7500] | 18560 | 636 | – |

**Table** 6.8: Optimum results of different subcases for new approach

| Sub-cases | Input | | Optimum results | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | M | N | $t_1^*$ | $[T_L^{'*}, T_R^{'*}]$ | $[ATP_{ijL}^*, ATP_{ijR}^*]$ | $[T_L^*, T_R^*]$ | Avg.Profit | Avg. Int. paid | Avg. Int. earned |
| 3.1 | 0.50 | 0.25 | 1.1471 | [0.7537, 0.8155] | [10802, 28420] | [1.4432, 1.6921] | 19611 | 412 | 443 |
| 3.2 | 0.25 | 0.50 | 1.4713 | [0.9664, 1.0135] | [11975, 27818] | [1.7607, 2.0275] | 19897 | 1207 | 361 |

### 6.4.6 Discussion

(i) It is revealed from Table 6.7 that the Subcase 1.3 is the most profitable scenario, since in this case, (M-N) is largest, so retailer can earn more interest than the other sub-cases and Subcase 2.5 is worst for the opposite reason.

(ii) Average profits for the Subcases 1.1, 1.2 and 1.3 gradually increase because of constant N and gradual increase of M. In these sub-cases, average interest paid to supplier decreases and average interest earned by retailer increases. These results are as per our expectation.

(iii) Comparing the optimum results between conventional and new approach, we observe from Table 6.8 that new approach gives better profits. Since all due payments are paid at a time $\widetilde{T}'$ before cycle length $\widetilde{T}$, the amount of interest paid to supplier is less as well as interest earned is more in new approach than the conventional approach.

(iv) For the given set of parameters in the above Experiment 1, Figs. 6.7-6.11 represent the concavity property of the average profit function against the production run time ($t_1$) for different subcases.

(v) $\alpha$-cuts of observed average profit, interest paid, interest earned and cycle time for the subcases 1.1 and 2.1 due to Experiment 1 are plotted in Figs. 6.12-6.18. It is interesting to note that the figures represent almost TFNs.

