Synopsis of the proposal for the research work towards Ph.D.

entitled

A Study on Algebraic Fuzzy Graphs

by

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1 Introduction

Nowadays, uncertainty and impreciseness present in almost all systems. Fuzzy set represents all systems with uncertainty and impreciseness perfectly. Thus fuzzy sets as well as fuzzy graphs are unavoidable research area. Applications of fuzzy graph include data mining, image segmentation, clustering, image capturing, networking, communication, planning, scheduling, etc.

Rosenfeld [17] introduced the concept of fuzzy graphs. After that, several researcher have used this concept in real field problems. McAllister [15] characterised the fuzzy intersection graphs. In this paper, fuzzy intersection graphs have been defined from the concept of the intersection of fuzzy sets. Bhutani and Battou [6] described about strongness of arcs in fuzzy graphs. Mathew and Sunitha [14] defined different types of arcs in fuzzy graphs. Akram and Deduk [1] introduced another important class of fuzzy graphs, namely, interval-valued fuzzy graphs. They also defined bipolar fuzzy graphs [4] as an extension of fuzzy graphs. Goetschel [10] introduced fuzzy hypergraphs as an extension of crisp hypergraphs.

In this thesis, some new area of fuzzy graphs are opened with applications. Fuzzy planar graphs, fuzzy tolerance graphs, fuzzy threshold graphs, fuzzy competition graphs are introduced here. Also, bipolar fuzzy intersection graphs, irregular bipolar fuzzy graphs and bipolar fuzzy hypergraphs are introduced and several properties of bipolar fuzzy graphs have been studied. Finally, some applications of these fuzzy graphs in social and telecommunication networks are described. The definition of fuzzy set is given below.

Definition 1.1 *[30] A fuzzy set A on an universal set X is characterized by a mapping $m : X \to [0,1]$, which is called the membership function. A fuzzy set is denoted by $A = (X, m)$.*

The definition of fuzzy graph is given below.

Definition 1.2 *[17] A fuzzy graph $\xi = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that for all $x, y \in V$, $\mu(x,y) \leq \min\{\sigma(x), \sigma(y)\}$, where $\sigma(x)$ and $\mu(x,y)$ represent the membership values of the vertex x and of the edge (x,y) in ξ respectively. A loop at a vertex x in a fuzzy graph is represented by $\mu(x,x) \neq 0$. An edge is non-trivial if $\mu(x,y) \neq 0$.*

Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. An edge (x,y) is called *strong* [8] if $\frac{1}{2}\min\{\sigma(x), \sigma(y)\} \leq \mu(x,y)$ and weak otherwise. The strength of the fuzzy edge (x,y) is represented by the value $\frac{\mu(x,y)}{\min\{\sigma(x),\sigma(y)\}}$. A fuzzy graph $\xi = (V, \sigma, \mu)$ is *complete* [16] if $\mu(u,v) = \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$, where (u,v) denotes the edge between the vertices u and v. A fuzzy graph $\xi = (V, \sigma, \mu)$

is said to be *bipartite* [16] if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further, if $\mu(v_1, v_2) = \min\{\sigma(v_1), \sigma(v_2)\}$ for all $v_1 \in V_1$ and $v_2 \in V_2$, then ξ is called a complete bipartite fuzzy graph.

The *underlying crisp graph* of the fuzzy graph $\xi = (V, \sigma, \mu)$ is denoted as $\xi^* = (V, \sigma^*, \mu^*)$ where $\sigma^* = \{u \in V | \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V | \mu(u, v) > 0\}$. Given $t \in [0, 1]$ and a fuzzy set A, we define the t-cut level set [16] of A to be the crisp set $A^t = \{x \in supp\ A \mid m(x) \geq t\}$. The t-cut level graph of ξ is the crisp graph $\xi^t = (\sigma^t, \mu^t)$.

The definition of fuzzy intersection graph is given below.

Definition 1.3 *[15] Let* $\mathcal{F}= \{A_1 = (X, m_1), A_2 = (X, m_2) \ldots, A_n = (X, m_n)\}$ *be a finite family of fuzzy sets defined on a set* X *and consider* $\mathcal{F}$ *as crisp vertex set* $V = \{v_1, v_2 \ldots, v_n\}$*. The fuzzy intersection graph of* $\mathcal{F}$ *is the fuzzy graph* $Int(\mathcal{F})= (V, \sigma, \mu)$ *where* $\sigma : V \to [0, 1]$ *is defined by* $\sigma(v_i) = h(A_i)$ *and* $\mu : V \times V \to [0, 1]$ *is defined by*

$$\mu(v_i, v_j) = \begin{cases} h(A_i \cap A_j), & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$$

Here $\mathcal{E}= \{(v_i, v_j) \mid \mu(v_i, v_j) > 0\}$ is called the edge set of the fuzzy graph. An edge (v_i, v_j) has a zero strength iff $m_i \cap m_j$ is zero function (empty intersection). A fuzzy interval graph is the fuzzy intersection graph of a finite family of fuzzy intervals.

The fuzzy set measures only the membership degree of elements. Here, bipolar fuzzy set measures both positive membership degree and negative membership degree. The definition of bipolar fuzzy set is given below.

Definition 1.4 *[32] Let* X *be nonempty set. A bipolar fuzzy set* B *on* X *is an object having the form* $B = \{(x, \mu^+(x), \mu^-(x)) | x \in X\}$*, where* $\mu^+ : X \to [0, 1]$ *and* $\mu^- : X \to [-1, 0]$ *are mappings.*

Motivated from the idea of bipolar fuzzy set, bipolar fuzzy graph is defined below.

Definition 1.5 *[1] A bipolar fuzzy graph with an underlying set* V *is defined to be the pair* $G = (A, B)$ *where* $A = (m_A^+, m_A^-)$ *is a bipolar fuzzy set on* V *and* $B = (m_B^+, m_B^-)$ *is a bipolar fuzzy set on* $E \subseteq V \times V$ *such that* $m_B^+(x, y) \leq min\{m_A^+(x), m_A^+(y)\}$ *and* $m_B^-(x, y) \geq max\{m_A^-(x), m_A^-(y)\}$ *for all* $(x, y) \in E$*. Here* A *is called bipolar fuzzy vertex set of* V*,* B *the bipolar fuzzy edge set of* E*.*

Let X be a finite set and let $\mathcal{E}$ be a finite family of nontrivial fuzzy sets on X (or subsets of X) such that $X = \bigcup\{suppA | A \in \mathcal{E}\}$. Then the pair $\mathcal{H}= (X, \mathcal{E})$ is a *fuzzy hypergraph* [16] on X.

A (crisp) *multiset* over a non-empty set V is simply a mapping $d : V \to N$, where N is the set of natural numbers. An element of non-empty set V may occur more than once with possibly the same or different membership values. A natural generalization of this interpretation of multiset leads to the notion of *fuzzy multiset* [29], or fuzzy bag, over a non-empty set V as a mapping $\tilde{C} : V \times [0,1] \to N$. The membership values of $v \in V$ are denoted as $v_{\mu^j}, j = 1, 2, \ldots, p$ where $p = \max\{j : v_{\mu^j} \neq 0\}$. So the fuzzy multiset can be denoted as $M = \{(v, v_{\mu^j}), j = 1, 2, \ldots, p | v \in V\}$.

2 Organization of the Thesis

The thesis is organized into eight chapters. Brief contents of all chapters are given below.

Chapter 1

Introduction

Chapter 1 presents the introduction of the thesis. In this chapter, some definitions which are used in different chapters of the thesis, are given. The definition of fuzzy set, bipolar fuzzy set, fuzzy graph, bipolar fuzzy graph, fuzzy hypergraph, etc. are given. This chapter presents a review of the literature and also the motivation of the work.

Chapter 2

Fuzzy Planar Graphs

In this chapter, fuzzy multi graph is introduced. Also fuzzy planar graph and fuzzy dual graph are defined. Some properties of the fuzzy planar graph are established.

The definition of fuzzy multi graph is given as follows. Let V be a non-empty set and $\sigma : V \to [0,1]$ be a mapping. Also let $E = \{((x,y),(x,y)_{\mu^j}), j = 1, 2, \ldots, p_{xy} | (x,y) \in V \times V\}$ be a fuzzy multiset of $V \times V$ such that $(x,y)_{\mu^j} \leq \min\{\sigma(x), \sigma(y)\}$ for all $j = 1, 2, \ldots, p_{xy}$, where $p_{xy} = \max\{j | (x,y)_{\mu^j} \neq 0\}$. Then $\psi = (V, \sigma, E)$ is denoted as *fuzzy multigraph* where $\sigma(x)$ and $(x,y)_{\mu^j}$ represent the membership value of the vertex x and the membership value of the edge (x,y) in ψ respectively.

As strength of the fuzzy edge (a,b) can be measured by the value $I_{(a,b)} = \frac{(a,b)_{\mu^k}}{\min\{\sigma(a),\sigma(b)\}}$. If $I_{(a,b)} \geq 0.5$, then the fuzzy edge is called strong otherwise weak.

In crisp planar graph, no edge crosses each other. But, if an edge crosses other edge, then we calculate intersecting value at the crossing point. Suppose two edges (a,b) and (c,d) in a fuzzy graph are crossed at the point P. The intersecting value at the point P is defined by $\mathcal{I}_P = \frac{I_{(a,b)}+I_{(c,d)}}{2}$. If the number of points of intersection in a fuzzy multigraph increases, planarity decreases. So for fuzzy multigraph, $\mathcal{I}_P$ is inversely proportional to the planarity. Based on this concept, a new terminology is introduced below for a fuzzy planar graph.

Definition 2.1 *[23] Let ψ be a fuzzy multigraph and for a certain geometrical representation $P_1, P_2, \ldots, P_z$ be the points of intersections between the edges. ψ is said to be fuzzy planar graph with fuzzy planarity value f, where*

$$f = \frac{1}{1 + \{\mathcal{I}_{P_1} + \mathcal{I}_{P_2} + \ldots + \mathcal{I}_{P_z}\}}.$$

For a fuzzy complete multigraph ψ, the fuzzy planarity value f of ψ is given by $f = \frac{1}{1+N_p}$, where N_p is the number of point of intersections between the edges in ψ.

The definition of strong fuzzy planar graph is mentioned below.

Definition 2.2 *A fuzzy planar graph ψ is called strong fuzzy planar graph if the fuzzy planarity value of the graph is greater than* 0.5.

Theorem 2.1 *The number of point of intersection can be stated from the following statement. Let ψ be a strong fuzzy planar graph. The number of point of intersections between strong edges in ψ is at most one.*

Proof. Let $\psi = (V, \sigma, E)$ be a strong fuzzy planar graph. Let, if possible, ψ has at least two point of intersections P_1 and P_2 between two strong edges in ψ.

For any strong edge $((a,b),(a,b)_{\mu^j})$, $(a,b)_{\mu^j} \geq \frac{1}{2}\min\{\sigma(a),\sigma(b)\}$. So $I_{(a,b)} \geq 0.5$.

Thus for two intersecting strong edges $((a,b),(a,b)_{\mu^j})$ and $((c,d),(c,d)_{\mu^i})$, $\frac{I_{(a,b)}+I_{(c,d)}}{2} \geq 0.5$, that is, $\mathcal{I}_{P_1} \geq 0.5$. Similarly, $\mathcal{I}_{P_2} \geq 0.5$. Then $1+\mathcal{I}_{P_1}+\mathcal{I}_{P_2} \geq 2$. Therefore, $f = \frac{1}{1+\mathcal{I}_{P_1}+\mathcal{I}_{P_2}} \leq 0.5$. It contradicts the fact that the fuzzy graph is a strong fuzzy planar graph.

