2014

M.Sc. Part-I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—II

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Algebra*)

[Marks : 50]

Answer *Q. No.* 6 and any *three* from the rest.

1. (a) Define binary tree. Show that the number of internal vertices in a binary tree is one less than the number of pendant vertices. 1+4

(b) Define automorphism. Let G be an abelian group. Show that the mapping $f : G \to G$ define by $f(x) = x^{-1}$ is an automorphism. 1+4

(c) Prove that in a lattice $(L, \leq)$ for any a, b $\in$ L,

$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$ if $a \leq b \leq c$ and

$a \vee b = a \wedge b$. 5

2. (a) Let G be a finite group and Z(G) be the centre of G. Then show that :

$$O(G) = O[Z(G)] + \sum_{a \notin Z(G)} [G : N(a)]$$

where N(a) is the normalizer of a. 5

(b) Define adjacency matrix of a graph. Find the adjacency matrix of the Peterson graph. 3

(c) Let G be a finite group and $O(G) = np^m$, where p is a prime number and p is not a divisor of n. Then show that there exists a subgroup H of G such that $O(H) = p^m$. 7

3. (a) If R is a commutative ring with unity and H is an ideal of R. Then show that R/H is an integral domain iff H is prime. 5

(b) Define dual of a graph. Show that the dual graph of any disconnected graph is connected graph. 1+4

(c) Show that every homomorphic image of a group G is isomorphic to some quotient group of G. 5

4. (a) Show that the mapping $f : C \to M_2(R)$ defined by

$f(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a homomorphism of rings.

Find ker f. 4+1

(b) Let A = {2, 3, 4, 6, 12}. Then show that (A, /) is a poset but not a lattice. 5

(c) Show that the chromatic polynomial of a graph C_n consisting of a single circuit of length n is

$$P_n(\lambda) = (\lambda - 1)^n + (-1)^n (\lambda - 1)^n.$$ 5

5. (a) Show that any circuit free graph with n vertices and (n–1) edges is a tree. 5

(b) Prove that any abelian group of order 15 is cyclic group. 5

(c) Show that in any Euclidean domain D

d(x, y) = d(x) if y is an unit in D

and d(x, y) > d(x) if y is not an unit in D

for any non-zero x, y $\in$ D. 5

6. Answer any *one* question : 5×1

(a) What do you mean by the term shortest path of a weighted graph ? Write an algorithm to find a shortest path from source vertex to destination vertex of a directed weighted graph. 1+4

(b) Let H be a subgroup of a group G. If $x^2 \in H$ for all $x \in G$, then show that H is a normal subgroup of G and G/H is commutative. 5

Group—B

(Functional Analysis)

[Marks : 50]

Answer Q. No. 7 and any *three* from the rest.

7. Answer any *one* : 2×1

(a) Define norm of a bounded linear operator between two normed spaces.

(b) Define compact metric space. Give an example of a metric space which is not compact.

8. (a) Show that the system of linear equations :

$$x_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_1$$
$$x_2 = a_{12}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_n$$

in n unknowns $x_1, x_2, \dots, x_n$ has a unique solution if $\sum_{k=1}^{n} |a_{jk}| < 1$ for all $j = 1, 2, \dots, n$.

(b) Define contraction mapping. Give an example of contraction mapping in an incomplete metric space without having any fixed point. 2+3

(c) Prove that any contraction mapping is uniformly continuous. 3

9. (a) Define Banach space with example.
Show that a normed space X is a Banach space if and only if every absolutely summable series of elements of X is summable in X. 2+8

(b) Let $\|\cdot\|$ be a norm on a linear space X. If $x, y \in X$ and $\|x+y\| = \|x\| + \|y\|$, then show that $\|sx + ty\| = s\|x\| + t\|y\|$ for all $s \geq 0$, $t \geq 0$. 6

10. (a) State Hahn-Banach theorem. Using Hahn-Banach theorem, show that if X is a non-zero normed linear space, then there exists a non-zero element of X^*. 2+8

(b) Define the following with examples — no where Dense set, First Category set, Seperable space. 6

11. (a) Establish polarization identity. 5

(b) If M is a non-empty, closed and convex set in a Hilbert space H, then show that there exists a unique element in M of smallest norm. 7

(c) Let $T \in B(H)$ where H is a Hilbert space. Then show that $\|T\|^2 = \|T^*T\|$.

12. (a) Let (X, d_1) and (Y, d_2) be metric space S and $f : X \to Y$ be a continuous mapping. Then show that f(A) is compact in Y if A is compact in X. Give an open cover for (0, 1). 4+2

(b) Let S be a non-empty subset of a Hilbert space H. Then show that $S^{\perp}$ is a closed linear subspace H. 4

(c) If $T \in B(H)$ is such that $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T = 0$, where H is a complex Hilbert space. 6