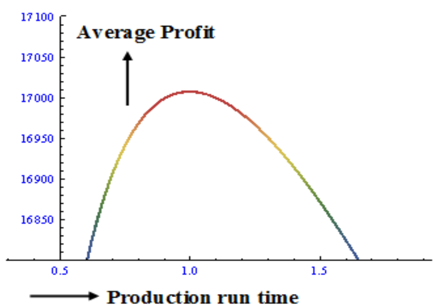



**Figure** 6.7: Average profit against production run time for subcase 1.1

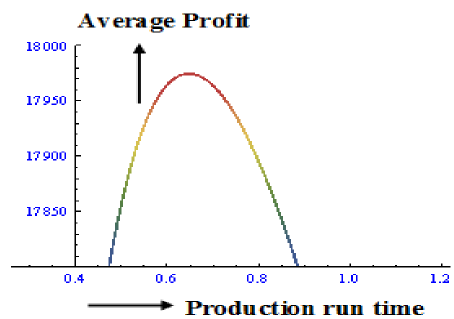



**Figure** 6.8: Average profit against production run time for subcase 1.2

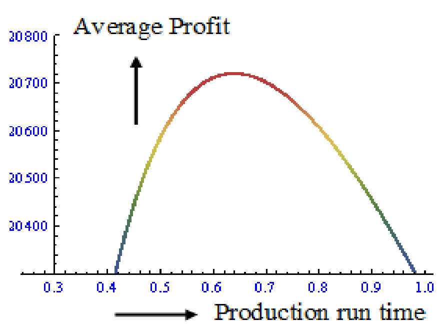



**Figure** 6.9: Average profit against production run time for subcase 1.3

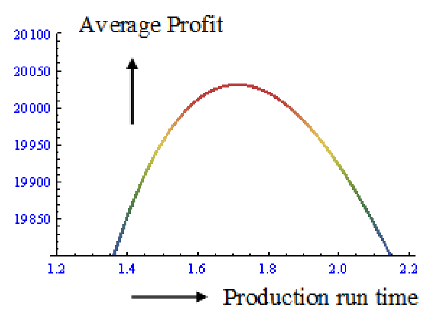



**Figure** 6.10: Average profit against production run time for subcase 1.4

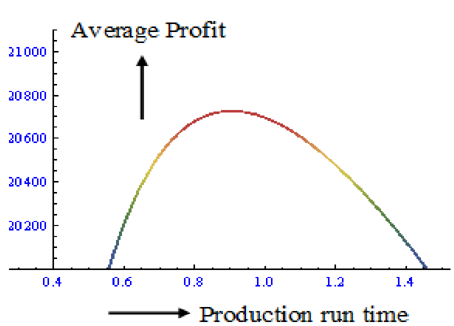



**Figure** 6.11: Average profit against production run time for subcase 1.5

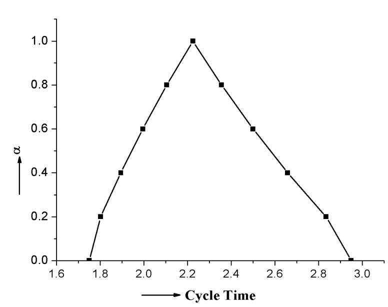



**Figure** 6.12: Cycle time against $\alpha$ for subcase 2.1

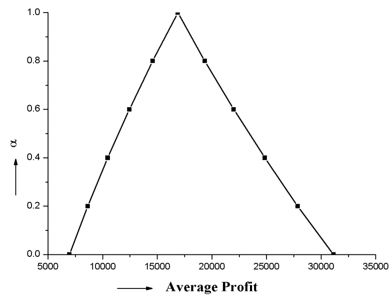



**Figure** 6.13: Average profit against $\alpha$ for subcase 1.1

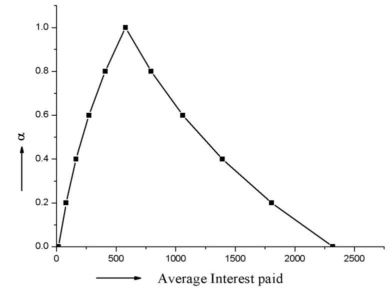



**Figure** 6.14: Average interest paid against $\alpha$ for subcase 1.1

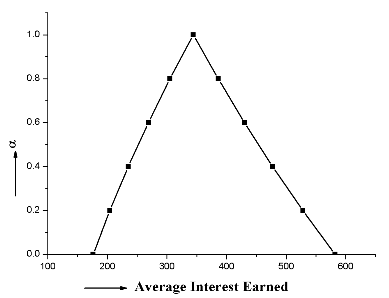



**Figure** 6.15: Average interest earned against $\alpha$ for subcase 1.1

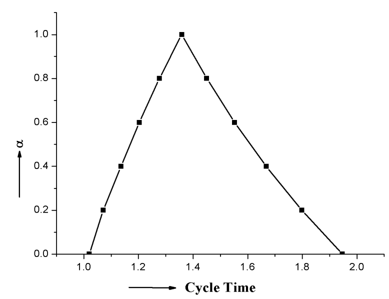



**Figure** 6.16: Cycle time against $\alpha$ for subcase 1.1

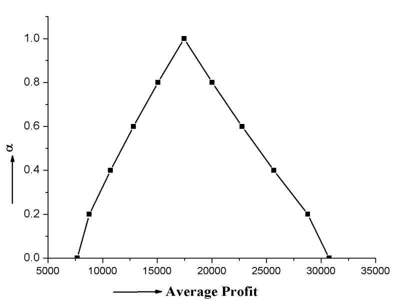



**Figure** 6.17: Average profit against $\alpha$ for subcase 2.1

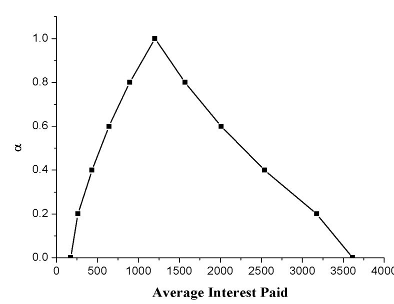



**Figure** 6.18: Average interest paid against $\alpha$ for subcase 2.1

## 6.5 Conclusion

The present investigation is proposed for the Trade Credit Policy in Fuzzy Environment using various inventory models and the observations are as follows:

• For the first model proposed here, by rigorous mathematical derivation, the concavity of the objective function of the model is depicted and the closed-form optimal solution of the model is obtained. The numerical illustration and sensitivity analysis of the problem are given on the framework described and an optimal solution procedure for finding optimal production policy is presented. Also, it is seen that changes in the different parameters lead to significant effects on the optimal production time and optimal total profit

At the end, we would like to point out that most of researchers on fuzzy inventory problems often employed the centroid method to obtain the estimate of total cost in the fuzzy sense. To achieve this task, the membership function of fuzzy total cost has to be found first using the extension principle, while the derivation is very complex. To avoid this complex mathematical deduction, we have used fuzzy expectation method for defuzzification in this study. The model presented in this study provides a basis for several possible extensions.

**For future research, this model can be extended such as**

(i) Shortages and variable ordering cost can be considered in more details.

(ii) Another possible extension would be with the time dependent or stock dependent holding cost and others.

(iii) The results presented here not only provide a valuable references for the decision-makers in planning and controlling the inventory but also furnish a useful model for many organisations such as shops dealing with domestic items and retail business industries. It can be used for domestic goods, fashion cloths, electronics components, medicines and other products.