So number of point of intersections between strong edges can not be two. It is clear that if the number of point of intersections of strong fuzzy edges increases, the fuzzy planarity value decreases. Similarly, if the number of point of intersection of strong edges is one, then the fuzzy planarity value $f > 0.5$. Any fuzzy planar graph without any crossing between edges is a strong fuzzy planar graph. Thus, we conclude that the maximum number of point of intersections between the strong edges in ψ is one. $\square$

Considerable edges are another kind of edges which is given below.

Definition 2.3 *Let* (V, σ, μ) *be a fuzzy graph and* $0 < c < 0.5$ *be a rational number. An edge* (x, y) *is said to considerable edge if* $\frac{\mu(x,y)}{\min\{\sigma(x),\sigma(y)\}} \geq c$*. If an edge is not considerable, it is called non-considerable edge. For fuzzy multigraph* (V, σ, E)*, a multi-edge* (x, y) *is said to be considerable edge if* $\frac{(x,y)_{\mu^j}}{\min\{\sigma(x),\sigma(y)\}} \geq c$ *for all* $j = 1, 2, \ldots, p_{xy}$*. The following theorem determines the upper bound of number of point of intersections between considerable edges. Let* ψ *be a strong fuzzy planar graph with considerable number c. The number of point of intersections between considerable edges in* ψ *is at most* $[\frac{1}{c}]$ *or* $\frac{1}{c} - 1$ *according as* $\frac{1}{c}$ *is not an integer or an integer respectively (here* $[x]$ *is greatest integer not exceeding* x*).*

Like a crisp planar graph, the fuzzy face can be defined in fuzzy planar graph. Let $\psi = (V, \sigma, E)$ be a fuzzy planar graph and

Definition 2.4 $E = \{((x, y), (x, y)_{\mu^j}), j = 1, 2, \ldots, p_{xy} |\, (x, y) \in V \times V\}$ *and* $p_{xy} = \max\{j |\, (x, y)_{\mu^j} \neq 0\}$*. A fuzzy face of* ψ *is a region, bounded by the set of fuzzy edges* $E' \subset E$*, of a geometric representation of* ψ*. The membership value of the fuzzy face is*

$$\min\left\{\frac{(x, y)_{\mu^j}}{\min\{\sigma(x), \sigma(y)\}}, j = 1, 2, \ldots, p_{xy} |\, (x, y) \in E'\right\}.$$

Theorem 2.2 *Let* ψ *be a fuzzy planar graph with fuzzy planarity value* f*. If* $f \geq 0.67$*,* ψ *does not contain any point of intersection between two strong edges.*

Motivated from this theorem, we introduce a special type of fuzzy planar graph called 0.67-fuzzy planar graph whose fuzzy planarity value is more than or equal to 0.67. This graph is important as it does not contain any crossing between strong edge.

Definition 2.5 *The fuzzy dual graph of 0.67-fuzzy planar graph is defined as follows. Let* $\psi = (V, \sigma, E)$ *be a* 0.67*-fuzzy planar graph and* $E = \{((x, y), (x, y)_{\mu^j}), j = 1, 2, \ldots, p_{xy} |\, (x, y) \in V \times V\}$*. Again, let* $F_1, F_2, \ldots, F_k$ *be the strong fuzzy faces of* ψ*. The fuzzy dual graph of* ψ *is a fuzzy planar graph* $\psi' = (V', \sigma', E')$*, where* $V' = \{x_i, i = 1, 2, \ldots, k\}$*, and the vertex* x_i *of* ψ' *is considered for the face* F_i *of* ψ*.*

The membership values of vertices are given by the mapping $\sigma' : V' \to [0, 1]$ *such that* $\sigma'(x_i) = \max\{(u, v)_{\mu^j}, j = 1, 2, \ldots, p_{uv} | (u, v)$ *is an edge of the boundary of the strong fuzzy face* $F_i\}$*.*

Between two faces F_i *and* F_j *of* ψ*, there may exist more than one common edge. Thus between two vertices* x_i *and* x_j *in fuzzy dual graph* ψ'*, there may be more than one edge. We denote* $(x_i, x_j)_{\nu^l}$ *be the membership value of the l-th edge between* x_i *and* x_j*. The membership values of the fuzzy edges of the fuzzy dual graph are given by* $(x_i, x_j)_{\nu^l} = (u, v)^l_{\mu^j}$ *where* $(u, v)^l$ *is* *an edge in the boundary between two strong fuzzy faces* F_i *and* F_j *and* $l = 1, 2, \ldots, s$*, where* s *is the number of common edges in the boundary between* F_i *and* F_j *or the number of edges between* x_i *and* x_j*.*

If there be any strong pendant edge in the 0.67*-fuzzy planar graph, then there will be a self loop in* ψ' *corresponding to this pendant edge. The edge membership value of the self loop is equal to the membership value of the pendant edge.*

Theorem 2.3 *Let* ψ *be a* 0.67*-fuzzy planar graph whose number of vertices, number of fuzzy edges and number of strong faces are denoted by* n*,* p*,* m *respectively. Also let* ψ' *be the fuzzy dual graph of* ψ*. Then*
(i) the number of vertices of ψ' *is equal to* m*,*
(ii) number of edges of ψ' *is equal to* p*,*
(iii) number of fuzzy faces of ψ' *is equal to* n*.*

Theorem 2.4 *Let* $\psi = (V, \sigma, E)$ *be a* 0.67*-fuzzy planar graph without weak edges and the fuzzy dual graph of* ψ *be* $\psi' = (V', \sigma', E')$*. The membership values of fuzzy edges of* ψ' *are equal to membership values of the fuzzy edges of* ψ*.*

In this chapter, application of fuzzy planar graph is described to fuzzy image contraction.

Chapter 3

Fuzzy Tolerance Graphs

In this chapter, fuzzy tolerance graph is introduced and some properties of it are proved. First of all, fuzzy tolerance is defined below, which is required to define fuzzy tolerance graph.

Definition 3.1 *Fuzzy tolerance of a fuzzy interval is denoted by* $\mathcal{T}$ *and defined by an arbitrary fuzzy interval whose core length is a positive real number. If the real number is taken* L *and* $|i_k - i_{k-1}| = L$ *where* $i_{k-1}, i_k \in R$ *then fuzzy tolerance is a fuzzy set of the interval* $[i_{k-1}, i_k]$*.*

The definition of *fuzzy tolerance graph* [19] is given below.

Definition 3.2 *Let* $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\}$ *be a finite family of fuzzy intervals on the real line and* $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ *be the corresponding fuzzy tolerances. We consider* $\mathcal{I}$ *as a crisp vertex set* $V = \{v_1, v_2, \ldots, v_n\}$*. The fuzzy tolerance graph is the fuzzy graph* (V, σ, μ) *where* $\sigma : V \to [0, 1]$ *is defined by* $\sigma(v_i) = h(\mathcal{I}_i) = 1$ *for all* $v_i \in V$ *and* $\mu : V \times V \to [0, 1]$ *is defined by*

$$\mu(v_i, v_j) = \begin{cases} 1, & \textit{if } c(\mathcal{I}_i \cap \mathcal{I}_j) \geq \; min\{c(\mathcal{T}_i), c(\mathcal{T}_j)\} \\ \frac{s(\mathcal{I}_i \cap \mathcal{I}_j) - min\{s(\mathcal{T}_i), s(\mathcal{T}_j)\}}{s(\mathcal{I}_i \cap \mathcal{I}_j)} h(\mathcal{I}_i \cap \mathcal{I}_j), & \textit{otherwise if } s(\mathcal{I}_i \cap \mathcal{I}_j) \geq min\{s(\mathcal{T}_i), s(\mathcal{T}_j)\} \\ 0, & \textit{otherwise.} \end{cases}$$

where $c(\mathcal{I}_i \cap \mathcal{I}_j)$ *is the core of the intersection of the intervals* $\mathcal{I}_i$ *and* $\mathcal{I}_j$*.* $min\{c(\mathcal{T}_i), c(\mathcal{T}_j)\}$ *is the minimum of the cores of the corresponding tolerances* $\mathcal{T}_i$ *and* $\mathcal{T}_j$*. Also* $s(\mathcal{I}_i \cap \mathcal{I}_j)$, $min\{s(\mathcal{T}_i), s(\mathcal{T}_j)\}$ *are the support of the intersection of intervals* $\mathcal{I}_i$, $\mathcal{I}_j$ *and minimum support the of tolerances* $\mathcal{T}_i$, $\mathcal{T}_j$ *respectively.*

Fuzzy bounded tolerance graph has been defined as follows.

Definition 3.3 *Let* $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\}$ *be a finite family of fuzzy intervals on real line and* $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ *be the corresponding fuzzy tolerances. If fuzzy interval* $\mathcal{I}_i$ *with core* $c(\mathcal{I}_i)$ *and support* $s(\mathcal{I}_i)$ *has tolerance with core* $c(\mathcal{T}_i)$ *and support* $s(\mathcal{T}_i)$ *such that* $c(\mathcal{I}_i) \geq c(\mathcal{T}_i)$ *and* $s(\mathcal{I}_i) \geq s(\mathcal{T}_i)$ *of a fuzzy tolerance graph then the corresponding representation is called fuzzy bounded tolerance representation and the fuzzy graph is called fuzzy bounded tolerance graph.*

The following result is about the core and support of fuzzy interval graph.

Theorem 3.1 *If* ξ *is a fuzzy interval graph, then* ξ *is a fuzzy tolerance graph with constant core and constant support of tolerance.*

Proof. Let ξ be a fuzzy interval graph with a representation $\mathcal{I}_v$ is assigned to the vertex v. Let the cores of fuzzy intervals $\mathcal{I}_x$ and $\mathcal{I}_y$ be denoted by $c(\mathcal{I}_x)$ and $c(\mathcal{I}_y)$ and that of supports be $s(\mathcal{I}_x)$ and $s(\mathcal{I}_y)$. Also $c(\mathcal{I}_x \cap \mathcal{I}_y) = c(\mathcal{I}_x) \wedge c(\mathcal{I}_y)$ and $s(\mathcal{I}_x \cap \mathcal{I}_y) = s(\mathcal{I}_x) \wedge s(\mathcal{I}_y)$. Let k_1 and k_2 be positive real numbers such that $k_1 < |c(\mathcal{I}_x \cap \mathcal{I}_y)|$ and $k_2 < |s(\mathcal{I}_x \cap \mathcal{I}_y)|$ for all $x, y \in V$ with $k_1 \leq k_2$.

Thus the intervals $\{\mathcal{I}_v \mid v \in V\}$ together with tolerances with core k_1 and support k_2 give a fuzzy tolerance representation. □

The following theorem is proved in the thesis.

Theorem 3.2 *If* ξ *is a fuzzy tolerance graph with constant core and constant support of tolerances then* ξ *is fuzzy bounded tolerance graph.*

Let ξ be a fuzzy tolerance graph. For each $t \in [0, 1]$, the cut level graphs are
(i) tolerance graph,
(ii) weakly chordal,

(iii) alternately orientable,

(iv) perfect graph.