• The second model proposed here addresses a production inventory model with fuzzy dynamic demand under bi-level trade credit policy in imprecise environment. However in a real market place, it is common that the retailer is not able to pay consistently at the fixed time point every time. Normally, under the condition of trade credit, it is beneficial to pay only at the trade credit limit point rather than before. Sometimes in extreme cases payment is made after trade credit period.

As explained under numerical illustration, there is difference in optimal policies which also result in significant differences in average profit in new approach. This new approach i.e. the clearance of all dues before the end of time period, as and when it is feasible is a new payment policy for the retailer for more profit.

**For future research, this model can be extended as**

(i) Retailer's model has wide range of applications in supplier-retail-customer business where the competition is stiff, especially in grocery, stationary goods, building materials shop etc.

(ii) Further, it is possible to incorporate the proposed model with more realistic assumptions, such as random demand, deteriorating items, allowable shortages, multi-supplier, multi-retailer, etc.

(iii) This two-level trade credit model can be extended to three-level trade credit model for supplier-wholesaler-retailer-customer supply chain system.

# Part III

# Summary of the Thesis

# Chapter 7

# Summary and Future Extension

## 7.1 Summary of the Thesis

In this thesis, total nine virgin uncertain inventory /production-inventory models, of which five in random and four in fuzzy environments are formulated and solved.

- The models are developed for different types of demands like stock dependent demand, price and quality dependent demand, promotional effort and advertisement dependent demand, credit period dependent dynamic demand and news-boy type probabilistic demand.

- The models are formulated with in-control and out-control states, effects of learning and forgetting, carbon emission, advanced payment, trade credit policy, inflation of money and many more criteria which are visible in recent management system.

- The models are simplified (may be converted from fuzzy to crisp, random to non-random, etc.) by using methods of Fuzzy Differential Equation (FDE), Possibility, Necessity, Credibility, Trust measures, method of chance constraint, etc.

- Different techniques are developed/presented to transform the imprecise parameters/objectives to corresponding deterministic ones. For the solution of single and multi-objective with/without constraints, different optimization techniques such as Generalised Reduced Gradient method (GRG), Genetic Algorithm (GA), Fuzzy Age based GA (FAGA), Multi-objective GA (MOGA), Rough Age based GA (RAGA), Teaching and Learning Based Optimization (TLBO), Intuitionistic Fuzzy Optimization Technique (IFOT), etc. are developed / presented and used.

- The models are illustrated with appropriate numerical examples and the optimum results are presented graphically. Moreover, the obtained results are discussed with respect to managerial insights.

- In some cases, sensitivity analyses are made with respect to some model parameters and presented graphically and in tabular forms to look deep into the models.

## 7.2 Future Extension

- Each model presented in the thesis has an impact to its future extensions. The models can be formulated in other type of uncertain environments, such as- fuzzy-rough, rough-fuzzy, random-fuzzy, etc.
- The models also can be extended with different types of uncertain parameters and/or variables like bi-fuzzy, type-2 fuzzy, rough, bi-rough etc.
- Different types of optimization techniques (Particle Swarm Optimization (PSO), Ant Colony Optimization (ACO), Geometric Programming (GP), etc) can also be applied to the models.
- Moreover, other recent development criteria like risk management, just-in-time inventory control, different forms of supply-chain, etc. also can be incorporated to the models.
- In the models, intension expression of customer to purchase the goods knowing the selling price at that time may be incorporated.
- CE due to transportation can be included in the models (presented in this dissertation) along with CE due to production.

Therefore, there is a huge scope to extend the research works presented in this thesis.

# Part IV

# Appendices, Bibliography and Indices

# Appendix A

# For Model 4.3

## A.1 Scenario 1

From Fig. 4.16, inventory area of OABC for item 1 is $\frac{1}{2}(AB + OC)OA$ (area of trapezium) $= \frac{1}{2}(Q_1 - \xi_1 D_1 + Q_1)T = \frac{1}{2}(2Q_1 - \xi_1 D_1)T$.
Similarly, inventory area of OADE for item 2 is $= \frac{1}{2}(2Q_2 - \xi_2 D_2)T$.

## A.2 Scenario 2

From Fig. 4.17, by the property of similar triangles $\Delta EOA$ and $\Delta EDC$, we have $\frac{EO}{ED} = \frac{OA}{DC}$ or, $\frac{Q_1}{\xi_1 D_1} = \frac{T_1}{T}$ or, $T_1 = \frac{Q_1 T}{\xi_1 D_1}$.

Inventory area of $\Delta EOA$ for item 1 is $\frac{1}{2}OA.OE = \frac{Q_1^2 T}{2\xi_1 D_1}$.

Shortage area of $\Delta ABC$ for item 1 is $\frac{1}{2}AB.BC = \frac{1}{2}(OB - OA).BC = \frac{1}{2}(T - T_1)(\xi_1 D_1 - Q_1) = \frac{(\xi_1 D_1 - Q_1)^2 T}{2\xi_1 D_1}$. [Using the value of $T_1$]

Inventory area of OBGF for item 2 is $\frac{1}{2}(Q_2 + Q_2 - \xi_1 D_1)T = \frac{1}{2}(2Q_2 - \xi_2 D_2)T$.