Definition 3.4 *Let* $\mathcal{I}= \{\mathcal{I}_1, \mathcal{I}_2, \dots \mathcal{I}_n\}$ *be a finite family of fuzzy intervals on real line. We consider the crisp vertex set* $V = \{v_1, v_2, \dots, v_n\}$*. The fuzzy interval containment graph is the fuzzy graph* (V, σ, μ) *where* $\sigma : V \to [0,1]$ *is defined by* $\sigma(v_i) = h(\mathcal{I}_i) = 1$ *for all* $i = 1, 2, ..., n$ *and* $\mu : V \times V \to [0,1]$ *is defined by*

$$\mu(v_i, v_j) = \begin{cases} 1, \textit{if core and support of one of } \mathcal{I}_i,\ \mathcal{I}_j \textit{ contain the other} \\ \frac{1}{2}\left(\frac{c(\mathcal{I}_i \cap \mathcal{I}_j)}{min\{c(\mathcal{I}_i), C(\mathcal{I}_j)\}} + \frac{s(\mathcal{I}_i \cap \mathcal{I}_j)}{min\{s(\mathcal{I}_i), S(\mathcal{I}_j)\}}\right) h(\mathcal{I}_i \cap \mathcal{I}_j), \textit{otherwise.} \end{cases}$$

where $c(\mathcal{I}_i \cap \mathcal{I}_j)$*,* $s(\mathcal{I}_i \cap \mathcal{I}_j)$ *are the core and support of the intersection of the intervals* $\mathcal{I}_i$*,* $\mathcal{I}_j$ *and* $min\{c(\mathcal{I}_i), c(\mathcal{I}_j)\}$*,* $min\{s(\mathcal{I}_i), s(\mathcal{I}_j)\}$ *are the minimum of core and support of the intervals* $\mathcal{I}_i$ *and* $\mathcal{I}_j$*.*

If ξ is a fuzzy interval containment graph with edge membership value 1 or 0 then ξ has a fuzzy tolerance representation with core of an interval equals to core of corresponding tolerance and support of the interval equals to support of the corresponding tolerance.

A fuzzy tolerance graph is said to have regular representation if it satisfies the following three properties

(1) Any fuzzy tolerance core or support lengths larger than the core and support lengths of the corresponding fuzzy interval can be set to infinity without changing the adjacencies of vertices in the fuzzy tolerance graph.

(2) All core and support lengths of fuzzy tolerances are distinct.

(3) No two different fuzzy interval cores and supports share an end point.

Theorem 3.3 *Every fuzzy tolerance graph has a regular representation.*

A fuzzy unit tolerance graph is a fuzzy tolerance graph that has tolerance representation in which all fuzzy interval core lengths are same and the support lengths are same. A fuzzy proper tolerance graph is one that has tolerance representation in which no fuzzy interval core and support properly contain another fuzzy interval core and support.

Remark: Any fuzzy unit or proper tolerance representation may be assumed to have fuzzy bounded tolerances.

Chapter 4

Fuzzy Threshold Graphs

In this chapter, we define fuzzy threshold graphs, fuzzy alternating 4-cycles, threshold dimension of fuzzy graphs, fuzzy Ferrers diagraph. Also some basic theorems related to the stated graphs have been presented.

The fuzzy threshold graph is defined below.

Definition 4.1 *A fuzzy graph $\xi = (V, \sigma, \mu)$ is called a fuzzy threshold graph [20] if there exists non negative real number T such that $\sum_{u \in U} \sigma(u) \leq T$ if and only if $U \subset V$ is stable set in ξ.*

Definition 4.2 *Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{a, b, c, d\}$. Also let $\sigma(a), \sigma(b), \sigma(c)$, $\sigma(d), \mu(a, b), \mu(c, d)$ be positive and $\mu(a, c) = \mu(b, d) = 0$. This configuration of four vertices is called fuzzy alternating 4-cycle. This fuzzy alternating 4-cycle induces a path $\mathcal{P}_4$ (when one of $\mu(a, d), \mu(b, c)$ is zero and other is non-zero), a square $\mathcal{C}_4$ (when both $\mu(a, d), \mu(b, c)$ are non-zero) or a matching $2\mathcal{K}_2$ (when both $\mu(a, d), \mu(b, c)$ are zero).*

Definition 4.3 *A strong alternating 4-cycle is an alternating 4-cycle if fuzzy $\mathcal{C}_4$ be induced from it.*

Degree partition is very important term in graph theory. Here we define degree partition in fuzzy graph.

Definition 4.4 *Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph whose distinct positive vertex degrees are $\lambda_1 < \lambda_2 < \ldots < \lambda_p$, and let $\lambda_0 = 0$ (even if no isolated vertex exists), $\lambda_{p+1} = |V| - \lambda_1$. Let $\mathcal{D}_i = \{v \in V : i \leq d(v) < i + 1\}$ for non negative integer $i \leq p$. The sequence $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_p$ is called degree partition of the fuzzy graph ξ.*

In the thesis, we have proved that, a fuzzy threshold graph does not have a strong fuzzy alternating 4-cycle and a fuzzy threshold graph is a fuzzy split graph.

Also, we have shown that if ξ is a fuzzy threshold graph, then ξ can be constructed from the one vertex graph by repeatedly adding an fuzzy isolated vertex or a fuzzy dominating vertex.

Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. Two vertices x and y are said to be strong comparable if there exists a path from x to y or y to x whose every arc is strong arc.

Definition 4.5 *The threshold dimension* $\tilde{t}(\xi)$ *of a fuzzy graph* $\xi = (V, \sigma, \mu)$ *is the minimum number* k *of fuzzy threshold subgraphs* $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ *of* ξ *that cover the edge set of* ξ *that is if* $\xi' = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \ldots \cup \mathcal{T}_k$ *then* $\xi' = (V, \sigma, \mu)$.

Theorem 4.1 *For every fuzzy graph* $\xi = (V, \sigma, \mu)$ *on* n *vertices we have* $\tilde{t}(\xi) \leq (n - \alpha(\xi))$. *Furthermore, if* ξ *is triangle-free, then* $\tilde{t}(\xi) = (n - |supp(\mathcal{S})|)$ *where* $\mathcal{S}$ *is the stable set with largest number of vertices.*

A fuzzy threshold graph can be partitioned into some threshold subgraphs. The fuzzy threshold partition number $\tilde{tp}(\xi)$ of the fuzzy graph ξ is the minimum number of fuzzy threshold subgraphs, not containing common strong arcs, cover edge set of ξ.

Theorem 4.2 *If* ξ *is a fuzzy triangle free graph, then* $\tilde{t}(\xi) = \tilde{tp}(\xi)$.

Proof. Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. We know that the edge set can be covered by $\tilde{t}(\xi)$ number of stars. If a strong arc belongs to more than one stars then we delete it from all but remain one of the stars. This gives a fuzzy threshold partition of size $\tilde{t}(\xi)$.

Definition 4.6 *A fuzzy diagraph* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *is said to be a fuzzy Ferrers diagraph if it does not contain vertices* x, y, z, w, *not necessarily distinct, satisfying* $\overrightarrow{\mu}(x, y), \overrightarrow{\mu}(z, w)$ *are non zero and* $\overrightarrow{\mu}(x, w), \overrightarrow{\mu}(z, y)$ *are zero.*

Definition 4.7 *Let* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *be a fuzzy diagraph. The underlying fuzzy graph of* $\overrightarrow{\xi}$ *is denoted by* $\mathcal{U}(\xi)$ *and is defined as* $\mathcal{U}(\xi) = (V, \sigma, \mu)$ *where* $\mu(u, v) = min\{\overrightarrow{\mu}(u, v), \overrightarrow{\mu}(v, u)\}$ *for all* $u, v \in V$.

Theorem 4.3 *If* $\overrightarrow{\xi}$ *is a symmetric fuzzy Ferrers diagraph, then its underlying fuzzy undirected loop less fuzzy graph* $\mathcal{U}(\xi)$ *may or may not be fuzzy threshold graph.*

Proof. If $\overrightarrow{\xi}$ is a symmetric fuzzy Ferrers diagraph, then its underlying fuzzy undirected loop less fuzzy graph $\mathcal{U}(\xi)$ has no fuzzy alternating 4 cycle. So it may contain fuzzy cycle. Hence $\mathcal{U}(\xi)$ may be fuzzy threshold graph. If $\mathcal{U}(\xi)$ does not have strong fuzzy alternating 4- cycle then $\mathcal{U}(\xi)$ must be fuzzy threshold graph using the result of previous theorem. □

Chapter 5

Bipolar Fuzzy Graphs

This chapter is divided into three sections. Some concepts of bipolar fuzzy set, cut level set of bipolar fuzzy set, truncations of bipolar fuzzy set are given here. In the first section, bipolar fuzzy hypergraph is defined and investigated its properties. The next section described about irregular bipolar fuzzy graphs. Bipolar fuzzy intersection graph is described in the last section.

Some basic concepts of bipolar fuzzy sets are discussed below.

Height [24] of a bipolar fuzzy set $B = \{(x, m^+(x), m^-(x))|x \in X\}$ on a nonempty set X is denoted by $h(B)$ and defined as $h(B) = max\{m^+(x)|x \in X\}$.

Depth [24] of a bipolar fuzzy set $B = \{(x, m^+(x), m^-(x))|x \in X\}$ on a nonempty set X is denoted by $d(B)$ and defined as $d(B) = min\{m^-(x)|x \in X\}$.

Let $B = \{(x, m^+(x), m^-(x))|x \in X\}$ be a bipolar fuzzy set on a nonempty set X. The support of B is denoted by $supp(B)$ and defined by $supp(B) = \{x|m^+(x) \neq 0 \text{ or } m^-(x) \neq 0\}$. The upper core of B is denoted by $\bar{c}(B)$ and defined by $\bar{c}(B) = \{x|m^+(x) = 1\}$. Similarly, the lower core of B is denoted by $\underline{c}(B)$ and defined by $\underline{c}(B) = \{x|m^-(x) = -1\}$.

Definition 5.1 *Let $t_1 \in (0, 1]$, $t_2 \in [-1, 0)$ and $B = (m^+, m^-)$ be a bipolar fuzzy set. We define $\{t_1, t_2\}$ cut level set of B to be the crisp set $B_{t_2}^{t_1} = \{x \in supp(B)|m^+(x) \geq t_1 \text{ and } m^-(x) \leq t_2\}$.*

Theorem 5.1 (*a*) *For any bipolar fuzzy sets A and B, $A \subseteq B$ if and only if $A_k^t \subseteq B_k^t$ for all $t \in (0, 1]$, $k \in [-1, 0)$.*

(*b*) *For any bipolar fuzzy sets A and B, $A = B$ if and only if $A_k^t = B_k^t$ for all $t \in (0, 1]$, $k \in [-1, 0)$.*

Proof. Let B_1 and B_2 be two bipolar fuzzy sets. Then $(B_1 \cap B_2)_{t_2}^{t_1} = (B_1)_{t_2}^{t_1} \cap (B_2)_{t_2}^{t_1}$. This result is proved as follows. Let $B_1 = (m_1^+, m_1^-)$ and $B_2 = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on X. Then $B_1 \cap B_2$ is a bipolar fuzzy set on X. Let $B_1 \cap B_2 = (m^+, m^-)$ where $m^+(x) = min\{m_1^+(x), m_2^+(x)\}$ and $m^-(x) = max\{m_1^-(x), m_2^-(x)\}$ for all $x \in X$.

Let y be an element of $(B_1 \cap B_2)_{t_2}^{t_1}$. So $m^+(y) \geq t_1$ and $m^-(y) \leq t_2$. This implies that $min\{m_1^+(y), m_2^+(y)\} \geq t_1$ and $max\{m_1^-(y), m_2^-(y)\} \leq t_2$. Clearly $m_1^+(y) \geq t_1$ and $m_2^+(y) \geq t_1$. Similarly, $m_1^-(y) \leq t_2$ and $m_2^-(y) \leq t_2$. Thus $m_1^+(y) \geq t_1$ and $m_1^-(y) \leq t_2$. So $y \in (B_1)_{t_2}^{t_1}$. Similarly, it can be stated that $y \in (B_2)_{t_2}^{t_1}$. So $y \in (B_1)_{t_2}^{t_1} \cap (B_2)_{t_2}^{t_1}$. Hence by the concept of set theory $(B_1 \cap B_2)_{t_2}^{t_1} \subset (B_1)_{t_2}^{t_1} \cap (B_2)_{t_2}^{t_1}$.

Conversely, it can be shown that $(B_1)_{t_2}^{t_1} \cap (B_2)_{t_2}^{t_1} \subset (B_1 \cap B_2)_{t_2}^{t_1}$. Finally we have the result $(B_1 \cap B_2)_{t_2}^{t_1} = (B_1)_{t_2}^{t_1} \cap (B_2)_{t_2}^{t_1}$. □

Theorem 5.2 *Let B_1 and B_2 be two bipolar fuzzy sets. Then $(B_1 \cup B_2)_{t_2}^{t_1} = (B_1)_{t_2}^{t_1} \cup (B_2)_{t_2}^{t_1}$ for all elements of $S \subseteq X$ if $t_1 \leq min\{m_1^+(x), m_2^+(x)\}$ and $t_2 \geq max\{m_1^-(x), m_2^-(x)\}$ for all $x \in S$.*

Definition 5.2 *Let $B = (m^+, m^-)$ be a bipolar fuzzy set on X. The middle truncation of B is the bipolar fuzzy subset $B_m = (m_m^+, m_m^-)$ on X where*

$$m_m^+(x) = \begin{cases} m^+(x), & \text{if } x \in B_{t_2}^{t_1} \\ 0, & \text{if } x \notin B_{t_2}^{t_1} \end{cases}$$

and

$$m_m^-(x) = \begin{cases} m^-(x), & \text{if } x \in B_{t_2}^{t_1} \\ 0, & \text{if } x \notin B_{t_2}^{t_1} \end{cases}$$

Definition 5.3 *Let $B = (m^+, m^-)$ be a bipolar fuzzy set on X. The side truncation of B is the bipolar fuzzy subset $B_s = (m_s^+, m_s^-)$ on X where*

$$m_s^+(x) = \begin{cases} t_1, & \text{if } x \in B_{t_2}^{t_1} \\ m^+(x), & \text{if } x \notin B_{t_2}^{t_1} \end{cases}$$

and

$$m_s^-(x) = \begin{cases} t_2, & \text{if } x \in B_{t_2}^{t_1} \\ m^-(x), & \text{if } x \notin B_{t_2}^{t_1} \end{cases}$$

5.1 Bipolar Fuzzy Hypergraphs

In this section, bipolar fuzzy hypergraph is defined. Also, several properties of it are studied.

Definition 5.4 *[22] Let X be a finite set and let ξ be a finite family of bipolar fuzzy subsets B on X (or subsets of X) such that $X = \bigcup_{B \in \xi} supp(B)$. Then $H = (X, \xi)$ is called a bipolar fuzzy hypergraph (on X) and ξ is called edge set of H, which are bipolar fuzzy sets on subsets of X.*

Definition 5.5 *A bipolar fuzzy hypergraph $H = (X, \xi)$ is support simple if whenever $A, B \in \xi$ and $A \subseteq B$ and $supp(A) = supp(B)$, then $A = B$.*

Definition 5.6 *Let $H_1 = (X_1, \xi_1)$ and $H_2 = (X_2, \xi_2)$ be two bipolar fuzzy hypergraphs. H_1 is called partial bipolar fuzzy hypergraph of H_2 if $\xi_1 \subseteq \xi_2$.*

Definition 5.7 *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a non-empty finite set and $B = \{B_1, B_2, \ldots, B_k\}$ be bipolar sets of subsets of X. (α, β)- cut of bipolar fuzzy hypergraph, $H = (X, B)$, denoted by $H_{(\alpha,\beta)}$, is an ordered pair $H_{(\alpha,\beta)} = (X_{(\alpha,\beta)}, \xi_{(\alpha,\beta)})$ where:*

(1) $X_{(\alpha,\beta)} = X$

(2) $\xi_{(\alpha,\beta)}) = \{B_{j,(\alpha,\beta)} | B_{j,(\alpha,\beta)} = \{x_i \in B_j | \mu^+(x_i) \geq \alpha, \mu^-(x_i) \leq \beta\}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, k\}$

(3) $B_{k+1,(\alpha,\beta)} = \{x_i \notin B_j, i = 1, 2, \ldots, n, j = 1, 2, ..., k\}$

Definition 5.8 *Let $H = (X, \xi)$ be a bipolar fuzzy hypergraph, and for $0 < \alpha \leq h(H), d(H) \leq \beta < 0$, let $h_{(\alpha,\beta)} = (X_{(\alpha,\beta)}, \xi_{(\alpha,\beta)})$ be the (α, β)-level hypergraph of H. The sequence of real numbers $\{s_k, s_{k-1}, \ldots, s_1, r_1, r_2, \ldots, r_n\}$ such that $d(H) = s_k < s_{k-1} < \ldots \leq s_1 < 0 < r_1 < r_2 < \ldots < r_n = h(H)$ which satisfies the following properties*

(1) If $s_{i+1} \leq l < s_i, r_i < k \leq r_{i+1}$, then $B_{(k,l)} = B_{(r_{i+1},s_{i+1})}$,

(2) $B_{(r_{i+1},s_{i+1})} \nsubseteq B_{(r_i,s_i)}$,

is called the fundamental sequence of H, and is denoted by $F(H)$.

Definition 5.9 *Let $H = (X, B)$ be a bipolar fuzzy hypergraph where $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $B = \{B_1, B_2, \ldots, B_n\}$ be a bipolar fuzzy sets on subsets of X. The bipolar fuzzy hypergraph $\bar{H} = (\bar{B}, \bar{X})$ is called dual bipolar fuzzy hypergraph of H if*

(1) $\bar{B} = \{b_1, b_2, \ldots, b_n\}$ is set of vertices of $\bar{H}$ corresponding to $B_1, B_2, \ldots, B_n$ respectively.

(2) $\bar{X} = \{\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}\}$ where $\bar{x_j} = \{(b_j, \mu_j^+(b_j), \mu_j^-(b_j)) | \mu_i^+(b_j) = \mu_j^+(x_i), \mu_i^-(b_j) = \mu_j^-(x_i)$.

Definition 5.10 *A bipolar fuzzy set $B = (\mu^+, \mu-)$ is called elementary bipolar fuzzy set if $\mu^+ : X \to [0, 1], \mu^- : X \to [-1, 0]$ are constant functions.*

Definition 5.11 *A bipolar fuzzy hypergraph ξ is called elementary bipolar fuzzy hypergraph if all bipolar fuzzy edges of ξ are elementary.*

In algebra, partial order is a binary relation " $\leq$ " over a set X which is reflexive, symmetric and transitive.

Definition 5.12 *Let $H = (X, \xi)$ be a bipolar fuzzy hypergraph and $C(H) = \{H_{(r_1,s_1)}, H_{(r_2,s_2)}, \ldots, H_{(r_k,s_k)}\}$. H is said to be ordered if $C(H)$ is ordered. The bipolar fuzzy hypergraph is simply ordered if $C(H)$ is simply ordered.*

Definition 5.13 *A bipolar fuzzy hypergraph $H = (X, \xi)$ is called $\{m^+, m^-\}$ tempered bipolar fuzzy hypergraph of a crisp hypergraph $H^* = (X, E)$ if there exists a bipolar fuzzy set $B = (m^+, m^-)$ such that $\xi = \{(\gamma_{E_i}^+, \gamma_{E_i}^-) | E_i \in E\}$ where*

$$\gamma_{E_i}^+(x) = \begin{cases} min\{m^+(e) | e \in E_i\} & \text{if } x \in E_i \\ 0, & \text{if otherwise} \end{cases}$$

and

$$\gamma^-_{E_i}(x) = \begin{cases} max\{m^-(e)|e \in E_i\} & if\ x \in E_i \\ 0, & if\ otherwise \end{cases}$$

Theorem 5.3 *A bipolar fuzzy hypergraph $H = (X, \xi)$ is a $\{m^+, m^-\}$ tempered bipolar fuzzy hypergraph of some crisp hypergraph H^* then H is elementary, support simple and simply ordered.*

Proof. Let $H = (X, \xi)$ is a $\{m^+, m^-\}$ tempered bipolar fuzzy hypergraph of some crisp hypergraph H^*. As it is $\{m^+, m^-\}$ tempered, the positive membership values and negative membership values of bipolar fuzzy edges of H are constant. Hence it is elementary. Clearly if support of two bipolar fuzzy edges of the $\{m^+, m^-\}$ tempered bipolar fuzzy hypergraph are equal then the bipolar fuzzy edges are equal. Hence it support simple. Let $C(H) = \{H_{(r_1,s_1)}, H_{(r_2,s_2)}, \ldots, H_{(r_k,s_k)}\}$ since H is elementary, it is ordered. Now we are to show that it is simple. Let $E \in H_{(r_{i+1},s_{i+1})} \setminus H_{(r_i,s_i)}$ then there exists $x^* \in E$ such that $\mu^+(x^*) = r_{i+1}$ and $\mu^-(x^*) = s_{i+1}$. Since $r_{i+1} < r_i, s_{i+1} > s_i$, it follows that $x^* \notin X_{(r_i,s_i)}$ and $E \nsubseteq X_{(r_i,s_i)}$. Hence H is simply ordered. □

Definition 5.14 *Let $H = (X, \xi)$ be a bipolar fuzzy hypergraph. A bipolar fuzzy transversal $T = (\tau^+, \tau^-)$ of H is a bipolar fuzzy set defined on X with the property that $T_{(h(B),d(B))} \cap B_{(h(B),d(B))} \neq \phi$ for each $B \in \xi$. A minimal bipolar fuzzy transversal T for H is a bipolar fuzzy transversal of H with the property that if $T_1 \subset T$, then T_1 is not a bipolar fuzzy transversal of H.*

Let M_H be an incidence matrix of a bipolar fuzzy hypergraph H.

$$M_H = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} (0.8, -0.7) & (0, 0) & (0.6, 0) \\ (0.5, -0.6) & (0.5, -0.3) & (0.4, -0.5) \\ (0.6, -0.4) & (0.4, -0.2) & (0.5, -0.4) \end{pmatrix} \end{array}$$

Here $B_1 = \{(x_1, 0.8, -0.7), (x_2, 0.5, -0.6), (x_3, 0.6, -0.4)\}$ is a minimal bipolar fuzzy transversal.

5.2 Irregular Bipolar Fuzzy Graphs

In this section, we define irregular bipolar fuzzy graphs and its various classifications. Size of regular bipolar fuzzy graphs is derived. The relation between highly and neighbourly irregular bipolar fuzzy graphs are established.