Since shortages of item 1 are fully substituted by item 2, holding inventory area of OBHIFO for item 2 is finally reduced to $\frac{1}{2}(2Q_2 - \xi_2 D_2)T - \frac{(\xi_1 D_1 - Q_1)^2 T}{2\xi_1 D_1}$.

## A.3 Scenario 4

From Fig. 4.18, by the property of similar triangles $\Delta HOA$ and $\Delta HFE$, we have $\frac{HO}{HF} = \frac{OA}{FE}$ or, $\frac{Q_1}{\xi_1 D_1} = \frac{T_1}{T}$ or, $T_1 = \frac{Q_1 T}{\xi_1 D_1}$ or, $T - T_1 = \frac{(\xi_1 D_1 - Q_1)T}{\xi_1 D_1}$.

In order to reserve the stock of item 2 to meet its usual demand $\xi_2 D_2/T$ in the market upto the pre-defined time T, the sale of the substitute item 1 is closed at B. This shortage rate $\xi_1 D_1/T$ is same as before. As $AE \parallel BD$, $\Delta ACE$ and $\Delta BCD$ are similar triangles. Thus, by the property of similar triangles we have, $\frac{AC}{BC} = \frac{CE}{CD}$ or, $BC = \frac{AC.CD}{CE}$ $= \frac{(T-T_1)\{(\xi_1 D_1-Q_1)-(Q_2-\xi_2 D_2)\}}{(\xi_1 D_1-Q_1)} = \frac{\{(\xi_1 D_1-Q_1)-(Q_2-\xi_2 D_2)\}T}{\xi_1 D_1}$.

Inventory area of $\Delta OHA$ for item 1 is $\frac{Q_1^2 T}{2\xi_1 D_1}$.

Shortage area of ABDEA for item 1 which is substituted by item 2 is $\Delta ACE - \Delta BCD$
$= \frac{1}{2}AC.CE - \frac{1}{2}BC.CD$
$= \frac{1}{2}(T-T_1)(\xi_1 D_1 - Q_1) - \frac{1}{2}\frac{\{(\xi_1 D_1-Q_1)-(Q_2-\xi_2 D_2)\}T}{\xi_1 D_1}\{(\xi_1 D_1 - Q_1) - (Q_2 - \xi_2 D_2)\}$
$= \frac{1}{2}[\frac{(\xi_1 D_1-Q_1)^2 T}{\xi_1 D_1} - \frac{\{(\xi_1 D_1-Q_1)-(Q_2-\xi_2 D_2)\}^2 T}{\xi_1 D_1}]$.

Hence, holding inventory area of OCJKIO for item 2 finally reduced to $\frac{1}{2}(2Q_2 - \xi_2 D_2)T - \frac{1}{2}[\frac{(\xi_1 D_1-Q_1)^2 T}{\xi_1 D_1} - \frac{\{(\xi_1 D_1-Q_1)-(Q_2-\xi_2 D_2)\}^2 T}{\xi_1 D_1}]$.

## A.4 Scenario 6

From Fig. 4.19, by the property of similar triangles ($\Delta OBE$ and $\Delta GCE$) and ($\Delta OAF$ and $\Delta HDF$), we have $T_1 = \frac{Q_1 T}{\xi_1 D_1}$ and $T_2 = \frac{Q_2 T}{\xi_2 D_2}$ respectively.

Hence, inventory area of $\Delta OBE$ and $\Delta OAF$ for item 1 and 2 are $\frac{Q_1^2 T}{2\xi_1 D_1}$ and $\frac{Q_2^2 T}{2\xi_2 D_2}$ respectively.

## A.5 Integral values in different regions

Substitute $D_1$ and $D_2$ by x and y variables respectively the integrations are calculated as follows:

$I_1^{(R_1)} = \int_a^{Q_1/\xi_1}\int_c^{Q_2/\xi_2} x dx dy = \frac{1}{2}(\frac{Q_1^2}{\xi_1^2} - a^2)(\frac{Q_2}{\xi_2} - c)$, $I_2^{(R_1)} = \int_a^{Q_1/\xi_1}\int_c^{Q_2/\xi_2} y dx dy = \frac{1}{2}(\frac{Q_1}{\xi_1} - a)(\frac{Q_2^2}{\xi_2^2} - c^2)$,
$I_3^{(R_1)} = \int_a^{Q_1/\xi_1}\int_c^{Q_2/\xi_2} dx dy = (\frac{Q_1}{\xi_1} - a)(\frac{Q_2}{\xi_2} - c)$,