The *regular bipolar fuzzy graph* [3] is defined as below.

Definition 5.15 *Let $G = (A, B)$ be a bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V and $E \subseteq V \times V$ respectively. If $d^+(u) = k_1, d^-(u) = k_2$ for all $u \in V$, k_1, k_2 are two real numbers, then the graph is called (k_1, k_2)-regular bipolar fuzzy graph*

Theorem 5.4 *Let G be a regular bipolar fuzzy graph where induced crisp graph G' is an even cycle. Then G is regular bipolar fuzzy graph if and only if either m_2^+ and m_2^- are constant functions or alternate edges have same positive membership values and negative membership values. The proof is given below.*

Proof. Let $G = (A, B)$ be a regular bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V and $E \subseteq V \times V$ respectively and underlying crisp graph G' of G be an even cycle. If either m_2^+, m_2^- are constant functions or alternate edges have same positive and negative membership values, then G is a regular bipolar fuzzy graph. Conversely, suppose G is a (k_1, k_2)-regular bipolar fuzzy graph. Let $e_1, e_2, \ldots, e_n$ be the edges of G' in order. As in the theorem 3,

$$m_2^+(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd,} \\ k_1 - c_1, & \text{if } i \text{ is even.} \end{cases}$$

$$m_2^-(e_i) = \begin{cases} c_2, & \text{if } i \text{ is odd,} \\ k_2 - c_2, & \text{if } i \text{ is even.} \end{cases}$$

If $c_1 = k_1 - c_1$, then m_2^+ is constant. If $c_1 \neq k_1 - c_1$, then alternate edges have same positive and negative membership values. Similarly for m_2^-. Hence the results. □

Definition 5.16 *The size of a (k_1, k_2)-regular bipolar fuzzy graph is $(\frac{pk_1}{2}, \frac{pk_2}{2})$ where $p = |V|$.*

If G is (k, k')-totally regular bipolar fuzzy graph, then $2S(G) + O(G) = (pk, pk')$ where $p = |V|$.

Definition 5.17 *Let $G = (A, B)$ be a bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V and $E \subseteq V \times V$ respectively. G is said to be irregular bipolar fuzzy graph if there exists a vertex which is adjacent to a vertex with distinct degrees.*

Definition 5.18 *Let G be a connected bipolar fuzzy graph. Then G is called neighbourly irregular bipolar fuzzy graph if for every two adjacent vertices of G have distinct degrees.*

Definition 5.19 *Let $G = (A, B)$ be a bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V and $E \subseteq V \times V$ respectively. G is said to be totally irregular bipolar fuzzy graph [21] if there exists a vertex which is adjacent to a vertex with distinct total degrees.*

Definition 5.20 *Let G be a connected bipolar fuzzy graph. Then G is called neighbourly totally irregular bipolar fuzzy graph if for every two adjacent vertices of G have distinct total degrees.*

Definition 5.21 *Let G be a connected bipolar fuzzy graph. Then G is called highly irregular bipolar fuzzy graph if every vertex of G is adjacent to vertices with distinct degrees.*

Theorem 5.5 *Let G be a bipolar fuzzy graph. Then G is highly irregular bipolar fuzzy graph and neighbourly irregular bipolar fuzzy graph if and only if the degrees of all vertices of G are distinct.*

Theorem 5.6 *Let G be a bipolar fuzzy graph. If G is neighbourly irregular and m_1^+, m_1^- are constant functions, then G is a neighbourly total irregular bipolar fuzzy graph.*

Theorem 5.7 *Let G be a bipolar fuzzy graph. If G is neighbourly total irregular and m_1^+, m_1^- are constant functions, then G is a neighbourly irregular bipolar fuzzy graph.*

5.3 Bipolar fuzzy intersection graph

In this section bipolar fuzzy hypergraphs have been defined. We showed that any bipolar fuzzy graph can be expressed as the bipolar fuzzy intersection graphs of some bipolar fuzzy sets.

Definition 5.22 *[24] Let $F = \{B_1 = (m_1^+, m_1^-), B_2 = (m_2^+, m_2^-), B_3 = (m_3^+, m_3^-), \ldots, B_n = (m_n^+, m_n^-)\}$ be a finite family of bipolar fuzzy sets defined on a non empty set X. Consider F as crisp vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. The bipolar fuzzy intersection graph of F is the bipolar fuzzy graph (A_1, A_2) where $A_1 = (\mu_1^+, \mu_1^-)$ and $A_2 = (\mu_2^+, \mu_2^-)$. $\mu_1^+ : V \to [0, 1]$ defined by $\mu_1^+(v_i) = h(B_i)$ and $\mu_1^- : V \to [-1, 0]$ defined by $\mu_1^-(v_i) = d(B_i)$. $\mu_2^+ : V \times V \to [0, 1]$ defined by*

$$\mu_2^+(v_i, v_j) = \begin{cases} h(B_i \cap B_j), & \text{if } i \neq j \\ 0, & \text{if } i = j, \end{cases}$$

Similarly, $\mu_2^- : V \times V \to [-1, 0]$ defined by

$$\mu_2^-(v_i, v_j) = \begin{cases} d(B_i \cap B_j), & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

Theorem 5.8 *If $G = (A_1, A_2)$ is a bipolar fuzzy graph, then some family of bipolar fuzzy subsets F, $G = Int(F)$.*

Proof. Let $G = (A_1, A_2)$ be a bipolar fuzzy graph on V where two bipolar fuzzy subsets be $A_1 = (\mu_1^+, \mu_1^-)$ on V and $A_2 = (\mu_2^+, \mu_2^-)$ on $V \times V$. We define bipolar fuzzy subsets $L_x = (m_x^+, m_x^-)$ as

$$m_x^+(y,z) = \begin{cases} \mu_1^+(x) & \text{if } y = x, z = x, \\ \mu_2^+(x,z) & \text{if } y = x, z \neq x, \\ \mu_2^+(y,x) & \text{if } y \neq x, z = x, \\ 0 & \text{if } y \neq x \text{ and } z \neq x. \end{cases}$$

and

$$m_x^-(y,z) = \begin{cases} \mu_1^-(x) & \text{if } y = x, z = x, \\ \mu_2^-(x,z) & \text{if } y = x, z \neq x, \\ \mu_2^-(y,x) & \text{if } y \neq x, z = x, \\ 0 & \text{if } y \neq x \text{ and } z \neq x. \end{cases}$$

We want to show that G is the bipolar fuzzy intersection graph of $F = \{L_x | x \in V\}$. By definition $m^+(x,x) = \mu_1^+(x) \geq \mu_2^+(x,y)$ and $m^-(x,x) = \mu_1^-(x) \leq \mu_2^-(x,y)$. So $h(L_x) = \mu_1^+(x)$ and $d(L_x) = \mu_1^-(x)$. For $x \neq y$ a nonzero value of $m_x^+(z,w) \wedge m_y^+(z,w)$ occurs if and only if $x = z$ and $y = w$ (or $y = z, x = w$). Thus $h(L_x \cap L_y) = m_x^+(x,y) \wedge m_y^+(x,y) = \mu_2^+(x,y)$. Similarly $d(L_x \cap L_y) = m_x^-(x,y) \wedge m_y^-(x,y) = \mu_2^-(x,y)$. Hence the desired result holds. □

Chapter 6

Fuzzy Competition Graphs

This chapter is divided into three sections namely, fuzzy k-competition graphs, p-competition fuzzy graphs and m-step fuzzy competition graphs.

6.1 Fuzzy k-Competition Graphs

Fuzzy competition graph as the generalization of competition graph is introduced here. One generalization of fuzzy competition graph as fuzzy k-competition graph is defined. At first, we define the out- and in- neighbourhood of a vertex.

Definition 6.1 *Fuzzy out-neighbourhood of a vertex v of a directed fuzzy graph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is the fuzzy set $\mathcal{N}^+(v) = (X_v^+, m_v^+)$ where $X_v^+ = \{u|\overrightarrow{\mu}(v, u) > 0\}$ and $m_v^+ : X_v^+ \to [0, 1]$ defined by $m_v^+(u) = \overrightarrow{\mu}(v, u)$. Similarly, fuzzy in-neighbourhood of a vertex v of a directed fuzzy graph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is the fuzzy set $\mathcal{N}^-(v) = (X_v^-, m_v^-)$ where $X_v^- = \{u|\overrightarrow{\mu}(u, v) > 0\}$ and $m_v^- : X_v^- \to [0, 1]$ defined by $m_v^-(u) = \overrightarrow{\mu}(u, v)$.*

The height of a fuzzy set A is denoted by $h(A)$ and it is defined by $h(A) = max\{\mu(x)| x \in SuppA\}$. Now, we define fuzzy competition graph as follows.

Definition 6.2 *[18] The fuzzy competition graph $\mathcal{C}(\overrightarrow{\xi})$ of a fuzzy digraph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is an undirected fuzzy graph $\xi = (V, \sigma, \mu)$ which has the same fuzzy vertex set as in $\overrightarrow{\xi}$ and has a fuzzy edge between two vertices $x,\ y \in V$ in $\mathcal{C}(\overrightarrow{\xi})$ if and only if $\mathcal{N}^+(x) \cap \mathcal{N}^+(y)$ is non-empty fuzzy set in $\overrightarrow{\xi}$ and the edge membership value between x and y in $\mathcal{C}(\overrightarrow{\xi})$ is $\mu(x, y) = (\sigma(x) \wedge \sigma(y))h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$.*

The particular type of fuzzy competition graph which is called fuzzy k-competition graph, is defined below.

Definition 6.3 *[18] Let k be a non-negative number. The fuzzy k-competition graph [18] $\mathcal{C}_k(\overrightarrow{\xi})$ of a fuzzy digraph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is an undirected fuzzy graph $\xi = (V, \sigma, \mu)$ which has the same fuzzy vertex set as $\overrightarrow{\xi}$ and has a fuzzy edge between two vertices $x,\ y \in V$ in $\mathcal{C}_k(\overrightarrow{\xi})$ if and only if $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > k$. The edge membership value between x and y in $\mathcal{C}_k(\overrightarrow{\xi})$ is $\mu(x, y) = \frac{(k'-k)}{k'}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$ where $k' = |\mathcal{N}^+(x) \cap \mathcal{N}^+(y)|$.*

Theorem 6.1 *Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph. If $\mathcal{N}^+(x) \cap \mathcal{N}^+(y)$ contains single element of $\overrightarrow{\xi}$, then the edge (x, y) of $C(\overrightarrow{\xi})$ is strong if and only if $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > 0.5$.*

Proof. Here $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is a fuzzy digraph. Let $\mathcal{N}^+(x) \cap \mathcal{N}^+(y) = \{(a, m)\}$, where m is the membership value of the element a. Here, $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| = m = h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$. So, $\mu(x, y) = m \times \sigma(x) \wedge \sigma(y)$. Hence the edge (x, y) in $C(\overrightarrow{\xi})$ is strong if and only if $m > 0.5$.

If all the edges of a fuzzy digraph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be strong, then $\frac{\mu(x,y)}{(\sigma(x) \wedge \sigma(y))^2} > 0.5$ for any edge (x, y) in $C(\overrightarrow{\xi})$.

Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph. If $h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y)) = 1$ and $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > 2k$, then the edge (x, y) is strong in $C_k(\overrightarrow{\xi})$.