$$I_1^{(R_2)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_c^{(Q_1+Q_2-\xi_1 x)/\xi_2} x dx dy = \frac{1}{\xi_2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} x(Q_1+Q_2-\xi_2 c-\xi_1 x)dx$$
$$= \frac{1}{\xi_1^2\xi_2}[\frac{(Q_1+Q_2-\xi_2 c)^3}{6} - \frac{Q_1^3}{6} - \frac{Q_1^2(Q_2-\xi_2 c)}{2}],$$
$$I_2^{(R_2)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_c^{(Q_1+Q_2-\xi_1 x)/\xi_2} y dx dy = \frac{1}{2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \{(\frac{Q_1+Q_2-\xi_1 x}{\xi_2})^2 - c^2\}dx$$
$$= \frac{\xi_2 c^3}{3\xi_1} + \frac{Q_2^3}{6\xi_1\xi_2^2} - \frac{c^2 Q_2}{\xi_1},$$
$$I_3^{(R_2)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_c^{(Q_1+Q_2-\xi_1 x)/\xi_2} dx dy = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} (\frac{Q_1+Q_2-\xi_1 x}{\xi_2} - c)dx$$
$$= \frac{1}{2\xi_1\xi_2}(Q_2-\xi_2 c)^2,$$
$$I_4^{(R_2)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_c^{(Q_1+Q_2-\xi_1 x)/\xi_2} \frac{1}{x} dx dy = \frac{1}{\xi_2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \{\frac{Q_1+Q_2-\xi_2 c}{x} - \xi_1\}dx$$
$$= \frac{1}{\xi_2}\{(Q_1+Q_2-\xi_2 c)log(\frac{Q_1+Q_2-\xi_2 c}{Q_1}) - (Q_2-\xi_2 c)\}$$

As in the region $R_2$, the integrals in region $R_3$ are evaluated interchanging the items. Hence,

$$I_1^{(R_3)} = \frac{\xi_1 a^3}{2\xi_2} + \frac{Q_1^3}{6\xi_2\xi_1^2} - \frac{a^2 Q_1}{\xi_2},\ I_2^{(R_3)} = \frac{1}{\xi_1^2\xi_2}[\frac{(Q_1+Q_2-\xi_1 a)^3}{6} - \frac{Q_2^3}{6} - \frac{Q_2^2(Q_1-\xi_1 a)}{2}],$$
$$I_3^{(R_3)} = \frac{1}{2\xi_1\xi_2}(Q_1-\xi_1 a)^2,\ I_4^{(R_3)} = \frac{1}{\xi_1}\{(Q_1+Q_2-\xi_1 a)log(\frac{Q_1+Q_2-\xi_1 a}{Q_2}) - (Q_1-\xi_1 a)\}$$

$$I_1^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} x dx dy = \frac{1}{\xi_2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} x(\xi_1 x - Q_1)dx$$
$$= \frac{1}{\xi_1^2\xi_2}[\frac{(Q_1+Q_2-\xi_2 c)^3}{3} - \frac{Q_1(Q_1+Q_2-\xi_2 c)^2}{2} + \frac{Q_1^3}{6}];$$
$$I_2^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} y dx dy = \frac{1}{2\xi_2^2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \{Q_2^2 - (Q_1+Q_2-\xi_1 x)^2\}dx$$
$$= \frac{1}{\xi_1\xi_2^2}[\frac{\xi_2^3 c^3}{6} + \frac{Q_2^3}{3} - \frac{\xi_2 c Q_2^2}{2}]$$
$$I_3^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} dx dy = \frac{1}{\xi_2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} (\xi_1 x - Q_1)dx$$
$$= \frac{1}{2\xi_1\xi_2}(Q_2-\xi_2 c)^2$$

$$I_4^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} \frac{1}{x} dx dy = \frac{1}{\xi_2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \frac{1}{x}(\xi_1 x - Q_1)dx$$
$$= \frac{1}{\xi_2}[(Q_2-\xi_2 c) - Q_1 log(\frac{Q_1+Q_2-\xi_2 c}{Q_1})]$$
$$I_5^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} \frac{y}{x} dx dy = \frac{1}{2\xi_2^2} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \frac{1}{x}\{Q_2^2 - (Q_1+Q_2-\xi_1 x)^2\}dx$$
$$= \frac{1}{\xi_2^2}[2(Q_1+Q_2)(Q_2-\xi_2 c) - \frac{1}{2}\{(Q_1+Q_2-\xi_2 c)^2 - Q_1^2\} - (Q_1^2+2Q_1Q_2)log(\frac{Q_1+Q_2-\xi_2 c}{Q_1})]$$
$$I_6^{(R_4)} = \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \int_{(Q_1+Q_2-\xi_1 x)/\xi_2}^{Q_2/\xi_2} \frac{y^2}{x} dx dy$$
$$= \frac{1}{3\xi_2^3} \int_{Q_1/\xi_1}^{(Q_1+Q_2-\xi_2 c)/\xi_1} \frac{1}{x}\{-Q_1^3 - 3Q_1Q_2(Q_1+Q_2) + 3(Q_1+Q_2)^2\xi_1 x - 3(Q_1+Q_2)\xi_1^2 x^2 + \xi_1^3 x^3\}dx$$
$$= \frac{1}{3\xi_2^3}[3(Q_1+Q_2)^2(Q_2-\xi_2 c) - \frac{3(Q_1+Q_2)}{2}\{(Q_1+Q_2-\xi_2 c)^2 - Q_1^2\} + \frac{1}{3}\{(Q_1+Q_2-\xi_2 c)^3 - Q_1^3\}$$
$$-\{Q_1^3 + 3Q_1Q_2(Q_1+Q_2)\}log(\frac{Q_1+Q_2-\xi_2 c}{Q_1})]$$