Proof. Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph. If $h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y)) = 1$ and $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| >$

$2k$, then the edge (x, y) is strong in $C_k(\overrightarrow{\xi})$. The proof of this fundamental result is established below. Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph and $C_k(\overrightarrow{\xi}) = (V, \sigma, \mu)$ be the corresponding fuzzy k-competition graph. Also let, $h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y)) = 1$ and $|\mathcal{N}^+(x) \cap \mathcal{N}^+(y)| > 2k$.

Now, $\mu(x, y) = \frac{k'-k}{k'}\sigma(x) \wedge \sigma(y)h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$, where $k' = |\mathcal{N}^+(x) \cap \mathcal{N}^+(y)|$. So, $\mu(x, y) = \frac{k'-k}{k'}\sigma(x) \wedge \sigma(y)$. Hence $\frac{\mu(x,y)}{\sigma(x)\wedge\sigma(y)} = \frac{k'-k}{k'} > 0.5$ as $k' > 2k$. Hence the edge (x, y) is strong. □

A special type of fuzzy graph based on fuzzy open neighbourhood is defined below.

Definition 6.4 *Fuzzy open neighbourhood of a vertex v of a fuzzy graph $\xi = (V, \sigma, \mu)$ is the fuzzy set $\mathcal{N}(v) = (X_v, m_v)$ where $X_v = \{u|\mu(v, u) > 0\}$ and $m_v : X_v \to [0, 1]$ defined by $m_v(u) = \mu(v, u)$. For each vertex $v \in V$, we define fuzzy singleton set, $A_v = (\{v\}, \sigma')$ such that $\sigma' : \{v\} \to [0, 1]$ defined by $\sigma'(v) = \sigma(v)$. Fuzzy closed neighbourhood of a vertex v is $\mathcal{N}[v] = \mathcal{N}(v) \cup A_v$.*

Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy open neighbourhood graph of ξ is a fuzzy graph $\mathcal{N}(\xi) = (V, \sigma, \mu')$ whose fuzzy vertex set is same as ξ and has a fuzzy edge between two vertices x and $y \in V$ in $\mathcal{N}(\xi)$ if and only if $\mathcal{N}(x) \cap \mathcal{N}(y)$ is non-empty fuzzy set in ξ and $\mu' : V \times V \to [0, 1]$ such that $\mu'(x, y) = [\sigma(x) \wedge \sigma(y)]h(\mathcal{N}(x) \cap \mathcal{N}(y))$.

Like open neighbourhood graph, fuzzy closed neighbourhood graph can be defined.

Definition 6.5 *Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy closed neighbourhood graph of ξ is a fuzzy graph $\mathcal{N}[\xi] = (V, \sigma, \mu')$ whose fuzzy vertex set is same as ξ and has a fuzzy edge between two vertices x and $y \in V$ in $\mathcal{N}[\xi]$ if and only if $\mathcal{N}[x] \cap \mathcal{N}[y]$ is non-empty fuzzy set in ξ and $\mu' : V \times V \to [0, 1]$ such that $\mu'(x, y) = [\sigma(x) \wedge \sigma(y)]h(\mathcal{N}[x] \cap \mathcal{N}[y])$.*

Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy (k)-neighbourhood graph (read as open fuzzy k-neighbourhood graph) of ξ is a fuzzy graph $\mathcal{N}_k(\xi) = (V, \sigma, \mu')$ whose vertex set is same as ξ and has an edge between two vertices x and $y \in V$ in $\mathcal{N}_k(\xi)$ if and only if $|\mathcal{N}(x) \cap \mathcal{N}(y)| > k$ in ξ and $\mu' : V \times V \to [0, 1]$ such that $\mu'(x, y) = \frac{(k'-k)}{k'}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}(x) \cap \mathcal{N}(y))$ where $k' = |\mathcal{N}(x) \cap \mathcal{N}(y)|$.

Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy $[k]$-neighbourhood graph (read as fuzzy closed k-neighbourhood graph) of ξ is a fuzzy graph $\mathcal{N}_k[\xi] = (V, \sigma, \mu')$ whose fuzzy vertex set is same as ξ and has a fuzzy edge between two vertices x and $y \in V$ in $\mathcal{N}_k[\xi]$ if and only if $|\mathcal{N}[x] \cap \mathcal{N}[y]| > k$ in ξ and $\mu' : V \times V \to [0, 1]$ such that $\mu'(x, y) = \frac{(k'-k)}{k'}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}[x] \cap \mathcal{N}[y])$ where $k' = |\mathcal{N}[x] \cap \mathcal{N}[y]|$.

Theorem 6.2 *For every edge of a fuzzy graph ξ, there exists one edge in $\mathcal{N}[\xi]$.*

Definition 6.6 *Let* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *be a fuzzy digraph. The underlying fuzzy graph of* $\overrightarrow{\xi}$ *is denoted by* $\mathcal{U}(\xi)$ *and is defined as* $\mathcal{U}(\xi) = (V, \sigma, \mu)$ *where* $\mu(u, v) = min\{\overrightarrow{\mu}(u, v), \overrightarrow{\mu}(v, u)\}$ *for all* $u, v \in V$.

Theorem 6.3 *If the symmetric fuzzy digraph* $\overrightarrow{\xi}$ *is loop less,* $\mathcal{C}_k(\overrightarrow{\xi}) = \mathcal{N}_k(\mathcal{U}(\xi))$ *where* $\mathcal{U}(\xi)$ *is the fuzzy graph underlying* $\overrightarrow{\xi}$.

Theorem 6.4 *If the symmetric fuzzy digraph* $\overrightarrow{\xi}$ *has loop at every vertex, then* $\mathcal{C}_k(\overrightarrow{\xi}) = \mathcal{N}_k[\mathcal{U}(\xi)]$ *where* $\mathcal{U}(\xi)$ *is the loop less fuzzy graph underlying* $\overrightarrow{\xi}$.

6.2 p-Competition Fuzzy Graphs

Here p-competition fuzzy graph is defined as another generalisation of competition graph.

Definition 6.7 *[18] Let* p *be a positive integer. The* p*-competition fuzzy graph* $\mathcal{C}^p(\overrightarrow{\xi})$ *of a fuzzy digraph* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *is an undirected fuzzy graph* $\xi = (V, \sigma, \mu)$ *which has same fuzzy vertex set as* $\overrightarrow{\xi}$ *and has a fuzzy edge between two vertices* x *and* $y \in V$ *in* $\mathcal{C}^p(\overrightarrow{\xi})$ *if and only if* $|supp(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))| \geq p$. *The edge membership value between* x *and* y *in* $\mathcal{C}^p(\overrightarrow{\xi})$ *is* $\mu(x, y) = \frac{(n-p)+1}{n}[\sigma(x) \wedge \sigma(y)]h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$ *where* $n = |supp(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))|$.

Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph. If $h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y)) = 1$ in $C^{[\frac{n}{2}]}(\overrightarrow{\xi})$, then the edge (x, y) is strong where $n = |supp(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))|$. (Note that for any real number x, $[x]$ = greatest integer not exceeding x). The statement can be proved as follows. Here $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is a fuzzy digraph. Let the corresponding $[\frac{n}{2}]$-fuzzy competition graph be $\xi = (V, \sigma, \mu)$ where $n = |supp(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))|$. Also assume that $h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y)) = 1, x, y \in V$. Now, $\mu(x, y) = \frac{n-[\frac{n}{2}]+1}{n}\sigma(x) \wedge \sigma(y)$. This gives the result, $\frac{\mu(x,y)}{\sigma(x)\wedge\sigma(y)} = \frac{n-[\frac{n}{2}]+1}{n} > 0.5$. Hence the edge (x, y) is strong.

6.3 m-Step Fuzzy Competition Graphs

Another generalization of fuzzy competition graph, called m-step fuzzy competition graph, is defined in this section. Some related fuzzy graphs including fuzzy m-step neighbourhood graph, fuzzy economic competition graphs and fuzzy m-step economic competition graphs are introduced. Some properties of these new graphs have been investigated.

Definition 6.8 *Let* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *be a fuzzy digraph. The* m*-step fuzzy digraph of* $\overrightarrow{\xi}$ *is denoted by* $\overrightarrow{\xi}_m = (\overrightarrow{\xi}, \sigma, \overrightarrow{\nu})$ *where fuzzy vertex set of* $\overrightarrow{\xi}$ *is same with the fuzzy vertex set of* $\overrightarrow{\xi}_m$ *and fuzzy edge* (x, y) *exists in* $\overrightarrow{\xi}_m$ *if there exists a fuzzy directed path* $\overrightarrow{P}^m_{(x,y)}$ *in* $\overrightarrow{\xi}$.

Definition 6.9 *Fuzzy m-step out-neighbourhood of a vertex v of a directed fuzzy graph* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *is the fuzzy set* $\mathcal{N}_m^+(v) = (X_v^+, \rho_v^+)$ *where*

$X_v^+ = \{u|$ *there exists a directed fuzzy path of length m from v to u,* $\overrightarrow{P}_{v,u}^m\}$ *and* $\rho_v^+ : X_v^+ \to [0,1]$ *defined by* $\rho_v^+(u) = min\{\overrightarrow{\mu}(x,y), (x,y)$ *is an edge of* $\overrightarrow{P}_{v,u}^m\}$.

Fuzzy m-step in-neighbourhood of a vertex y of a directed fuzzy graph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *is the fuzzy set* $\mathcal{N}_m^-(y) = (X_y^-, \rho_y^-)$ *where*

$X_y^- = \{x|$ *there exists a directed fuzzy path of length m from x to y,* $\overrightarrow{P}_{x,y}^m\}$ *and* $\rho_y^- : X_y^- \to [0,1]$ *defined by* $\rho_y^-(x) = min\{\overrightarrow{\mu}(u,v), (u,v)$ *is an edge of* $\overrightarrow{P}_{x,y}^m\}$.

Definition 6.10 *Let* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *be a fuzzy digraph. The m-step fuzzy competition graph [26] of* ξ *is denoted by* $C_m(\overrightarrow{\xi})$ *and defined by* $C_m(\overrightarrow{\xi}) = (V, \sigma, \nu)$ *where* $\nu : V \times V \to [0,1]$ *such that* $\nu(x,y) = \sigma(x) \wedge \sigma(y)\ h(\mathcal{N}_m^+(x) \cap \mathcal{N}_m^+(y))$ *for all* $x, y \in V$*. Here* $\nu(x,y), x, y \in V$ *represents the edge membership value of the edge* (x,y) *in m-step fuzzy competition graph.*

Definition 6.11 *Let* $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ *be a fuzzy digraph. Let v be a common vertex of m-step out-neighbourhoods of vertices* $x_1, x_2, \ldots, x_k$*. Also let* $\overrightarrow{\mu}(y_1, z_1), \overrightarrow{\mu}(y_2, z_2), \ldots, \overrightarrow{\mu}(y_k, z_k)$ *be the minimum membership values of edges of the paths* $\overrightarrow{P}_{(x_1,v)}^m, \overrightarrow{P}_{(x_2,v)}^m, \ldots, \overrightarrow{P}_{(x_n,v)}^m$ *respectively. The m-step prey* $v \in V$ *is strong prey if* $\overrightarrow{\mu}(y_i, z_i) > 0.5$ *for all* $i = 1, 2, \ldots, k$.