$$I_1^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} x dx dy = \frac{1}{\xi_2} \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} x(Q_2-\xi_2 c) = \frac{Q_2-\xi_2 c}{2\xi_2}\{b^2 - \frac{(Q_1+Q_2-\xi_2 c)^2}{\xi_1^2}\}$$
$$I_2^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} y dx dy = \frac{Q_2^2-c^2}{2\xi_2^2}\{b - \frac{Q_1+Q_2-\xi_2 c}{\xi_1}\}$$
$$I_3^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} dx dy = \frac{1}{\xi_1\xi_2}\{\xi_1 b - (Q_1+Q_2-\xi_2 c)\}(Q_2-\xi_2 c)$$
$$I_4^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} \frac{1}{x} dx dy = (\frac{Q_2}{\xi_2} - c)log(\frac{\xi_1 b}{Q_1+Q_2-\xi_2 c})$$
$$I_5^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} \frac{y}{x} dx dy = \frac{1}{2\xi_2^2}(Q_2^2 - \xi_2^2 c^2)log(\frac{\xi_1 b}{Q_1+Q_2-\xi_2 c})$$
$$I_6^{(R_5)} = \int_{(Q_1+Q_2-\xi_2 c)/\xi_1}^{b} \int_c^{Q_2/\xi_2} \frac{y^2}{x} dx dy = \frac{1}{3\xi_2^3}(Q_2^3 - \xi_2^3 c^3)log(\frac{\xi_1 b}{Q_1+Q_2-\xi_2 c})$$

As in the region $R_4$, the integrals in region $R_6$ are evaluated interchanging the items. Hence,

$$I_1^{(R_6)} = \frac{1}{\xi_2\xi_1^2}[\frac{\xi_1^3 a^3}{6} + \frac{Q_1^3}{3} - \frac{\xi_1 c Q_1^2}{2}], \qquad I_2^{(R_6)} = \frac{1}{\xi_2^2\xi_1}[\frac{(Q_1+Q_2-\xi_1 a)^3}{3} - \frac{Q_1(Q_1+Q_2-\xi_1 a)^2}{2} + \frac{Q_2^3}{6}]$$
$$I_3^{(R_6)} = \frac{1}{2\xi_1\xi_2}(Q_1-\xi_1 a)^2 \qquad I_4^{(R_6)} = \frac{1}{\xi_1}[(Q_1-\xi_1 a) - Q_2 log(\frac{Q_1+Q_2-\xi_1 a}{Q_2})]$$
$$I_5^{(R_6)} = \frac{1}{\xi_1^2}[2(Q_1+Q_2)(Q_1-\xi_1 a) - \frac{1}{2}\{(Q_1+Q_2-\xi_1 a)^2 - Q_2^2\} - (Q_2^2+2Q_1Q_2)log(\frac{Q_1+Q_2-\xi_1 a}{Q_2})]$$
$$I_6^{(R_6)} = \frac{1}{3\xi_1^3}[3(Q_1+Q_2)^2(Q_1-\xi_1 a) - \frac{3(Q_1+Q_2)}{2}\{(Q_1+Q_2-\xi_1 a)^2 - Q_2^2\} + \frac{1}{3}\{(Q_1+Q_2-\xi_1 a)^3 - Q_2^3\}$$
$$-\{Q_2^3 + 3Q_1Q_2(Q_1+Q_2)\}log(\frac{Q_1+Q_2-\xi_1 a}{Q_2})]$$

As in the region $R_5$, the integrals in region $R_7$ are evaluated interchanging the items. Hence,

$$I_1^{(R_7)} = \frac{Q_1^2-a^2}{2\xi_1^2}\{d - \frac{Q_1+Q_2-\xi_1 a}{\xi_2}\}, \qquad I_2^{(R_7)} = \frac{Q_1-\xi_1 a}{2\xi_1}\{d^2 - \frac{(Q_1+Q_2-\xi_1 a)^2}{\xi_2^2}\}$$
$$I_3^{(R_7)} = \frac{1}{\xi_1\xi_2}\{\xi_2 d - (Q_1 + Q_2 - \xi_1 a)\}(Q_1 - \xi_1 a), \qquad I_4^{(R_7)} = (\frac{Q_1}{\xi_1} - a)log(\frac{\xi_2 d}{Q_1+Q_2-\xi_1 a})$$
$$I_5^{(R_7)} = \frac{1}{2\xi_1^2}(Q_1^2 - \xi_1^2 a^2)log(\frac{\xi_2 d}{Q_1+Q_2-\xi_1 a}), \qquad I_6^{(R_7)} = \frac{1}{3\xi_1^3}(Q_1^3 - \xi_1^3 a^3)log(\frac{\xi_2 d}{Q_1+Q_2-\xi_1 a})$$

$$I_1^{(R_8)} = \int_{Q_1/\xi_1}^{b} \int_{Q_2/\xi_2}^{d} xdxdy = \frac{1}{2}(b^2 - \frac{Q_1^2}{\xi_1^2})(d - \frac{Q_2}{\xi_2}), \qquad I_2^{(R_8)} = \int_{Q_1/\xi_1}^{b} \int_{Q_2/\xi_2}^{d} ydxdy = \frac{1}{2}(d^2 - \frac{Q_2^2}{\xi_2^2})(b - \frac{Q_1}{\xi_1})$$
$$I_3^{(R_8)} = \int_{Q_1/\xi_1}^{b} \int_{Q_2/\xi_2}^{d} dxdy = (b - \frac{Q_1}{\xi_1})(d - \frac{Q_2}{\xi_2}), \qquad I_4^{(R_8)} = \int_{Q_1/\xi_1}^{b} \int_{Q_2/\xi_2}^{d} \frac{1}{x}dxdy = (d - \frac{Q_2}{\xi_2})log(\frac{\xi_1 b}{Q_1})$$
$$I_5^{(R_8)} = \int_{Q_1/\xi_1}^{b} \int_{Q_2/\xi_2}^{d} \frac{1}{y}dxdy = (b - \frac{Q_1}{\xi_1})log(\frac{\xi_2 d}{Q_2})$$

# Appendix B

# For Model 5.2

## B.1 Solution Procedure of FDE

**In the interval** $[0, \tilde{\tau}]$**:**
$\frac{dI(t)}{dt} = P - D$ with the initial condition I(0)=0. Solving this equation in crisp environment we get, $I(t) = (P - D)t$.
**In the interval** $[\tilde{\tau}, t_1]$**:**
The fuzzy differential equation is $\frac{d\widetilde{I}(t)}{dt} = P - D - (1 - \theta)\gamma P(t - \widetilde{\tau})$ with fuzzy initial condition $\widetilde{I}(\tilde{\tau}) = (P - D)\tilde{\tau}$. Solving this fuzzy differential equation we get,

$$\begin{cases} I_L(\alpha, t) = (P - D)t - \frac{(1-\theta)\gamma P}{2}(t - \tau_L)^2 \\ I_R(\alpha, t) = (P - D)t - \frac{(1-\theta)\gamma P}{2}(t - \tau_R)^2 \end{cases} \tag{B.1}$$

**In the interval** $[t_1, \tilde{T}]$**:**
The differential equation is $\frac{dI(t)}{dt} = -D$ with boundary condition $I(T) = 0$. Solving the above equation we get, $I(t) = D(T - t)$. If $\tilde{T}$ is fuzzy in nature then using the Jadeh's Extension Principle we have,

$$\begin{cases} I_L(\alpha, t) = D(T_L - t) \\ I_R(\alpha, t) = D(T_R - t) \end{cases} \tag{B.2}$$

## B.2 Checking of Buckley-Feuring Conditions

**In the interval** $[\tilde{\tau}, t_1]$**:**
Here $\tilde{\tau} = (\tau_1, \tau_2, \tau_3)$ is a tri-angular fuzzy number having $\alpha$-cut $\tilde{\tau}[\alpha] = [\tau_L, \tau_R]$, where $\tau_L = \tau_1 + \alpha(\tau_2 - \tau_1)$ and $\tau_R = \tau_3 - \alpha(\tau_3 - \tau_2)$. Also, $\tilde{I}(t)[\alpha] = [I_L(\alpha, t), I_R(\alpha, t)]$, where $I_L(\alpha, t) = (P - D)t - \frac{(1-\theta)\gamma P}{2}(t - \tau_L)^2$ and $I_R(\alpha, t) = (P - D)t - \frac{(1-\theta)\gamma P}{2}(t - \tau_R)^2$.
Therefore, $\frac{dI_L(\alpha,t)}{dt} = (P-D)-(1-\theta)\gamma P(t-\tau_L)$ and $\frac{dI_R(\alpha,t)}{dt} = (P-D)-(1-\theta)\gamma P(t-\tau_R)$.
Differentiating the above equations with respect to $\alpha$ we get,