The strength of the prey v is measured from the function $s : V \to [0,1]$ such that

$$s(v) = \frac{\sum_{i=1}^{k} \overrightarrow{\mu}(y_i, z_i)}{k}.$$

It is proved that if a prey v of $\overrightarrow{\xi}$ is strong, then the strength of v, $s(v) > 0.5$. Also, if all preys of $\overrightarrow{\xi}$ are strong, then all the edges of $C_m(\overrightarrow{\xi})$ are strong.

Theorem 6.5 *If* $\overrightarrow{\xi}$ *is a fuzzy digraph and* $\overrightarrow{\xi}_m$ *is the m-step fuzzy digraph of* $\overrightarrow{\xi}$*, then* $C(\overrightarrow{\xi}_m) = C_m(\overrightarrow{\xi})$.

Proof. Here $\overrightarrow{\xi}$ is a fuzzy digraph and $\overrightarrow{\xi}_m$ is the m-step fuzzy digraph of $\overrightarrow{\xi}$. Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ and $\overrightarrow{\xi}_m = (V, \sigma, \overrightarrow{\nu})$. Also let $C(\overrightarrow{\xi}_m) = (V, \sigma, \nu)$ and $C_m(\overrightarrow{\xi}) = (V, \sigma, \mu)$. It can be easily verified that the fuzzy vertex sets of these graphs are same. So we have to show that the fuzzy edge sets of $C(\overrightarrow{\xi}_m)$ and $C_m(\overrightarrow{\xi})$ are equal. Let (x,y) be an edge in $C(\overrightarrow{\xi}_m)$. So, there exists fuzzy directed edges $\overrightarrow{(x,a_1)}, \overrightarrow{(y,a_1)}; \overrightarrow{(x,a_2)}, \overrightarrow{(y,a_2)}; \ldots \overrightarrow{(x,a_k)}, \overrightarrow{(y,a_k)}$ for some positive integer k in $\overrightarrow{\xi}_m$. Now, in $\overrightarrow{\xi}_m$, $N^+(x) \cap N^+(y) = \{(a_i, m_i) | i = 1, 2, \ldots, k\}$ where $m_i = \overrightarrow{\nu}(x, a_i) \wedge \overrightarrow{\nu}(y, a_i)$. Let $M = max\{m_i | i = 1, 2, \ldots, k\}$. Hence, $\nu(x,y) = \sigma(x) \wedge \sigma(y) h(N^+(x) \cap N^+(y)) = M \times \sigma(x) \wedge \sigma(y)$. An edge $\overrightarrow{(x,a_i)}$ exists in $\overrightarrow{\xi}_m$ that implies there exists a fuzzy directed path from x to a_i of length m, $\overrightarrow{P}^m_{(x,a_i)}$ in $\overrightarrow{\xi}$ and $\overrightarrow{\nu}(x,a_i) = min\{\overrightarrow{\mu}(u,v) \,|\, (u,v)$ is an edge in $\overrightarrow{P}^m_{(x,a_i)}\}$. Thus the edge (x,y) is also available in $C_m(\overrightarrow{\xi})$. Also, $h(N^+_m(x) \cap N^+_m(y)) = M$ in $\overrightarrow{\xi}$. Hence finally, $\mu(x,y) = \sigma(x) \wedge \sigma(y) h(N^+_m(x) \cap N^+_m(y)) = M \times \sigma(x) \wedge \sigma(y)$. This proves that there exists an edge in $C_m(\overrightarrow{\xi})$ for every edge in $C(\overrightarrow{\xi}_m)$. Similarly, for every edge in $C_m(\overrightarrow{\xi})$ there exists an edge in $C(\overrightarrow{\xi}_m)$. This proves that $C(\overrightarrow{\xi}_m) = C_m(\overrightarrow{\xi})$. □

Theorem 6.6 *Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be directed fuzzy graph. If $m > |V|$ then $C_m(\overrightarrow{\xi})$ has no edge.*

Definition 6.12 *Fuzzy m-step neighbourhood of a vertex v of the fuzzy graph $\xi = (V, \sigma, \mu)$ is the fuzzy set $\mathcal{N}_m(v) = (X_v, \rho_v)$ where*

$X_v = \{u|$ *there exists a fuzzy path of length m from v to u, $P^m_{v,u}\}$ and $\rho_v : X_v \to [0,1]$ is defined by $\rho_v(u) = min\{\mu(x,y), (x,y)$ is an edge of $P^m_{v,u}\}$.*

Definition 6.13 *Let $\xi = (V, \sigma, \mu)$ be a fuzzy graph. The m-step fuzzy neighbourhood graph is denoted by $N_m(\xi)$ and defined by $N_m(\xi) = (V, \sigma, \eta)$ where $\eta : V \times V \to [0,1]$ such that $\eta(x,y) = \sigma(x) \wedge \sigma(y)\ h(\mathcal{N}_m(x) \cap \mathcal{N}_m(y))$ for all $x, y \in V$. Here $\eta(x,y)$ represents the membership value of edge (x,y) of $N_m(\xi)$.*

Theorem 6.7 *If a fuzzy digraph $\overrightarrow{\xi}$ does not contain any parallel edge, then $C_m(\overrightarrow{\xi}) = N_m(\xi)$ for $m > 1$ where ξ is the underlying fuzzy graph of $\overrightarrow{\xi}$.*

Definition 6.14 *The fuzzy economic competition graph $E(\overrightarrow{\xi})$ of a fuzzy digraph $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ is an undirected fuzzy graph $\xi = (V, \sigma, \phi)$ which has the same fuzzy vertex set as in $\overrightarrow{\xi}$ and has a fuzzy edge between two vertices x, $y \in V$ in $E(\overrightarrow{\xi})$ if and only if $\mathcal{N}^-(x) \cap \mathcal{N}^-(y)$ is non-empty fuzzy set in $\overrightarrow{\xi}$ and the membership value of the edge (x,y) in $\mathcal{C}(\overrightarrow{\xi})$ is $\phi(x,y) = (\sigma(x) \wedge \sigma(y))\ h(\mathcal{N}^-(x) \cap \mathcal{N}^-(y))$.*

Definition 6.15 *Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph. The m-step fuzzy economic competition graph is denoted by $E_m(\overrightarrow{\xi})$ and is defined by $E_m(\overrightarrow{\xi}) = (V, \sigma, \phi)$ where $\phi : V \times V \to [0,1]$ such that $\phi(x,y) = \sigma(x) \wedge \sigma(y)\ h(\mathcal{N}^-_m(x) \cap \mathcal{N}^-_m(y))$ for all $x, y \in V$. Here $\phi(x,y)$ represents the membership value of the edge (x,y) of m-step fuzzy economic competition graph.*

Theorem 6.8 *Fuzzy competition graphs and fuzzy economic competition graphs of any complete fuzzy digraph are same.*

Proof. Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ be a fuzzy digraph and corresponding fuzzy competition graph be $\mathcal{C}(\overrightarrow{\xi}) = (V, \sigma, \mu)$ and the corresponding fuzzy economic graph be $E_m(\overrightarrow{\xi}) = (V, \sigma, \phi)$.

Here we have to show that $\mu(u,v) = \phi(u,v)$ for all $u,v \in V$. Now, $\mu(x,y) = (\sigma(x) \wedge \sigma(y))\ h(\mathcal{N}^+(x) \cap \mathcal{N}^+(y))$ and $\phi(x,y) = \sigma(x) \wedge \sigma(y)\ h(\mathcal{N}_m^-(x) \cap \mathcal{N}_m^-(y))$. As $\overrightarrow{\xi}$ is fuzzy complete digraph, $\mathcal{N}^+(x) \cap \mathcal{N}^+(y) = \mathcal{N}_m^-(x) \cap \mathcal{N}_m^-(y)$. Hence the result. □

Theorem 6.9 *If $\overrightarrow{\xi_1}$ is the fuzzy sub-digraph of $\overrightarrow{\xi}$. Then*
(i) $\mathcal{C}_m(\overrightarrow{\xi_1}) \subset \mathcal{C}_m(\overrightarrow{\xi})$.
(ii) $E_m(\overrightarrow{\xi_1}) \subset E_m(\overrightarrow{\xi})$.

Proof. Let $\overrightarrow{\xi} = (V, \sigma, \overrightarrow{\mu})$ and $\overrightarrow{\xi_1} = (V_1, \sigma_1, \overrightarrow{\mu_1})$ where $V_1 \subset V$, $\sigma_1(x) \leq \sigma(x)$ for all $x \in V_1$ and $\overrightarrow{\mu_1}(u,v) \leq \overrightarrow{\mu}(u,v)$ for all $u,v \in V_1$.

(i) The fuzzy vertex set of $\mathcal{C}_m(\overrightarrow{\xi_1})$ is the subset of $\mathcal{C}_m(\overrightarrow{\xi})$ as $V_1 \subset V$. Now for any fuzzy edge (u,v) in $\mathcal{C}_m(\overrightarrow{\xi_1})$, $\mathcal{N}_m^+(u) \cap \mathcal{N}_m^+(v)$ is fuzzy subset of the same in $\mathcal{C}_m(\overrightarrow{\xi})$. So $\overrightarrow{\mu_1}(u,v) \leq \overrightarrow{\mu}(u,v)$ for all $u,v \in V_1$. Hence the result.

(ii) The proof is similar to (i). □

Chapter 7

Application of Fuzzy Graphs

Applications of fuzzy graph are shown in two fields. In first section of this chapter a social network is formed with the help of fuzzy graphs. In last section, the application of fuzzy graph is described in telecommunication.

7.1 A New Approach to Social Networks Based on Fuzzy Graphs

Social networks are the areas in which a huge number of people are connected. In this section, a new social network called fuzzy social network (FSN) has been introduced based on fuzzy graph. For this network, centrality, single and multi-bridges and transfer value of the bridges are newly defined and illustrated by examples. In this network, the strength of relationship can be graded by different values between 0 and 1, and we have shown that this representation is more realistic. Also, we have introduced a new concept of registration for a new user so that the chance of fake user may be reduced.

Here we assume all social actors as fuzzy vertices and all linkages as fuzzy edges. Moreover, linkages may be of different kinds. So we represent the social network as fuzzy multigraph (V, σ, E) where $V = \{P_1, P_2, \ldots, P_\lambda\}$, λ is a large positive integer, be a set of social units. $\sigma : V \to [0,1]$ be a mapping and $E = \{((P_i, P_j), \mu^r(P_i, P_j)), r = 1, 2, \ldots, g | (P_i, P_j) \in V \times V\}$

be a fuzzy multiset of $V \times V$ such that $\mu^r(P_i, P_j) \leq \sigma(P_i) \wedge \sigma(P_j)$ for all $P_i, P_j \in V$ and for all $r = 1, 2, \ldots, g$ where $g = max\{r | \mu^r(P_i, P_j) \neq 0\}$.

Without loss of generality, we assume that all linkages are of same kind. In this case, unit membership value of a social unit is same as unit membership value of a social unit, represented in fuzzy multi-graph representation but link membership values are little different. Then the FSN can be represented by a fuzzy graph $FSN = (V, \sigma, \mu)$ where $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ are mappings such that $\mu(P_i, P_j) \leq \sigma(P_i) \wedge \sigma(P_j)$ for all $P_i, P_j \in V$.

Centrality is one of the most studied concepts in social network analysis. Numerous measures have been developed, including degree centrality, closeness, betweenness, eigenvector centrality, information centrality, the rush index, etc.