$\frac{d}{d\alpha}\left(\frac{dI_L(\alpha,t)}{dt}\right) = (1-\theta)\gamma P \frac{d}{d\alpha}(\tau_L) = (1-\theta)\gamma P \frac{d}{d\alpha}\{\tau_1+\alpha(\tau_2-\tau_1)\} = (1-\theta)\gamma P(\tau_2-\tau_1) > 0.$
and
$\frac{d}{d\alpha}\left(\frac{dI_R(\alpha,t)}{dt}\right) = (1-\theta)\gamma P \frac{d}{d\alpha}(\tau_R) = (1-\theta)\gamma P \frac{d}{d\alpha}\{\tau_3-\alpha(\tau_3-\tau_2)\} = -(1-\theta)\gamma P(\tau_3-\tau_2) < 0.$
Also, $\frac{dI_L(1,t)}{dt} = \frac{dI_R(1,t)}{dt} = (P-D)-(1-\theta)\gamma P(t-\tau_2)$.
Hence, all the equations and conditions defined by (2.20) and (2.15) respectively are satisfied.

## B.3 IFOT for minimization problem

Individual minimum of the objective functions $ACEC_k$ for all $k = L, C, R$ are obtained and given in Table B.1. Now we calculate $L_L = 4862.07$, $L_C = 5057.22$, $L_R = 5243.88$, $U_L = 4876.38$, $U_C = 5063.69$, $U_R = 5265.32$. we formulate the following problem as :

$$\left.\begin{array}{lll} & \max\ (\mu-\nu) & \\ \text{sub to} & \mu \leq \frac{e^{-w\left(\frac{ACEC_L-4862.07}{4876.38-4862.07}\right)}-e^{-w}}{1-e^{-w}}; & \nu \geq \left(\frac{ACEC_L-4862.07}{4876.38-4862.07}\right)^2 \\ & \mu \leq \frac{e^{-w\left(\frac{ACEC_C-5057.22}{5063.69-5057.22}\right)}-e^{-w}}{1-e^{-w}}; & \nu \geq \left(\frac{ACEC_C-5057.22}{5063.69-5057.22}\right)^2 \\ & \mu \leq \frac{e^{-w\left(\frac{ACEC_R-5243.88}{5265.32-5243.88}\right)}-e^{-w}}{1-e^{-w}}; & \nu \geq \left(\frac{ACEC_R-5243.88}{5265.32-5243.88}\right)^2 \\ & t_1 \geq \begin{cases} \beta+\tau_1+\rho_1(\tau_2-\tau_1), \\ \beta+\tau_3-(1-\rho_2)(\tau_3-\tau_2), \end{cases} & \begin{array}{l}\text{in possibility sense} \\ \text{in necessity sense.}\end{array} \\ & \mu \geq \nu \text{ and } \mu+\nu \leq 1; \quad \mu,\ \nu \geq 0. & \end{array}\right\} \quad (B.3)$$

Table B.1: Individual minimum and maximum of objective functions

| Objective functions | Minimize $ACEC_L$ | Minimize $ACEC_C$ | Minimize $ACEC_R$ |
|---|---|---|---|
| $ACEC_L$<br>$ACEC_C$<br>$ACEC_R$ | $ACEC_L^* = 4862.07$<br>$ACEC_C = 5063.69$<br>$ACEC_R = 5265.32$ | $ACEC_L = 4867.14$<br>$ACEC_C^* = 5057.22$<br>$ACEC_R = 5247.29$ | $ACEC_L = 4876.38$<br>$ACEC_C = 5060.13$<br>$ACEC_R^* = 5243.88$ |
| Variables $(P^*, t_1^*)$ | (572.27, 3.23) | (521.85, 3.66) | (491.60, 3.98) |

The solutions obtained for Eq. (B.3) are given in Table B.2.

**Table** B.2: Optimum results of Eq. (B.3) for w=0.10

| $\mu^*$ | $\nu^*$ | $P^*$ | $t_1^*$ | $ACEC_L^*$ | $ACEC_C^*$ | $ACEC_R^*$ |
|---|---|---|---|---|---|---|
| 0.7431 | 0.0612 | 529.67 | 3.59 | 4865.61 | 5057.39 | 5249.18 |

Now we perform the Pareto-Optimal Solution test for strong or weak solutions. The Pareto-Optimal results are presented in Table B.3. In Table B.3, the value of $V^*$ is quite small hence, the optimal results in Table B.3 are strong Pareto-optimal solution and can be accepted.

**Table** B.3: Pareto-Optimal results

| $V^*$ | $P^*$ | $t_1^*$ | $ACEC_L^*$ | $ACEC_C^*$ | $ACEC_R^*$ |
|---|---|---|---|---|---|
| 0.0000 | 529.67 | 3.59 | 4865.61 | 5057.39 | 5249.18 |

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