Degree centrality measures the direct linkage of a social unit to the others, betweenness centrality measures the number of paths between any pair of social units through the social unit. In all kinds of networks, central persons are more valuable than others. They can send messages to more people, can collect more information. Degree centrality measures the number of direct friends (or linkages). So it does not measure the number of connected people by a path. It may be noted that friend of a friend in Facebook shares information to the person. So betweenness centrality is also useful. But friend of friend does not share so much information like the direct friend. So importance of the linkage will gradually decrease from a person to another person by a connected path.

In FSN, if a unit P_f is directly connected with the unit P, then we say that P_f is distance-1 friend of P. The set of all distance-1 friends of P be denoted by $d_1(P)$. That is,

$$d_1(P) = \{P_i \in V : P_i \text{ is a distance-1 friend of } P\}.$$

Similarly, if there is a shortest path (i.e. minimum number of links) between P and P_f containing k egdes or links, then P_f is a distance-k friend of P. That is,

$$d_k(P) = \{P_i \in V : P_i \text{ is a distance-}k \text{ friend of } P\}.$$

Now, let $d'_k(P) = d_k(P) - d'_{k-1}(P)$, where $k = 2, 3, \ldots$ and $d'_1(P) = d_1(P)$ (Note that for classical sets A, B, $A - B = \{x \in A \text{ and } x \notin B\}$).

It is natural that the distance-1 friends are more important than distance-2 friends, distance-2 friends are more important than distance-3 friends, and so on. The linguistic term "more important" can be represented by weights. Let w_k, $0 \leq w_k \leq 1$ be the weight which represents the importance between the distance-k friends. The weights gradually decreases if the distance between the friends increases. Thus $w_1 \geq w_2 \geq \ldots \geq w_k \geq \ldots$

Let $u_1(=P_i), u_2, u_3, \ldots, u_k(=P_j)$ be the vertices on the path between P_i and P_j. We define fuzzy distance $D_f(P_i, P_j)$ between P_i and P_j along this path as

$$D_f(P_i, P_j) = \sum_{l=1}^{k-1} \mu(u_l, u_{l+1}).$$

In a network, it may be observed that there are multiple paths between two vertices. In FSN, we consider those paths of same length whose fuzzy distance D_f is maximum. If there are k edges in this path of maximum fuzzy distance, then we denote this distance by D_f^k i.e, $D_f^k(P_i, P_j)$ represents the fuzzy distance between the vertices P_i and P_j in FSN along a certain path containing exactly k edges. We assumed that a social unit P has at most distance-p friends.

Now we define the *centrality* [25] $C(P)$ of a social unit P of FSN as follows

$$C(P) = \sum_{u_1 \in d_1'(P)} w_1 D_f^1(P, u_1) + \sum_{u_2 \in d_2'(P)} w_2 D_f^2(P, u_2) + \ldots + \sum_{u_p \in d_p'(P)} w_p D_f^p(P, u_p).$$

In this measurement, the importance of close friend is given more than the next to close friend and gradually decreases for the furthest friend. The importance are introduced by incorporating the weight w_i, for distance-i friend, $i = 1, 2, 3, \ldots$

If a broker or structural hole is strongly connected to two groups of people, connected to each other, the transfer of information will be very speedy. But, if the connection is weak, information flow will not be fast. So not only broker or bridge is an important unit in a social network, but also its strength. Depending on the strength, a bridge can be classified as strong and weak.

If the link membership values of all the links in a bridge are greater than 0.5, then the bridge is called strong bridge. On the other hand, if link membership value of at least one link is less than or equal to 0.5, then the bridge is called weak bridge.

Let a bridge B be constructed by social units $P_i, i = 1, 2, \ldots, z$ between two groups of people A, connected with P_1 and B, connected with P_z. Let the link membership value between P_i and P_j be $\mu(P_i, P_j)$. Let minimum of the link membership values of the links in B be m. The transfer value of B be denoted by $T(B)$ and defined by the value $T(B) = m \times C(P_1) \wedge C(P_z)$ where $C(P)$ denotes the centrality of a person in FSN. For multi-bridge the transfer value is defined by the sum of the transfer values of each bridges.

7.2 Telecommunication System Based on Fuzzy Graphs

Telecommunication is one of the unavoidable utilities of daily life. Telecommunication service providers have data base of their users. These records are represented by crisp graphs. Certain parameters like center persons, churn prediction, etc. are more perfectly calculated if the data are represented by fuzzy graphs. In this section, we have introduced a fuzzy telecommunication network (FTN) using fuzzy graph theory. Churn prediction is a big issue for telecom service providers. New idea on measurement of churn prediction is presented here.

Let $V_1 = \{C_1, C_2, \dots, C_\lambda\}$, where λ is very large integer, be the set of all registered customers in the telecommunication network FTN and $V_2 = \{C_{\lambda+1}, C_{\lambda+2}, \dots, C_\varphi\}$ be the out side customers connected to the members of FTN. Let $V = V_1 \cup V_2$. The membership values of the customers are given by $\phi : V \to [0,1]$ and the membership values of the links between the customers is given by $\overrightarrow{\mu} : V \times V \to [0,1]$. Then the telecommunication system is represented by a directed fuzzy graph $\overrightarrow{\xi} = (V, \phi, \overrightarrow{\mu})$.

The underlying fuzzy graph of $\overrightarrow{\xi}$ is denoted by $\xi = (V, \sigma, \mu)$, where $\mu : V \times V \to [0,1]$ such that $\mu(a,b) = \frac{\overrightarrow{\mu}(a,b)+\overrightarrow{\mu}(b,a)}{2}$ for all $a, b \in V$.

In a telecommunication network, a person is called star person if his/her number of friends is more than M, where M is an integer, decided by the service provider.

In general, friend (F_2) of friend (F_1) is important to a person (P) in telecommunication if the friend (F_2) of friend is star person in the network. We now define centrality of a person in the telecommunication system FTN below.

In FTN, if a unit P_f is directly connected with the unit P, then we say that P_f is distance-1 friend of P. The set of all distance-1 friends of P be denoted by $d_1(P)$. That is,

$$d_1(P) = \{P_i \in V : P_i \text{ is a distance-1 friend of } P\}.$$

If there is a shortest path (i.e. minimum number of links) between P and $P_f^\star$, a star customer, containing k egdes or links, then $P_f^\star$ is a distance-k star friend of P. Let

$$d_k^\star(P) = \{P_i \in V : P_i \text{ is a distance-}k \text{ star friend of } P\}.$$

Now, we define $d_k^{\star\prime}(P) = d_k^\star(P) - d_{k-1}^{\star\prime}(P)$, where $k = 3, 3, \dots$ and $d_2^{\star\prime}(P) = d_2^\star(P)$ (Note that for classical sets A, B, $A - B = \{x \in A \text{ and } x \notin B\}$).

It is natural that the distance-2 star friends are more important than distance-3 star friends, distance-3 star friends are more important than distance-4 star friends, and so on. The linguistic term "more important" can be represented by weights. Let w_k, $0 \leq w_k \leq 1$ be the weight which represents the importance between the distance-k friends. The weights gradually decreases if the distance between the friends increases. Thus $w_1 \geq w_2 \geq \ldots \geq w_k \geq \ldots$

Let $u_1(= P_i), u_2, u_3, \ldots, u_k(= P_j)$ be the vertices on the path between P_i and P_j. We define fuzzy distance $D_f(P_i, P_j)$ between P_i and P_j along this path as

$$D_f(P_i, P_j) = \sum_{l=1}^{k-1} \mu(u_l, u_{l+1}).$$

In a network, it may be observed that there are multiple paths between two vertices. In FTN, we consider those paths of same length whose fuzzy distance D_f is maximum. If there are k edges in this path of maximum fuzzy distance, then we denote this distance by D_f^k i.e, $D_f^k(P_i, P_j)$ represents the fuzzy distance between the vertices P_i and P_j in FTN along a certain path containing exactly k edges. For simplicity, we consider the friends of a customer of distance p, i.e, we take upto distance-p friends.

Now we define the centrality $C(P)$ of a social unit P of FTN as follows

$$C(P) = \sum_{u_1 \in d_1'(P)} w_1 D_f^1(P, u_1) + \sum_{u_2 \in d_2^{\star\prime}(P)} w_2 D_f^2(P, u_2) + \ldots + \sum_{u_p \in d_p^{\star\prime}(P)} w_p D_f^p(P, u_p).$$

In this measurement, the importance of close friend is given more than the next to close friend and gradually decreases the furthest friend. The importance are introduced by incorporating the weight w_i, for distance-i friend, $i = 1, 2, 3, \ldots$.

In fuzzy social network (FSN), centrality of a person was defined as the weighted sum of fuzzy distances of connected persons along certain paths. Here centrality of a customer in FTN is the weighted sum of fuzzy distances of directly connected customers and star customers connected by a certain path.

Churn of customers in telecommunication system is a big problem for service providers. Churn problem occurs in prepaid mobile system mostly. So making a list of churning persons is an important task for service providers. Now, people feel luxury with many sim cards. So better offers from any of telecom service providers are accepted easily by people. Besides portability is easy now. So people change their mobile service provider due to minor causes.

We are aware of the fact that calling within same service provider has more facilities. So if strong persons decide to change their mobile service providers, then sometimes their followers do the same. Besides, out going or incoming calls of a phone number measure the stability in the network. If a person's outgoing calls increase or remain the same compared to the previous interval of time, then the service providers have nothing to worry.

Similarly, one of other factors which indicates the activities of a customer is number of distinct phone numbers to which the customer is connected. If the number of connected customers in particular interval of time is rapidly decreases, then the customer may be churned in future.

If the number of outgoing calls per unit interval of time decrease, we calculate the decrease rate as

$$D_O = \frac{\text{Reduction of time of out going calls from previous interval of time}}{\text{Total time of outgoing calls in previous interval of time}}$$

If the number of friends decrease, then the rate of decrease is denoted as D_F and is defined as

$$D_F = \frac{\text{Reduction of number of friends from previous interval of time}}{\text{Total number of friends in previous interval of time}}$$

If the number of incoming calls decrease, decrease rate is denoted as D_I and is defined as

$$D_I = \frac{\text{Reduction of time of incoming calls from previous interval of time}}{\text{Total time of incomming calls in previous interval of time}}$$

The *measure of churn prediction* [27] of a customer P is denoted by $\chi(P)$ and defined by $\chi(P) = \frac{w_1 D_F + w_2 D_O + w_3 D_I}{w_1 + w_2 + w_3}$ where w_1, w_2, w_3 represent the weights associated with the significance of D_O, D_F and D_I. Generally $w_1 \geq w_2 \geq w_3$ as reduction of calling time is more significant than that of D_F and D_I. Note that value of $\chi(P)$ lies between 0 and 1. If this value of a customer is nearer to 1, the customer is going to churn. Similarly, if the value is less than 0.5, the service provider is nothing to worry.

Chapter 8

Conclusion

In this chapter, a conclusion is drawn about the several fuzzy graphs which are discussed in the thesis. In the literature of graph theory, there are several concepts of strong edges. We have used one concept of strong edge. This concept is independent as it does not depend on the membership values as well as strengths of other edges. But, other concepts can be used in fuzzy graph theory which are developed here. Bipolar fuzzy graphs can be used in different real field problems. Fuzzy competition graphs can be used in different ecological systems. In future, we will prepare a model in ecology based on fuzzy competition graphs. The applications of fuzzy graphs are shown in image contraction, social networks and telecommunications. Fuzzy graph theory can be applied in large number of branches. In future, we use fuzzy graph theory in traffic light problems, scheduling problems, etc.

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